Fundamentals of Low-Noise Electronic Analysis and Design

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Contents

Preface

This book is the outgrowth of a graduate level course in low-noise electronic design that the author has taught at the Georgia Institute of Technology. It is an introductory level text that covers the basic mathematics of noise analysis, the types of electronic noise, the mathematical representation of noise signals, noise models of amplifiers, noise specifications, and noise analysis of circuits containing diodes, bipolar junction transistors, and field effect transistors.

Noise is not a topic that is covered in much depth in most electronic texts. For this reason, engineers who have not had training in noise often have the mistaken impression that noise analysis of circuits is an esoteric area that is based on strange formulas having obscure origins. Indeed, technical papers on noise often foster this impression because they are so difficult to understand if one does not have an understanding of the fundamentals. One of the objectives of this book is to describe these fundamentals.

Because noise is random, it must be described by statistical methods. Chapter 1 reviews the concept of a probability density function and how it is used to describe the amplitude distribution of random noise voltage and current signals. The concept of a spectral density is introduced as a way to describe the distribution of noise in the frequency domain. The key to the spectral density is the autocorrelation function and its relation to the spectral density is described.

Chapter 2 describes the types of random noise that occur in electronic systems. These are thermal noise, shot noise, flicker noise, and burst noise. A description of the physical origin of each is given. Both the Nyquist formula for the thermal noise voltage generated by a resistor and the Schottky formula for shot noise in a current are derived.

The time domain and phasor representations of noise signals and how signals from more than one source are combined are covered in Chapter 3. The important concept of noise bandwidth is defined and some of the pitfalls in measuring noise are described.

The basic amplifier noise model is described in Chapter 4. The equivalent noise input voltage and current are defined. Applications of the model to the noise analysis of multi-stage amplifiers and op amps are described. The effects on noise of parallel input devices and an input transformer are investigated.

Although the basic amplifier noise models of Chapter 4 are applicable to rf amplifiers, there are special considerations when transmission lines are employed. Chapter 5 contains a review of transmission line theory and the Smith chart. The signal-flow graph is introduced as an analysis tool for transmission line problems. The s-parameter amplifier model is covered and the amplifier noise sources are incorporated into it. Fundamentals of rf matching networks are covered.

Chapter 6 covers amplifier noise specifications. This includes signal-to-noise ratio, noise factor, noise figure, and noise temperature. The effect of a matching network on noise is investigated. The concept of Smith chart noise circles and gain circles for rf design are covered. Methods of measuring the noise factor are described.

Chapters 7, 8, and 9 cover intrinsic noise sources in the diode, the bjt, the jfet, and the mosfet. The device equations and small-signal models of each are reviewed. The input noise voltage and current are derived for the bjt, the jfet, and the mosfet. Methods of designing for minimum noise with each device are described.

Chapter 10 covers noise in feedback amplifiers. The effect of feedback on noise is investigated. Conditions that the feedback network not increase noise are derived.

Example problems are presented in each chapter to illustrate applications of the theory.

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Chapter 1

Noise as a Random Process

Noise is any unwanted signal that interferes with a desired signal. Most electronic noise can be categorized as being either random or deterministic. An example of deterministic noise is the hum in a loudspeaker caused by an ac ripple on a power supply voltage in an amplifier. Another example is the interference with television reception caused by a nearby short wave transmitter. Deterministic noise is caused by an identifiable external source and can usually be eliminated by proper methods of grounding, shielding, etc. The control of such noise is a topic of the discipline known as electromagnetic compatibility.

Random noise is generated by almost everything in nature. In video signals, noise appears as snow on the screen of a television set. In audio signals, noise can be heard as a background hiss. In electronic circuits, noise is controlled by careful design. Because it cannot be described by any mathematical function, it must be described by probability and statistics. Some of the important ideas of these disciplines that are applicable to noise analysis are described in this chapter.

1.1 Notation and Numerical Values of Constants

The notations used in this text for voltages and currents correspond to the following conventions: dc bias values are indicated by an upper case letter with upper case subscripts, e.g. V_{DS}, I_C. Instantaneous values of small-signal variables are indicated by a lower-case letter with lower-case subscripts, e.g. v_s, i_c. Total values, i.e. dc bias values plus small-signal ac values, are indicated by a lower-case letter with upper-case subscripts, e.g. v_S, i_C. Mean-square values of small-signal variables are represented by a bar over the square of the variable, e.g. $\overline{v_s^2}$, $\overline{i_c^2}$, where the bar indicates an arithmetic average of a statistical ensemble of functions. The time average of a small-signal variable is represented by enclosing the variable with the $\langle\cdot\rangle$ symbols, e.g. $\langle v_s^2\rangle$, $\langle i_c^2\rangle$. For an ergodic process, time averages are equal to ensemble averages, i.e. $\langle v_s^2\rangle = \overline{v_s^2}$. The root-mean square or rms value of a variable is the square root of the mean-square value, e.g. $\sqrt{\overline{v_s^2}}$, $\sqrt{\overline{i_c^2}}$. Phasors are indicated by an upper-case letter and lower-case subscripts, e.g. V_s, I_c.

Circuit symbols for independent sources are circular, symbols for controlled sources have a diamond shape, and symbols for noise sources are square. Voltage sources have a $\pm$ sign within the symbol and current sources have an arrow. In the numerical evaluation of noise equations, Boltzmann's constant is $k = 1.38 \times 10^{-23}$ J/K and the electronic charge is $q = 1.60 \times 10^{-19}$ C. The standard temperature is denoted by T_0 and is taken to be $T_0 = 290^{\text{o}}$ K. For this value, $4kT_0 = 1.60 \times 10^{-20}$ and the thermal voltage is $V_T = kT_0/q = 25.0$ mV.

1.2 The Probability Density Function

A deterministic process is one whose values are known for all time. For example, the voltage $v(t) = V_1 + V_2 \sin\omega t$, where V_1, V_2, and ω are constants, is a deterministic process. By definition, random noise is non-deterministic. It cannot be represented by any mathematical function. For this reason, it must be characterized by a probability density function. To illustrate how such a function is defined, let x represent a random variable. For example, x might be the measured value of a voltage or a current. Suppose that N measurements are made. Let us divide the range of observed values into subintervals of width Δx and count the number of times that a measured value occurred in each. The probability density function $p(x)$ is defined by the limit

$$p(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(\lim_{N \to \infty} \frac{\#\text{ in the interval } [x, x + \Delta x]}{N} \right) \tag{1.1}$$

It follows from this definition that $p(x)$ is a positive real function.

From the above definition, we can conclude that the fraction of total measurements that lie in the range from x_1 to x_2 is the area under the curve of $p(x)$ between these values. Alternately, this area represents the probability that a single observed value of x lies in this range. Symbolically, we write

$$\text{Prob}\,(x_1 \le x \le x_2) = \int_{x_1}^{x_2} p\,(x)\,dx \tag{1.2}$$

Because x must satisfy $-\infty < x < \infty$, it follows that $p\,(x)$ must satisfy

$$\int_{-\infty}^{\infty} p\,(x)\,dx = 1 \tag{1.3}$$

In other words, it is certain that x falls in the range from $-\infty$ to ∞.

The distribution function, or integrated probability density function, $P\,(x)$ is defined by

$$P\,(x) = \int_{-\infty}^{x} p\,(x)\,dx \tag{1.4}$$

This function gives the probability that a single observed value lies in the range from $-\infty$ to x. From this definition, it follows that the probability that a single observed value lies in the range from x_1 to x_2 is

$$\text{Prob}\,(x_1 \le x \le x_2) = P\,(x_2) - P\,(x_1) \tag{1.5}$$

In addition, $P\,(x)$ has the properties

$$P\,(-\infty) = 0 \tag{1.6}$$

$$P\,(\infty) = 1 \tag{1.7}$$

$$\frac{d}{dx} P\,(x) = p\,(x) \tag{1.8}$$

1.3 Statistical Averages

There are two basic ways that the average or mean value of a random variable can be defined. One is called a time average. The other is called an ensemble average. The two are equal if the process which generates the random variable is an ergodic process.

1.3.1 Time Averages

The definition of a time average is based on the definition of the average value of a function in calculus. If $x\,(t)$ is a function of time, then the time average is denoted by $\langle x \rangle$ and is given by

$$\langle x \rangle = \frac{1}{T} \int_0^T x\,(t)\,dt \tag{1.9}$$

where T is called the averaging time. It must be chosen long enough so that the result would not be significantly changed if T is increased. If x is a voltage or a current, then $\langle x\rangle$ is simply its dc component. When this expression is used to solve for the average value of a periodic signal, T is usually chosen to be the period of the signal.

Example 1 *A voltage is given by $v(t) = V_1 \sin(2\pi t/T)$, where V_1 is a positive real constant. Calculate the average value if $v(t)$ is full-wave rectified and if $v(t)$ is half-wave rectified.*

Solution. The average value of the full-wave rectified voltage is calculated as follows:

$$\langle |v(t)|\rangle = \frac{1}{T}\int_0^T |V_1 \sin(2\pi t/T)|\, dt = \frac{V_1}{2\pi}\int_0^{2\pi} |\sin\theta|\, d\theta = \frac{2V_1}{\pi}$$

The average value of the half-wave rectified voltage is 1/2 of this or V_1/π.

Example 2 *Solve for $\left\langle v^2(t)\right\rangle$, where $v(t)$ is the voltage given in Example 1.*

Solution. The average value of the squared voltage is calculated as follows:

$$\left\langle v^2(t)\right\rangle = \frac{1}{T}\int_0^T V_1^2 \sin^2(2\pi t/T)\, dt = \frac{V_1^2}{2\pi}\int_0^{2\pi} \sin^2\theta d\theta = \frac{V_1^2}{2}$$

It follows from this result that the rms value of $v(t)$ is $V_1/\sqrt{2}$.

1.3.2 Ensemble Averages

To illustrate the definition of an ensemble average, suppose that we have N sources of a random variable. We assume that the sources are identical in that each is governed by the same stationary process. Thus the output of each source is described by the same probability density function. In addition, we assume that the sources are independent in the sense that the value of the output of one does not depend on the value of the output of any other. Let the output of all N sources be observed at the same time t_1. If x_i is the output of the ith source, the ensemble average of x at $t = t_1$ is defined by

$$\overline{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \tag{1.10}$$

We assume that N is sufficiently large so that the result does not significantly change if N is increased. If $\overline{x}$ is independent of the time at which the N observations are made, the random process is said to be stationary.

For a stationary random process, the average value is not a function of the time at which it is calculated. That is, the average is the same if it is repeated an hour later, a day later, or a week later. We will assume here that all random processes are stationary. Although it is difficult to postulate any process that is truly stationary for all past time and all future time, the concept of stationarity over a time interval of interest in a system is appropriate here.

A random process is said to be ergodic if the time average given by Eq. (1.9) can be replaced with the ensemble average given by Eq. (1.10). For our purposes, we will assume that all random processes used to model electronic noise are ergodic. Although the mathematical details of ergodic processes are too involved to be investigated here, it can be shown that an ergodic process is stationary but a stationary process is not necessarily ergodic.

The ensemble average in Eq. (1.10) can be calculated from the probability density function for x. Out of N observed values, the number of observations that lie in the interval from x to $x + \Delta x$ is $Np(x)\Delta x$, where we assume that Δx is small. We multiply this number by x to obtain the contribution of these values to the summation. Thus for each interval of width Δx, the contribution to the summation is $Nxp(x)\Delta x$. To complete the summation, we sum over all x and let $\Delta x \to 0$ to obtain an integral over x. It follows that the ensemble average is given by

$$\overline{x} = \frac{1}{N}\left[\int_{-\infty}^{\infty} Nxp(x)\,dx\right] = \int_{-\infty}^{\infty} xp(x)\,dx \qquad (1.11)$$

This equation can be generalized to obtain the ensemble average of any function of a random variable. If $f(x)$ is a function of x, the ensemble average of $f(x)$ is given by

$$\overline{f(x)} = \int_{-\infty}^{\infty} f(x)\,p(x)\,dx \qquad (1.12)$$

Example 3 *Calculate the mean square value of the function* $v(t) = V_1 \cos\omega_1 t + V_2 \cos\omega_2 t$, *where* $\omega_1 \neq \omega_2$.

Solution. First, we start with the ensemble average of v^2 to obtain

$$\begin{aligned}\overline{v^2} &= \overline{\left(V_1 \cos \omega_1 t + V_2 \cos \omega_2 t\right)^2} \\ &= \overline{V_1^2 \cos^2 \omega_1 t + 2V_1 V_2 \cos \omega_1 t \cos \omega_2 t + V_2^2 \cos^2 \omega_2 t}\end{aligned}$$

The ensemble average operation obeys the distributive law so that $\overline{v^2}$ can be written

$$\begin{aligned}\overline{v^2} &= V_1^2 \overline{\cos^2 \omega_1 t} + 2V_1 V_2 \overline{\cos \omega_1 t \cos \omega_2 t} + V_2^2 \overline{\cos^2 \omega_2 t} \\ &= V_1^2 \overline{\cos^2 \omega_1 t} + V_1 V_2 \left[\overline{\cos\left(\omega_1 - \omega_2\right) t} + \overline{\cos\left(\omega_1 + \omega_2\right) t}\right] \\ &\quad + V_2^2 \overline{\cos^2 \omega_2 t}\end{aligned}$$

Because a deterministic process is ergodic, the four ensemble averages in the above expression can be replaced with time averages. The time average of each term is evaluated over its respective period, which is different for each term. The expression reduces to

$$\begin{aligned}\overline{v^2} &= V_1^2 \left\langle \cos^2 \omega_1 t \right\rangle + V_1 V_2 \left[\left\langle \cos\left(\omega_1 - \omega_2\right) t \right\rangle + \left\langle \cos\left(\omega_1 + \omega_2\right) t \right\rangle\right] \\ &\quad + V_2^2 \left\langle \cos^2 \omega_2 t \right\rangle \\ &= \frac{V_1^2}{2} + \frac{V_2^2}{2}\end{aligned}$$

1.4 Variance

The variance of a random variable x is defined as the average value of $(x - \overline{x})^2$. For example, if x is a voltage or a current, the variance is simply the mean-square value of its ac component. We denote the variance by σ^2. It is given by

$$\sigma^2 = \overline{(x - \overline{x})^2} = \int_{-\infty}^{\infty} (x - \overline{x})^2 \, p(x) \, dx \tag{1.13}$$

The square-root of the variance, i.e. σ, is called the standard deviation. If x is a voltage or a current, its standard deviation is simply the rms value of its ac component.

1.5 The Gaussian Probability Function

The Gaussian or normal probability density function is the most important probability function in the study of noise. If the variation in a process is caused by a large number N of random and unrelated occurrences, it can be shown that the probability density function for the process tends to a Gaussian function in the limit as $N \to \infty$. This is a statement of what is known as the central limit theorem. Because electronic noise is generated by the randomness in the flow of many current carriers, it can be described by the normal density function.

Let x be a Gaussian random variable. Denote its mean or average value by $\overline{x}$ and its mean-square value or variance by σ^2. The probability density function is given by

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-(x-\overline{x})^2}{2\sigma^2}\right] \tag{1.14}$$

A plot of $p(x)$ as a function of x is shown in Fig. 1.1. Because of its shape, this plot is often referred to as a "bell-shaped curve." The maximum value occurs at $x = \overline{x}$ and is inversely proportional to σ. If $\sigma \to 0$, the maximum value approaches ∞, the width of the curve approaches zero, and the function becomes a unit impulse.

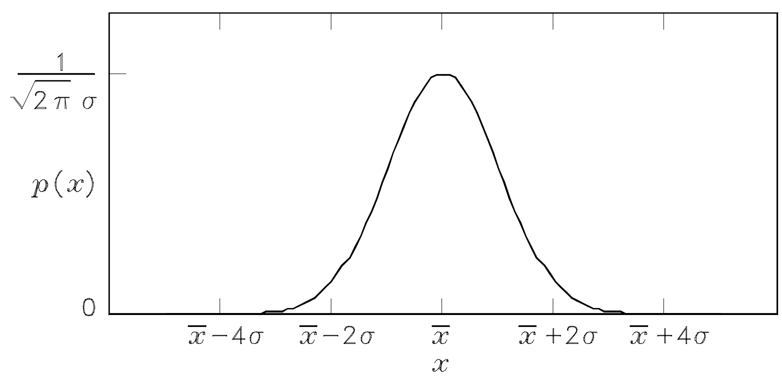

Figure 1.1: Plot of the Gaussian probability density function.

The probability that a single observation of x lies between the limits $\overline{x} - \sigma$ and $\overline{x} + \sigma$ is given by the area under the curve of $p(x)$ between these limits. It is equal to 0.683 or 68.3%. Because this is approximately equal to 2/3, it has given rise to the saying that two-thirds of the time, a random variable lies within plus or minus one standard deviation from its average value. For example, if it is assumed that normal statistics apply to the quiz grades of a class, then approximately 2/3 of the members of the class should have a grade within one standard deviation of the average and 1/3 should have a grade outside this range, 1/6 on the upper end and 1/6 on the lower end. In such applications, Eq. (1.10) is used to calculate the quiz average $\overline{x}$, where N is equal to the number of students. The variance is calculated from

$$\sigma^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \overline{x})^2 \tag{1.15}$$

Although it might seem that the $N-1$ in this equation should be N, it can be shown that $N-1$ gives a better estimate of σ^2 when the sample size is limited, i.e. when N is not large.

A plot of the distribution function given by Eq. (1.4) for the Gaussian probability density function is shown in Fig. 1.2. The plot shows that $P(\overline{x}) = 0.5$. This means that the probability that $x < \overline{x}$ is 0.5. The probability that $x > \overline{x}$ is $1 - P(\overline{x}) = 0.5$.

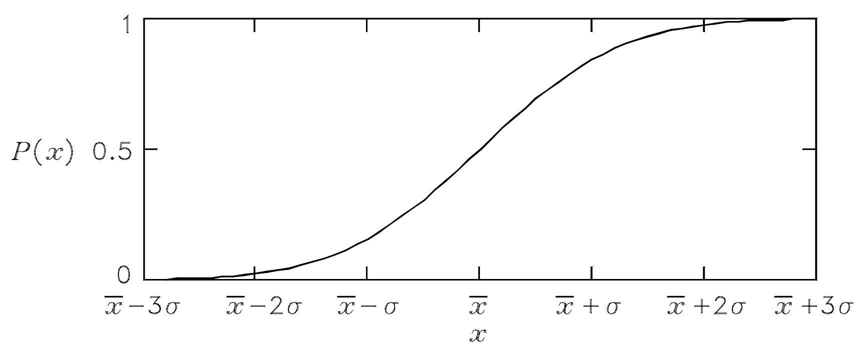

Figure 1.2: Plot of the Gaussian distribution function.

1.5.1 Crest Factor

In circuit analysis, it is common to write voltages and currents as the sum of a dc component and an ac component. The dc component is called the bias or quiescent value. Because this is deterministic, it is not considered to be a part of random electrical noise. Thus electronic noise is usually modeled as a zero mean process. Let $v(t)$ be a zero-mean random noise voltage having a variance $\sigma = v_{\mathrm{rms}}$. If $v(t)$ is modeled as

Table 1.1: Prob($|v| > v_1$)

v_1	Prob($\|v\| > v_1$)	% of time that $\|v\| > v_1$
$0.5v_{\text{rms}}$	0.617	61.7%
v_{rms}	0.317	31.7%
$2v_{\text{rms}}$	4.55×10^{-2}	4.55%
$2.576v_{\text{rms}}$	1×10^{-2}	1%
$3v_{\text{rms}}$	2.70×10^{-3}	0.270%
$4v_{\text{rms}}$	6.33×10^{-5}	0.00633%

a Gaussian random variable, its probability density function is

$$p(v) = \frac{1}{\sqrt{2\pi}v_{\text{rms}}} \exp\left[\frac{-v^2}{2v_{\text{rms}}^2}\right] \tag{1.16}$$

We desire the probability that $|v| > v_1$, where $v_1 > 0$. This is given by

$$\begin{aligned} \text{Prob}(|v| > v_1) &= 1 - \int_{-v_1}^{v_1} p(x)\, dx \\ &= 1 - [P(v_1) - P(-v_1)] \\ &= 2[1 - P(v_1)] \end{aligned} \tag{1.17}$$

where $P(v)$ is the distribution function and we have used the relation $P(-v_1) = 1 - P(v_1)$ for a zero-mean process. Table 1.1 gives this probability for several values of v_1. When the probability is multiplied by 100%, the percent of time that $|v| > v_1$ is obtained. It can be seen from the table that the probability that $|v| > 4v_{\text{rms}}$ is essentially zero. This means that if v is observed on an oscilloscope, the probability that it will have a peak value less than $4v_{\text{rms}}$ is almost unity.

The crest factor of a waveform is defined as the ratio of its peak value to its rms value. This is a well defined quantity for a deterministic process. In the case of the waveform of a random variable, however, it cannot be defined with certainty. Instead, it is commonly specified as a function of the percent of the time that the peak value is greater than (or less than) the specified value. For example, our voltage v has a crest factor that is greater than 2.576 only 1% of the time. This means, for example, that if we observe the waveform of v on an oscilloscope for 100 seconds, the total time that peak value exceeds $2.576 v_{\mathrm{rms}}$ is only 1 second out of the 100.

1.6 Spectral Density

Let $v(t)$ be a zero-mean random voltage that is defined over the time interval $-T/2 \leq t \leq T/2$. We assume that $v(t) = 0$ outside this interval and that the process is ergodic. The mean-square value of $v(t)$ can be written as the time average

$$\overline{v^2} = \frac{1}{T} \int_{-T/2}^{T/2} v^2(t)\, dt \tag{1.18}$$

We wish to define a function of frequency $S_v(f)$ such that $\overline{v^2}$ is also given by

$$\overline{v^2} = \int_0^{\infty} S_v(f)\, df \tag{1.19}$$

where we interpret $S_v(f)\, \Delta f$ as being the amount of $\overline{v^2}$ that is contained in the frequency band from f to $f + \Delta f$, where Δf is small. In this case, $S_v(f)$ is called the spectral density of $v(t)$. The spectral density defined here is called the one-sided density because it is defined only for positive frequencies.

Let $F(f)$ be the Fourier transform of $v(t)$. It is given by

$$F(f) = \int_{-T/2}^{T/2} v(t)\, e^{-j2\pi f t} dt \tag{1.20}$$

It follows that $v(t)$ is given by the inverse transform

$$v(t) = \int_{-\infty}^{\infty} F(f)\, e^{j2\pi f t} df \tag{1.21}$$

Thus the mean-square value of $v(t)$ over the interval from $-T/2$ to $T/2$

can be written

$$\begin{aligned}
\overline{v^2} &= \frac{1}{T}\int_{-T/2}^{T/2} v(t)\,dt \int_{-\infty}^{\infty} F(f)\,e^{j2\pi ft}df \\
&= \frac{1}{T}\int_{-\infty}^{\infty} F(f)\,df \int_{-T/2}^{T/2} v(t)\,e^{j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} \frac{F(f)\,F(-f)}{T}df \\
&= 2\int_{0}^{\infty} \frac{F(f)\,F(-f)}{T}df \qquad (1.22)
\end{aligned}$$

where the factor of 2 arises because the integration limits have been changed from $(-\infty, \infty)$ to $(0, \infty)$. This is possible because $F(f)\,F(-f)$ is an even function of f.

Because $F(-f) = F^*(f)$, an alternate way of writing the above result is

$$\overline{v^2} = \int_0^{\infty} \frac{2\,|F(f)|^2}{T}df \qquad (1.23)$$

When we compare this equation to Eq. (1.19), it follows that the spectral density is given by

$$S_v(f) = \frac{2\,|F(f)|^2}{T} \qquad (1.24)$$

The units of $S_v(f)$ are V^2/Hz. It is tempting to say that we can take a limit in this equation as $T \to \infty$ to obtain the spectral density for a continuing random function of time that is defined over the interval $-\infty < t < \infty$. However, there are mathematical problems associated with this. These problems can be avoided if we use the autocorrelation function to evaluate $S_v(f)$. This is discussed in the following section.

1.7 Autocorrelation and the Spectral Density

The autocorrelation function $\varphi(\tau)$ for a random voltage function $v(t)$ is defined by

$$\varphi(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} v(t)\,v(t+\tau)\,dt \qquad (1.25)$$

With the aid of the inverse Fourier transform of Eq. (1.21), we can write

$$\begin{aligned}
\varphi(\tau) &= \frac{1}{T}\int_{-T/2}^{T/2} v(t)\,dt \int_{-\infty}^{\infty} F(f)\,e^{j2\pi f(t+\tau)}df \\
&= \frac{1}{T}\int_{-\infty}^{\infty} F(f)\,e^{j2\pi f\tau}df \int_{-T/2}^{T/2} v(t)\,e^{j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} \frac{F(f)\,F(-f)}{T} e^{j2\pi f\tau}df \\
&= \int_{-\infty}^{\infty} \frac{|F(f)|^2}{T} e^{j2\pi f\tau}df
\end{aligned} \tag{1.26}$$

We now take the inverse Fourier transform of this equation to obtain

$$\frac{|F(f)|^2}{T} = \int_{-\infty}^{\infty} \varphi(\tau)\,e^{-j2\pi f\tau}d\tau \tag{1.27}$$

But $S_v(f) = 2|F(f)|^2/T$, where we assume $f \geq 0$. It thus follows that the spectral density is given by

$$S_v(f) = 2\int_{-\infty}^{\infty} \varphi(\tau)\,e^{-j2\pi f\tau}d\tau \tag{1.28}$$

Note that the time T does not appear in this equation. Therefore, the problems associated with taking the limit as $T \to \infty$ have disappeared. This is somewhat an oversimplification of the problems involved in taking this limit, but it suffices to say that the autocorrelation function of a random variable is the key to its spectral density function.

The factor of 2 in Eq. (1.28) arises because we have defined the spectral density as a "one-sided function," i.e. only for $f \geq 0$. In some texts, the spectral density is defined for both negative and positive f. In this case, the 2 is absent from Eq. (1.28) and the lower limit on the integral in Eq. (1.19) is $-\infty$. The one-sided definition simplifies noise analyses by eliminating the need for negative frequencies.

Chapter 2

Types of Noise

2.1 Thermal Noise

Thermal noise, also called Johnson noise, is generated by the random thermal motion of electrons in a resistive material. It is present in all circuit elements containing resistance and is independent of the composition of the resistance. It is modeled the same way in discrete-circuit resistors and in integrated-circuit monolithic and thin-film resistors. The phenomenon was discovered (or anticipated) by Schottky in 1918 when he called it the "wärmeffect.," in English now commonly called "thermal noise." It was first measured by Johnson in 1928. Several months after Johnson described his work to Nyquist, Nyquist derived his famous formula, based on the thermodynamics of a telephone line. Nyquist showed that the mean-square open-circuit thermal noise voltage across a resistor is given by

$$\overline{v_t^2} = 4kTR\Delta f \tag{2.1}$$

where k is Boltzmann's constant, T is the absolute or Kelvin temperature, R is the resistance, and Δf is the bandwidth in Hz over which the noise is measured. This equation is referred to as the *Nyquist formula.* The corresponding mean-square short-circuit thermal noise current is given by

$$\overline{i_t^2} = \frac{\overline{v_t^2}}{R^2} = \frac{4kT\Delta f}{R} \tag{2.2}$$

The Thévenin and Norton thermal noise models of a resistor are given in Fig. 2.1. Because noise is random, the source polarities are arbitrary. In general, the polarities must be labeled when writing circuit equations that involve small-signal or phasor voltages and currents. The mean-square noise is independent of the assumed polarities.

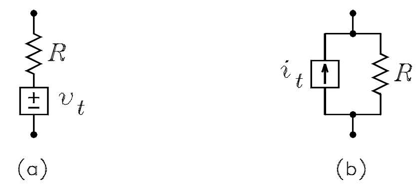

Figure 2.1: Resistor noise models. (a) Th évenin. (b) Norton.

2.1.1 Derivation of the Nyquist Formula

Consider an ideal, lossless transmission line terminated at each end in its characteristic resistance R as shown in Fig. 2.2. Because the line is matched at each end, the thermal noise power fed into the line by one of the resistors is completely absorbed by the other resistor. If both resistors are at the same temperature, the system is said to be in thermal equilibrium. Under this condition, each resistor must feed the same noise power into the line. Otherwise, one would be heating up and the other would be cooling off.

Let the switches at the ends of the line be closed after equilibrium is established. The shorted line then becomes a resonator with an infinite number of resonance modes. Let λ be the wavelength of any mode. Because the voltage at each end of the line is zero, it follows that the length of the line must be equal to multiples of $\lambda/2$, i.e. $L = \lambda/2$, λ, $3\lambda/2$, etc. The frequency of each mode is given by $f = c/\lambda$, where c is the phase velocity of the line. It follows that the resonance frequencies are $c/2L$, c/L, $3c/2L$, etc. Let the difference between two adjacent resonance

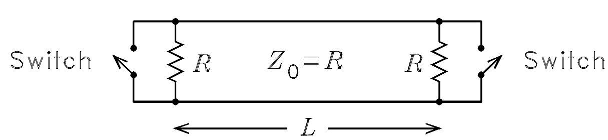

Figure 2.2: Circuit for derivation of the Nyquist formula.

frequencies be denoted by Δf. It follows that

$$\Delta f = \frac{c}{2L} \tag{2.3}$$

This can be made as small as desired by increasing L.

According to the equipartition theorem of statistical mechanics, there is an energy of $kT/2$ joules associated with each degree of freedom in a system. Each resonance mode in the transmission line has energy storage in both the electric field and in the magnetic field. For a lossless line, these two energies are equal. Thus, there is an energy $2 \times kT/2 = kT$ stored in each mode. The energy per unit bandwidth is given by $kT/\Delta f = 2kTL/c$ J/Hz. One-half of this energy comes from each resistor, thus the energy per unit bandwidth per resistor is

$$w = \frac{kT}{2\Delta f} = \frac{kTL}{c} \tag{2.4}$$

Before the switches are closed, there are two independent waves propagating on the line, one to the right and one to the left. The velocity of propagation of each is the line phase velocity c. At the instant the switches are shorted, the total energy in each wave is equal to the energy delivered by either resistor in the immediately preceding time interval given by $t_d = L/c$. It follows that the power per unit bandwidth in W/Hz delivered by each resistor to the line during that interval is given by

$$p = \frac{w}{t_d} = kT \tag{2.5}$$

Let N_a be the noise power over the band Δf that is generated by one resistor and absorbed by the other. This is called the available noise power. It follows that N_a is given by

$$N_a = p\Delta f = kT\Delta f \tag{2.6}$$

Let each resistor be modeled by a noise voltage v_t in series with the resistor. The corresponding current which flows on the line is $v_t/2R$. The available noise power from either resistor can thus be written

$$N_a = \overline{\left(\frac{v_t}{2R}\right)^2} \times R = \frac{\overline{v_t^2}}{4R} \tag{2.7}$$

By equating the two expressions for N_a, we obtain

$$\overline{v_t^2} = 4kTR\Delta f \tag{2.8}$$

This is the Nyquist formula for the mean-square thermal noise voltage in the band Δf.

An application of the Nyquist formula at arbitrarily high frequencies predicts that the total mean-square noise voltage diverges as $\Delta f \to \infty$. This is called the *ultraviolet catastrophe*. However, it can be shown that the expression is valid only as long as $f \ll kT/h$, where $h = 6.62 \times 10^{-34}$ J·s is Plank's constant. For $T = 290$ K, this gives $f \ll 6 \times 10^{12}$ Hz. In general, the mean-square noise voltage is given by

$$\overline{v_t^2} = \frac{4hfR\Delta f}{\exp(hf/kT) - 1} \tag{2.9}$$

For $f \ll kT/h$, this reduces to Eq. (2.8). Fig. 2.3 shows the variation with frequency of the normalized available thermal noise power per unit bandwidth $N_a/kT\Delta f = \overline{v_t^2}/4kTR\Delta f$ for three temperatures: 290 K – the standard temperature, 78 K – the temperature of liquid nitrogen, and 4 K – the temperature of liquid helium. Eq. (2.8) holds only in the regions where the curves have a value of unity. It can be seen that the Nyquist formula can be in error for very high frequencies and/or very low temperatures.

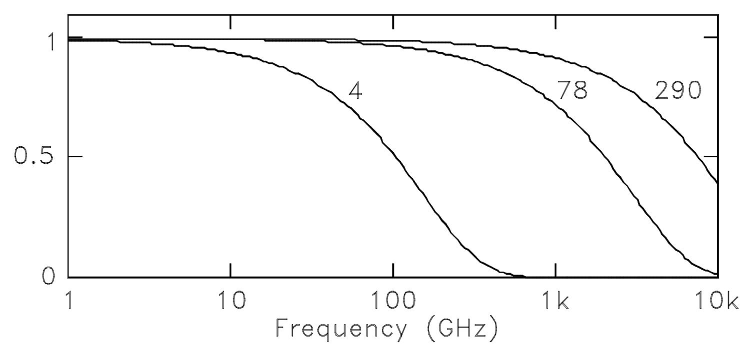

Figure 2.3: Plots of normalized available thermal noise power as a function of frequency for $T = 290$ K, 78 K, and 4 K.

2.1.2 Crest Factor

The crest factor of thermal noise is commonly taken to be 3 times the rms value, i.e. $3\sqrt{\overline{v_t^2}}$. To calculate the probability that the instantaneous value is greater than the crest factor, a statistical model for the amplitude distribution is required. It is common to use a Gaussian or normal probability density function to calculate this. For a Gaussian random variable, the probability that the magnitude of the instantaneous value exceeds 3 times the rms value is 0.27%. Some references take the crest value to be 4 times the rms value. In this case, the probability is 0.0063%.

2.1.3 Nyquist Formula for a Complex Impedance

Let Z be the complex impedance of a two terminal network. The mean-square, open-circuit thermal noise voltage generated by the impedance in the band Δf is given by

$$\overline{v_t^2} = 4kT\,\mathrm{Re}\,(Z)\,\Delta f \tag{2.10}$$

Because Z is a function of frequency, Δf must be small enough so that $\mathrm{Re}\,(Z)$ is approximately a constant over the band. Otherwise, the noise must be expressed by an integral. In the frequency band from f_1 to f_2, the mean-square, open-circuit thermal noise voltage is given by

$$\overline{v_t^2} = 4kT\int_{f_1}^{f_2}\mathrm{Re}\,(Z)\,df \tag{2.11}$$

Let $Y = 1/Z$ be the complex admittance of the network. The mean-square, short-circuit thermal noise current in the band Δf is given by

$$\overline{i_t^2} = 4kT\,\mathrm{Re}\,(Y)\,\Delta f \tag{2.12}$$

Over the band from f_1 to f_2 the mean-square noise current is

$$\overline{i_t^2} = 4kT\int_{f_1}^{f_2}\mathrm{Re}\,(Y)\,df \tag{2.13}$$

2.1.4 Spectral Density

The spectral density of a noise source is defined as the mean-square value of the source per unit bandwidth. In general, the spectral density is a function of frequency. The voltage and current spectral densities, respectively, for the thermal noise generated by a complex impedance Z are given by

$$S_v\,(f) = \frac{\overline{v_t^2}}{\Delta f} = 4kT\,\mathrm{Re}\,(Z) \tag{2.14}$$

$$S_i(f) = \frac{\overline{i_t^2}}{\Delta f} = 4kT\,\mathrm{Re}(Y) \tag{2.15}$$

where $Y = 1/Z$ is the admittance.

The spectral density of the thermal noise generated by a resistor is independent of frequency. In this case, the spectral density is often said to be uniform or flat. It is also called white noise by analogy to white light which also has a flat spectral density in the optical band.

2.1.5 Spot Noise Voltage and Current

The spot value of a noise voltage or current is the rms value of the noise in a band of 1 Hz. It is given by the square root of the spectral density. For example, the thermal spot noise voltage generated by a complex impedance Z is given by

$$\sqrt{S_v(f)} = \sqrt{\frac{\overline{v_t^2}}{\Delta f}} = \sqrt{4kT\,\mathrm{Re}(Z)} \tag{2.16}$$

The units are read "volts per root Hz." The spot noise current generated by the impedance is given by

$$\sqrt{S_i(f)} = \sqrt{\frac{\overline{i_t^2}}{\Delta f}} = \sqrt{4kT\,\mathrm{Re}(Y)} \tag{2.17}$$

where $Y = 1/Z$. The units are read "amperes per root Hz."

2.1.6 Noise Resistance

A mean-square noise voltage can be represented in terms of an equivalent noise resistance R_n. Let $\overline{v_n^2}$ be the mean-square noise voltage in the band Δf. The noise resistance R_n is defined as the value of a resistor at the standard temperature $T_0 = 290^{\mathrm{o}}$ K which generates the same noise voltage. It is given by

$$R_n = \frac{\overline{v_n^2}}{4kT_0\Delta f} \tag{2.18}$$

2.1.7 Noise Conductance

A mean-square noise current can be represented in terms of an equivalent noise conductance G_n. Let $\overline{i_n^2}$ be the mean-square noise current in the band Δf. The noise conductance G_n is defined as the value of a conductance at the standard temperature which generates the same noise current. It is given by

$$G_n = \frac{\overline{i_n^2}}{4kT_0\Delta f} \tag{2.19}$$

2.1.8 Example Thermal Noise Calculations

Resistor Shunted by Capacitor

Figure 2.4(a) shows a resistor shunted by a capacitor. The impedance is given by

$$Z = R \| \frac{1}{j\omega C} = \frac{R}{1 + j\omega RC} = \frac{R}{1 + (\omega RC)^2} - j\frac{\omega R^2 C}{1 + (\omega RC)^2} \tag{2.20}$$

The total open-circuit noise voltage across the resistor is given by

$$\begin{aligned} \overline{v_t^2} &= \int_0^\infty \frac{4kTR}{1 + (2\pi f RC)^2} df = \frac{2kT}{\pi C} \int_0^\infty \frac{dx}{1 + x^2} \\ &= \frac{2kT}{\pi C} \left[\tan^{-1} x\right]_0^\infty = \frac{kT}{C} \end{aligned} \tag{2.21}$$

Note that this is independent of R. If $C \to 0$, it predicts that $\overline{v_t^2} \to \infty$. This is another statement of the ultraviolet catastrophe, which is physically impossible because the Nyquist formula does not hold at arbitrarily high frequencies. All physical resistors exhibit a shunt capacitance between the leads. For example, if $C = 1$ pF, Eq. (2.21) predicts that $\sqrt{\overline{v_t^2}} = 6.3 \times 10^{-5}$ V. Noise calculated from Eq. (2.21) is often referred to as "kT over C" noise.

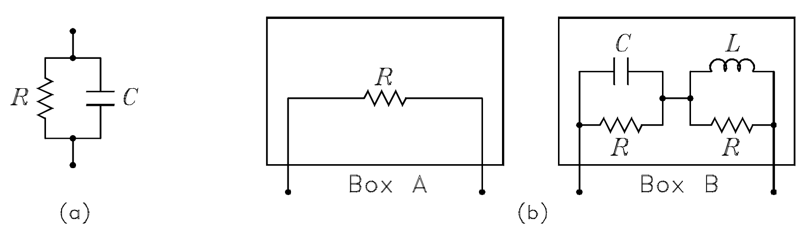

Figure 2.4: (a) Parallel RC network. (b) Slepian-Goldner black-box identification problem.

The Slepian-Goldner Problem

Figure 2.4(b) show two "black boxes," each containing a two terminal impedance. The impedance of the network in Box B is given by

$$Z = R\|\frac{1}{Cs} + R\|Ls = \frac{R}{1+RCs} + \frac{Ls}{1+(L/R)\,s} \tag{2.22}$$

If $RC = L/R$, or equivalently $R = \sqrt{L/C}$, it follows that $Z = R$. The problem is as follows: given the two black boxes shown in Fig. 2.4(b) containing ideal R, L, and C elements, if $R = \sqrt{L/C}$, how can the boxes be distinguished between each other by terminal measurements alone?

Because the terminal impedances are the same, no transient or steady-state measurement can be used to solve the problem. However, the two networks can be made to have different noise characteristics. If a dc battery is connected across the terminals of each network, the current in Box B would heat only the resistor in parallel with the capacitor. If the mean-square noise voltage generated by each network is then measured, it will be found that Box A generates more noise than Box B. This is because the high frequency noise generated by the heated resistor in Box B is shunted by the capacitor. Alternately, the spectral density of the noise generated by Box A is flat while the spectral density of the noise from Box B has a low-pass shelving characteristic, i.e. it is larger at low frequencies than at high frequencies.

2.2 Shot Noise

Shot noise is generated by the random emission of electrons or by the random passage of electrons and holes across a potential barrier. The shot noise generated in a device is modeled by a parallel noise current source. The mean-square shot noise current in the frequency band Δf is given by

$$\overline{i_{sh}^2} = 2qI\Delta f \tag{2.23}$$

where q is the electronic charge and I is the dc current flowing through the device. This equation was derived by Schottky in 1928 and is known as the *Schottky formula*. The spectral density of shot noise is flat, thus shot noise is white noise. It is commonly assumed that the amplitude distribution can be modeled by a Gaussian or normal distribution. Thus the relation between the crest factor and rms value for shot noise is the same as it is for thermal noise.

2.2.1 Derivation of the Schottky Formula

The derivation given here is not mathematically rigorous, but it illustrates the major steps. Consider the flow of a current $i(t)$ across a potential barrier to consist of the random arrival of charges of value q. Let the average number of charges per unit time be n so that the number of charges in time T is given by $N = nT$. If the current associated with each charge is modeled by a rectangular pulse of width Δt, the current waveform might appear as the one shown in Fig. 2.5. We assume that Δt is small enough so that the probability of more than one charge in time Δt is zero.

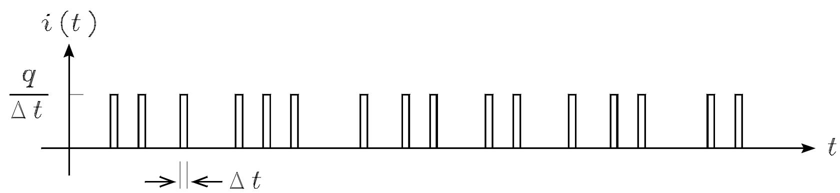

Figure 2.5: Current waveform.

Because the area under a pulse must be equal to the charge q, the height of each pulse is $q/\Delta t$. The dc component of the current is given by

$$I = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} i(t)\, dt = \lim_{T\to\infty} \frac{1}{T} (q \times nT) = nq \tag{2.24}$$

The integral in this equation represents the total charge arriving in the time interval $-T/2 \le t \le T/2$. This is given by charge per pulse q multiplied by the number of pulses $N = nT$ in the interval.

The autocorrelation function for $i(t)$ is the key to its spectral density function. It is given by

$$\varphi(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} i(t)\, i(t+\tau)\, dt \tag{2.25}$$

Let us first consider the case where $\tau = 0$. Let the number of pulses in the interval $-T/2 \le t \le T/2$ be N. There are N contributions to the integral, each having the value $(q/\Delta t)^2 \times \Delta t$. The autocorrelation function is

$$\begin{aligned}\varphi_0(0) &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} i^2(t)\,dt = \lim_{T\to\infty}\frac{1}{T}\left[N\left(\frac{q}{\Delta t}\right)^2\Delta t\right]\\ &= n\frac{q^2}{\Delta t}\end{aligned} \tag{2.26}$$

In the limit as $\Delta t \to 0$, $\varphi_0(0)$ can be written

$$\varphi_0(0) = nq^2\delta(0) = qI\delta(0) = qI\delta(\tau) \tag{2.27}$$

where $\delta(\tau)$ is the unit impulse function defined by the two properties

$$\delta(x) = 0 \qquad \text{for } x \neq 0 \tag{2.28}$$

$$\int_{-\infty}^{\infty}\delta(x)\,dx = 1 \tag{2.29}$$

Note that the units of $\delta(x)$ are the units of the reciprocal of its argument x. Otherwise, the integral would not be dimensionless.

Now consider the case where $\tau \neq 0$. Let the interval $-T/2 \le t \le T/2$ be divided into subintervals of width Δt. If N pulses of width Δt occur in time T, then the probability that a single pulse of $i(t)$ occurs in Δt is $N\Delta t/T = n\Delta t$. We assume that $i(t)$ and $i(t+\tau)$ are independent. Thus the probability that a pulse of $i(t)$ and $i(t+\tau)$ simultaneously occurring in Δt is $(n\Delta t)^2$. The number of subintervals is $T/\Delta T$. Thus the number of subintervals in which a pulse of $i(t)$ and $i(t+\tau)$ occurs simultaneously is $(n\Delta t)^2 \times T/\Delta T$. The contribution to the autocorrelation integral for each of these is $(q/\Delta t)^2 \times \Delta t$. Thus the autocorrelation function for $\tau > 0$ is given by

$$\varphi_1(\tau) = \lim_{T\to\infty}\frac{1}{T}\left[(n\Delta t)^2 \times \frac{T}{\Delta t} \times \left(\frac{q}{\Delta t}\right)^2 \times \Delta t\right] = (nq)^2 = I^2 \tag{2.30}$$

It follows that the overall autocorrelation function is given by

$$\varphi(\tau) = \varphi_0(0) + \varphi_1(\tau) = qI\delta(\tau) + I^2 \tag{2.31}$$

The two-sided spectral density of $i(t)$ is given by the Fourier transform of the autocorrelation function. It is

$$S_i(f) = \int_{-\infty}^{\infty}\left[Iq\delta(\tau) + I^2\right]e^{j2\pi f\tau}d\tau = qI + I^2\delta(f) \tag{2.32}$$

where the sifting property of the impulse function has been used to obtain the first term. The second term follows from the integral

$$\delta(x) = \int_{-\infty}^{\infty} e^{j2\pi xy} dy \tag{2.33}$$

The spectral density consists of two components, one is a constant and the other is an impulse at $f = 0$. The impulse term represents the spectral density of the dc component of $i(t)$. The constant represents the spectral density of the shot noise component. When the latter is multiplied by a bandwidth Δf, the mean-square shot noise current in the band from f to $f + \Delta f$ is obtained. It is given by $qI\Delta f$. However, because $S(f)$ is two sided, f can be either positive or negative. If f is restricted to be positive, the contribution due to the band from $-f$ to $-f+\Delta f$ must be added to the contribution in the band from f to $f+\Delta f$. This introduces a factor of 2 in the expression. Thus the mean-square shot noise current in the band Δf is given by

$$\overline{i_{sh}^2} = 2qI\Delta f \tag{2.34}$$

where we consider the frequency be to positive. This is the Schottky formula.

The Schottky formula predicts that the mean-square shot noise diverges in the limit as $\Delta f \to \infty$. This is similar to the ultraviolet catastrophe for thermal noise. It is a result of our letting $\Delta t \to 0$ in Eq. (2.26) to obtain an impulse function. Because of the discrete nature of charge, it is impossible to model the flow of current by a series of zero width pulses. When the finite width of the current pulses is taken into account, the low frequency limit of the spectral density is found to be given by Eq. (2.32). As frequency is increased, the spectral density approaches zero so that the total mean-square noise cannot diverge. However, Eq. (2.34) is sufficiently accurate for frequencies of interest in electronic circuits.

2.3 Flicker Noise

Flicker noise is a noise that has a spectral density that is proportional to $1/f^n$, where $n \simeq 1$. It is also called "$1/f$ noise." This type of noise seems to be a systematic effect inherent in electrical conduction. Various explanations of its origins have been made, but it remains an ill-understood phenomenon. In resistive materials, its origin seems to be caused by a fluctuation of the mobility of the free charge carriers.

In semiconductors, it is generated by tunnelling effects in the surface oxide layer of the material. It is also generated by the imperfect contact between two conducting materials. In this case, it is called contact noise.

Flicker noise not only occurs in electrical systems. It has been observed in biological systems, in fluctuations of the frequency of rotation of the earth, in the fluctuations of variables in economics, and in the power fluctuations of nuclear reactors. Indeed, it seems to be a systematic effect inherent in all physical processes.

The $1/f$ spectral density of flicker noise has been shown to hold down to extremely low frequencies. With some devices, it has been shown to hold down to frequencies as low as one cycle per month, where the noise merges with the natural drift of the device. Measurements at frequencies this low are very difficult because of the long times required.

Flicker noise is modeled by a noise current source in parallel with a device. In general, the mean-square flicker noise current in the frequency band Δf is given by

$$\overline{i_f^2} = \frac{K_f I^m \Delta f}{f^n} \tag{2.35}$$

where I is the dc current, $n \simeq 1$, K_f is the flicker noise coefficient, and m is the flicker noise exponent.

2.3.1 Flicker Noise in Resistors

Flicker noise in resistors is called excess noise. It is caused by fluctuations in the conductivity in the presence of a dc current. The fluctuations are caused by the time variation in the path that the current takes through the resistor. For example, if a high enough voltage is applied between two electrodes, an arc develops. The path that the arc takes moves randomly with time in a similar way that the path of current through a resistor varies.

Let $G = 1/R$ be the conductivity of a resistor R. If a dc voltage V is applied to the resistor, a dc current flows that is given by $I = GV$. If G changes by an amount ΔG, it follows that I changes by an amount $\Delta I = V\Delta G$. If we let $i_{ex} = \Delta I$, it follows that the mean-square value of i_{ex} is given by

$$\overline{i_{ex}^2} = V^2 \overline{\Delta G^2} = G^2 V^2 \overline{\left[\frac{\Delta G}{G}\right]^2} \tag{2.36}$$

Experimental evidence suggests the relation

$$\overline{\left[\frac{\Delta G}{G}\right]^2} = K_f \frac{\Delta f}{f} \tag{2.37}$$

where K_f is the flicker noise coefficient. It follows that

$$\overline{i_{ex}^2} = G^2 V^2 K_f \frac{\Delta f}{f} = \frac{K_f I^2 \Delta f}{f} \tag{2.38}$$

Carbon composition resistors are fabricated as a carbon rod with electrodes attached to the ends. Of all resistor types, these exhibit the most excess noise. Carbon film resistors exhibit less excess noise. These are fabricated as a glass rod with a carbon film deposited on it. Because the film is thin, there is less volume through which the current path can vary. Metal film resistors generate less excess noise. These are fabricated as a glass rod with a metal film deposited on it. For a given resistance, a metal film must be thinner than a carbon film, thus decreasing the volume through which the current path can vary. Wire wound resistors exhibit the lowest excess noise. However, the inductance of these resistors limit their application to low frequency design.

Because i_{ex} is defined with a dc voltage across the resistor, it represents an ac short-circuit noise current. Thus it is modeled by a current source in parallel with the resistor. The circuit is shown in Fig. 2.6(a), where i_t represents the thermal noise current. Fig. 2.6(b) shows the circuit obtained when a Thévenin equivalent is made. In this circuit v_t represents the thermal noise voltage and v_{ex} represents the excess noise voltage. Its mean square value is given by

$$\overline{v_{ex}^2} = \overline{i_{ex}^2} R^2 = \frac{K_f I^2 R^2 \Delta f}{f} = \frac{K_f V^2 \Delta f}{f} \tag{2.39}$$

where $V.$ is the dc voltage across the resistor.

Figure 2.7 shows the plot of the spectral density for either $\overline{i_{ex}^2}$ or $\overline{v_{ex}^2}$ as a function of frequency. The flicker noise corner frequency is the frequency at which the low-frequency value is twice the high-frequency thermal noise limit. At this frequency, the noise is 3 dB higher than the thermal limit. The flicker noise corner frequency is given by

$$f_{\text{flk}} = \frac{K_f I^2 R^2}{4kT} = \frac{K_f V^2}{4kT} \tag{2.40}$$

Note that this is a function of the dc bias.

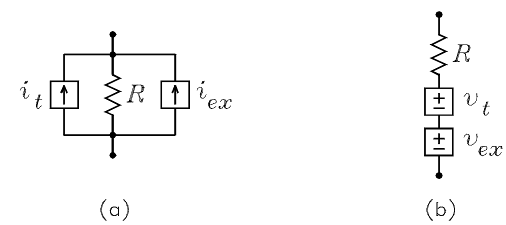

Figure 2.6: Resistor noise models. (a) Norton. (b) Thévenin.

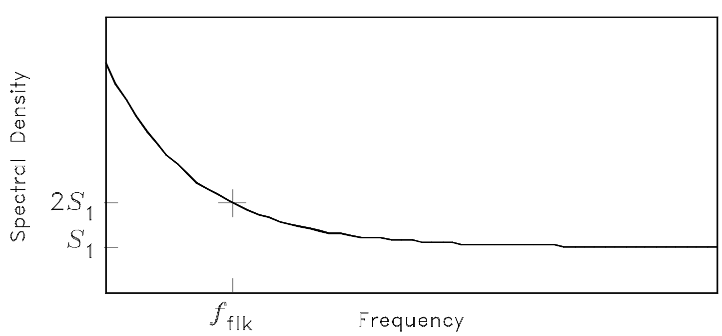

Figure 2.7: Plot of spectral density as a function of frequency showing flicker noise corner frequency.

In the band from f_1 to f_2 the total excess noise current and voltage are obtained by integrating the spectral densities over the band to obtain

$$\overline{i_{ex}^2} = K_f I^2 \int_{f_1}^{f_2} \frac{df}{f} = K_f I^2 \ln\left(\frac{f_2}{f_1}\right) \tag{2.41}$$

$$\overline{v_{ex}^2} = K_f V^2 \int_{f_1}^{f_2} \frac{df}{f} = K_f V^2 \ln\left(\frac{f_2}{f_1}\right) \tag{2.42}$$

The noise index of a resistor in dB is defined by

$$NI = 10\log\left(\frac{\overline{v_{ex}^2}}{V^2}\right) = 10\log\left(\frac{\overline{i_{ex}^2}}{I^2}\right) \tag{2.43}$$

where $f_2 = 10f_1$, $\overline{v_{ex}^2}$ is expressed in $(\mu V)^2$, and $\overline{i_{ex}^2}$ is expressed in $(\mu A)^2$. The noise index can be expressed in terms of the flicker noise coefficient by

$$NI = 10\log\left(10^{12} K_f \ln 10\right) \tag{2.44}$$

Given the noise index NI, the values of $\overline{v_{ex}^2}$ and $\overline{i_{ex}^2}$ in the range from f_1 to f_2 are given by

$$\overline{v_{ex}^2} = 10^{NI/10} \times V^2 \times \log\left(\frac{f_2}{f_1}\right)\ (\mu\mathrm{V})^2 \tag{2.45}$$

$$\overline{i_{ex}^2} = 10^{NI/10} \times I^2 \times \log\left(\frac{f_2}{f_1}\right)\ (\mu\mathrm{A})^2 \tag{2.46}$$

Note that $\log(f_2/f_1)$ is the number of decades in the range from f_1 to f_2.

Example 1 *A resistor of value* 10 *k*Ω *has a dc voltage across it of* 10 *V. The noise index is* $NI = 2$ *dB. Calculate the flicker noise coefficient and the rms excess noise voltage and current generated by the resistor over the band from* 10 *Hz to* 10 *kHz.*

Solution.

$$K_f = \frac{10^{NI/10-12}}{\ln 10} = 6.88 \times 10^{-13}$$

$$\overline{v_{ex}^2} = 10^{2/10} \times 10^2 \times \log\left(\frac{10000}{10}\right) = 475\ (\mu\mathrm{V})^2$$

$$\sqrt{\overline{v_{ex}^2}} = 21.8\ \mu\mathrm{V}$$

The dc current through the resistor is $I = 10/10,000 = 0.001$ A. Thus we have

$$\overline{i_{ex}^2} = 10^{2/10} \times 0.001^2 \times \log\left(\frac{10000}{10}\right) = 4.75 \times 10^{-6} \ (\mu\text{A})^2$$

$$\sqrt{\overline{i_{ex}^2}} = 2.18 \text{ nA}$$

2.4 Burst Noise

Burst noise is caused by a metallic impurity in a pn junction caused by a manufacturing defect. It is minimized by improved fabrication processes. When burst noise is amplified and reproduced by a loudspeaker, it sounds like corn popping. For this reason, it is also called popcorn noise. When viewed on an oscilloscope, burst noise appears as fixed amplitude pulses of randomly varying width and repetition rate. The rate can vary from less than one pulse per minute to several hundred pulses per second. Typically, the amplitude of burst noise is 2 to 100 times that of the background thermal noise.

Chapter 3

Characteristics of Noise

3.1 Addition of Noise Signals

The addition deterministic signals is accomplished mathematically by simply adding the functions which describe the individual signals. It is straightforward then to calculate the peak, the average, the mean-square, and the root-mean square values from the function representing the sum. When random signals are added, however, there are no functions which describe the signals. If the statistics of the signals are known, then the statistics of the sum signal can be determined. In noise analyses, the mean-square sum and the root-mean square (rms) sum of signals are the quantities that are commonly of interest. The rms sum is simply the square root of the mean-square sum. In the following, methods of calculating the mean-square sum of both real signals and signals represented by phasors are described.

3.1.1 Real Signals

Let $v_a(t)$ and $v_b(t)$ be two noise voltages having the mean-square values $\overline{v_a^2}$ and $\overline{v_b^2}$. The sum voltage is given by $v_{\text{sum}}(t) = v_a(t) + v_b(t)$. The mean-square value of the sum is calculated as follows:

$$\begin{aligned}\overline{v_{\text{sum}}^2} &= \overline{[v_a(t) + v_b(t)]^2} = \overline{v_a^2(t)} + 2\overline{v_a(t)\,v_b(t)} + \overline{v_b^2(t)} \\ &= \overline{v_a^2} + 2\rho\sqrt{\overline{v_a^2}}\sqrt{\overline{v_b^2}} + \overline{v_b^2} \qquad (3.1)\end{aligned}$$

where ρ is the correlation coefficient defined by

$$\rho = \frac{\overline{v_a(t)\, v_b(t)}}{\sqrt{\overline{v_a^2}}\sqrt{\overline{v_b^2}}} \tag{3.2}$$

The correlation coefficient can take on values in the range $-1 \le \rho \le +1$. For the case $\rho = 0$, the voltages $v_a(t)$ and $v_b(t)$ are said to be statistically independent or uncorrelated.

Example 1 *Figure 3.1(a) shows two resistors in series. The thermal noise of each is modeled by a series voltage source. Calculate the mean-square thermal noise voltage across the circuit and show that it is the same as that generated by a resistor of value $R_1 + R_2$. Assume that the two resistors generate independent noise.*

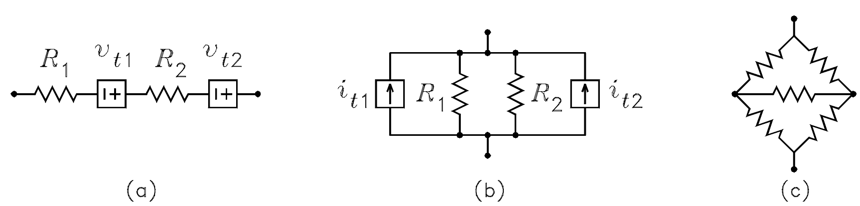

Figure 3.1: (a) Figure for Example 1. (b) Figure for Example 2. (c) Example circuit requiring a $Y - \Delta$ transformation.

Solution. The instantaneous thermal noise voltage is

$$v_t = v_{t1} + v_{t2}$$

The mean-square value is

$$\overline{v_t^2} = \overline{(v_{t1} + v_{t2})^2} = \overline{v_{t1}^2} + 2\overline{v_{t1} v_{t2}} + \overline{v_{t2}^2} = 4kT(R_1 + R_2)\,\Delta f$$

This is the mean-square thermal noise voltage generated by a resistor of value $R_1 + R_2$.

Example 2 *Figure 3.1(b) shows two resistors in parallel. The thermal noise of each is modeled by a parallel current source. Calculate the mean-square thermal noise voltage across the circuit and show that it is the same as that generated by a resistor of value $R_1 \| R_2$. Assume that the two resistors generate independent noise.*

Solution. The instantaneous thermal noise voltage is

$$v_t = (i_{t1} + i_{t2}) R_1 \| R_2$$

The mean-square value is

$$\begin{aligned} \overline{v_t^2} &= \overline{(i_{t1} + i_{t2})^2} (R_1 \| R_2)^2 = \left(\overline{i_{t1}^2} + 2\overline{i_{t1} i_{t2}} + \overline{i_{t2}^2} \right) (R_1 \| R_2)^2 \\ &= \left(\frac{4kT\Delta f}{R_1} + \frac{4kT\Delta f}{R_1} \right) (R_1 \| R_2)^2 = 4kT\Delta f \left(\frac{1}{R_1 \| R_2} \right) (R_1 \| R_2)^2 \\ &= 4kT (R_1 \| R_2) \Delta f \end{aligned}$$

This is the mean-square thermal noise voltage generated by a resistor of value $R_1 \| R_2$.

It follows from the above two examples that the thermal noise generated by any resistive network which can be reduced to a single resistor by making series and parallel combinations is the same as the thermal noise generated by the equivalent resistance. An example network which cannot be reduced to a single resistor by making series and parallel combinations is shown in Fig. 3.1(c). However, a $Y - \Delta$ transformation can be used to convert the circuit into one which can.

3.1.2 Phasor Signals

Let $v_s(t)$ be a sinusoidal voltage given by

$$v_s(t) = V_p \cos(\omega t + \varphi) \tag{3.3}$$

The signal is said to have a peak value V_p and a phase φ. The mean-square value is given by

$$\overline{v_s^2(t)} = V_p^2 \overline{\cos^2(\omega t + \varphi)} = \frac{V_p^2}{2} \tag{3.4}$$

An alternate expression for $v_s(t)$ is

$$v_s(t) = \text{Re}\left[V_p e^{j(\omega t + \varphi)} \right] = \text{Re}\left(\sqrt{2} V_s e^{j\omega t} \right) \tag{3.5}$$

where V_s is the phasor given by

$$V_s = \frac{V_p}{\sqrt{2}} e^{j\varphi} \tag{3.6}$$

The mean-square value of $v_s(t)$ can be calculated from V_s as follows:

$$\overline{v_s^2(t)} = |V_s|^2 = V_s V_s^* = \frac{V_p}{\sqrt{2}} e^{j\varphi} \times \frac{V_p}{\sqrt{2}} e^{-j\varphi} = \frac{V_p^2}{2} \tag{3.7}$$

where the $*$ denotes a complex conjugate. We can conclude that the mean-square value of a sinusoidal signal is simply the square magnitude of its phasor representation. The magnitude of the phasor represents the rms value of the signal. With deterministic signals, the magnitude of the phasor is often taken to represent the peak value of the variable rather than the rms value. However, with non-deterministic noise signals, there is no definable peak value. Therefore, it is convenient to take the magnitude of the phasor to be the rms value.

It is often necessary to use phasor representations of noise voltages and currents in writing equations for circuits containing capacitors and/or inductors. In this case, both the magnitude and the phase of the phasor are random variables. The ensemble average of the square magnitude of the phasor represents the mean-square value of the noise voltage or current in the band Δf centered on the frequency of analysis. To illustrate the addition of noise phasors, let V_a and V_b be the phasor representations of two noise voltages in the band Δf at a particular frequency. The sum is given by $V_{\text{sum}} = V_a + V_b$. The mean-square sum is denoted by $\overline{v_{\text{sum}}^2}$ and is calculated as follows:

$$\begin{aligned} \overline{v_{\text{sum}}^2} &= \overline{(V_a + V_b)\left(V_a^* + V_b^*\right)} = \overline{V_a V_a^*} + 2\,\text{Re}\left(\overline{V_a V_b^*}\right) + \overline{V_b V_b^*} \\ &= \overline{v_a^2} + 2\sqrt{\overline{v_a^2}}\sqrt{\overline{v_b^2}}\,\text{Re}\,(\gamma) + \overline{v_b^2} \end{aligned} \tag{3.8}$$

where γ is the complex correlation coefficient defined by

$$\gamma = \gamma_r + j\gamma_i = \frac{\overline{V_a V_b^*}}{\sqrt{\overline{v_a^2}}\sqrt{\overline{v_b^2}}} \tag{3.9}$$

Equation (3.9) seems to imply that only the real part of γ needs to be known. In general, it is necessary to know both the real and imaginary parts. To illustrate this, consider the sum $V_{\text{sum}} = V_n + I_n Z$, where Z is a complex impedance. The mean-square sum is given by

$$\begin{aligned} \overline{v_{\text{sum}}^2} &= \overline{V_n V_n^*} + 2\,\text{Re}\left(\overline{V_n I_n^* Z^*}\right) + \overline{(I_n Z)\left(I_n^* Z^*\right)} \\ &= \overline{v_n^2} + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\text{Re}\,(\gamma Z^*) + \overline{i_n^2}\,|Z|^2 \end{aligned} \tag{3.10}$$

where the correlation coefficient γ is given by

$$\gamma = \gamma_r + j\gamma_i = \frac{\overline{V_n I_n^*}}{\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}} \tag{3.11}$$

For arbitrary Z, both γ_r and γ_i must be known to evaluate the term $\mathrm{Re}\,(\gamma Z^*)$. Some references define the correlation coefficient to be $\gamma = \overline{V_n^* I_n}/\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}$. In this case, the correlation term in Eq. (3.10) is $\mathrm{Re}\,(\gamma Z)$.

Noise formulas derived by a phasor analysis of circuits containing complex impedances can be converted into formulas for circuits containing real impedances, i.e. resistors, by setting to zero in the formulas the reactive or imaginary part of all impedances and setting $\gamma_i = 0$ and $\gamma_r = \rho$, where ρ is real. However, the procedure cannot be done in reverse. For this reason, noise formulas derived by a phasor analysis are the more general form of the formulas.

3.1.3 Correlation Impedance and Admittance

The concepts of a correlation impedance Z_γ and a correlation admittance Y_γ between a noise voltage V_n and a noise current I_n are often used in the noise literature. These are defined by

$$Z_\gamma = R_\gamma + jX_\gamma = \frac{\overline{V_n I_n^*}}{\overline{i_n^2}} = \gamma \frac{\sqrt{\overline{v_n^2}}}{\sqrt{\overline{i_n^2}}} = (\gamma_r + j\gamma_i)\,\frac{\sqrt{\overline{v_n^2}}}{\sqrt{\overline{i_n^2}}} \tag{3.12}$$

$$Y_\gamma = G_\gamma + jB_\gamma = \frac{\overline{V_n^* I_n}}{\overline{v_n^2}} = \gamma^* \frac{\sqrt{\overline{i_n^2}}}{\sqrt{\overline{v_n^2}}} = (\gamma_r - j\gamma_i)\,\frac{\sqrt{\overline{i_n^2}}}{\sqrt{\overline{v_n^2}}} \tag{3.13}$$

It follows that $Z_\gamma Y_\gamma = |\gamma|^2$. With these definitions, Eq. (3.10) can be written in the alternate forms

$$\begin{aligned} \overline{v_{\mathrm{sum}}^2} &= \overline{v_n^2} + \overline{i_n^2}\left[2\,\mathrm{Re}\,(Z_\gamma Z^*) + |Z|^2\right] \\ &= \overline{v_n^2}\left[1 + 2\,\mathrm{Re}\left(Y_\gamma^* Z^*\right)\right] + \overline{i_n^2}\,|Z|^2 \end{aligned} \tag{3.14}$$

3.1.4 Example Noise Phasor Calculations

Parallel RC Network

Consider the parallel RC circuit shown in Fig. 3.2(a). The thermal noise generated by the resistor is modeled by a series voltage source. Voltage division can be used to solve for the open circuit noise voltage across the network. It is given by

$$V_n = V_t \frac{1/j\omega C}{R + 1/j\omega C} = \frac{V_t}{1 + j\omega RC} \tag{3.15}$$

The mean square voltage in the band Δf is

$$\begin{aligned} \overline{v_n^2} &= \overline{\left(\frac{V_t}{1 + j\omega RC} \times \frac{V_t^*}{1 - j\omega RC} \right)} \\ &= \frac{\overline{V_t V_t^*}}{1 + (\omega RC)^2} = \frac{4kTR\Delta f}{1 + (2\pi f RC)^2} \end{aligned} \tag{3.16}$$

Because this is a function of frequency, the band Δf must be small.

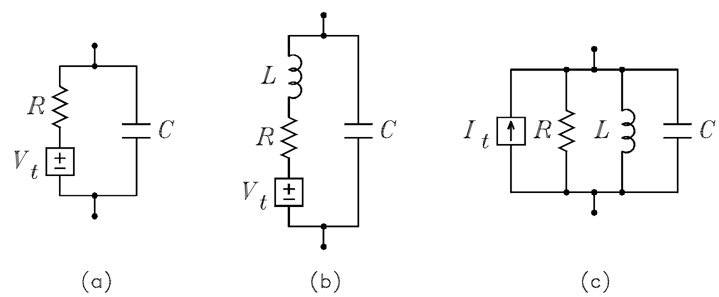

Figure 3.2: Circuits for example noise phasor calculations.

The total noise is obtained by replacing Δf with df and integrating. It is

$$\overline{v_n^2} = \int_0^\infty \frac{4kTR}{1 + (2\pi f RC)^2} df \tag{3.17}$$

This integral is evaluated in the Section 2.1.8 where it is shown that

$$\overline{v_n^2} = \frac{kT}{C} \tag{3.18}$$

This example illustrates a second method of solving for the total noise generated by a parallel RC network.

Correlation Example

Consider the RLC network shown in Fig. 3.2(b). The thermal noise generated by the resistor is modeled by a series voltage source. Voltage division can be used to solve for the open-circuit noise voltage. It is given by

$$V_{n(\mathrm{oc})} = V_t \frac{1/j\omega C}{R + j\omega Ls + 1/j\omega C} = \frac{V_t}{1 - \omega^2 LC + j\omega RC} \tag{3.19}$$

The short-circuit noise current is given by

$$I_{n(\mathrm{sc})} = \frac{V_t}{R + j\omega L} = \frac{V_t/R}{1 + j\omega L/R} \tag{3.20}$$

With $s = j\omega$ and $\omega = 2\pi f$, the mean-square values are

$$\overline{v_{n(\mathrm{oc})}^2} = \frac{\overline{V_t V_t^*}}{(1 - \omega^2 LC)^2 + (\omega RC)^2} = \frac{4kTR\Delta f}{(1 - \omega^2 LC)^2 + (\omega RC)^2} \tag{3.21}$$

$$\overline{i_{n(\mathrm{sc})}^2} = \frac{\overline{V_t V_t^*}/R^2}{1 + (\omega L/R)^2} = \frac{4kT\Delta f/R}{1 + (\omega L/R)^2} \tag{3.22}$$

To solve for the correlation coefficient between $V_{n(\mathrm{oc})}$ and $I_{n(\mathrm{sc})}$, it is necessary to evaluate $V_{n(\mathrm{oc})} I_{n(\mathrm{sc})}^*$. It is given by

$$\begin{aligned} V_{n(\mathrm{oc})} I_{n(\mathrm{sc})}^* &= \frac{V_t}{1 - \omega^2 LC + j\omega RC} \frac{V_t^*/R}{1 - j\omega L/R} \\ &= \frac{V_t V_t^*/R}{1 + j\omega \left(RC - L/R + \omega^2 L^2 C/R\right)} \end{aligned} \tag{3.23}$$

The correlation coefficient is given by

$$\begin{aligned} \gamma &= \frac{\overline{V_{n(\mathrm{oc})} I_{n(\mathrm{sc})}^*}}{\sqrt{\overline{v_{n(\mathrm{oc})}^2}}\sqrt{\overline{i_{n(\mathrm{sc})}^2}}} \\ &= \frac{1}{\sqrt{\overline{v_{n(\mathrm{oc})}^2}}\sqrt{\overline{i_{n(\mathrm{sc})}^2}}} \frac{\overline{V_t V_t^*}/R}{1 + j\omega \left(RC - L/R + \omega^2 L^2 C/R\right)} \\ &= \frac{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}\sqrt{1 + (\omega L/R)^2}}{1 + j\omega \left(RC - L/R + \omega^2 L^2 C/R\right)} \end{aligned} \tag{3.24}$$

Correlation Impedance and Admittance Example

The correlation impedance of $V_{n(\mathrm{oc})}$ and $I_{n(\mathrm{sc})}$ for the circuit in Fig. 3.2(b) is given by

$$Z_\gamma = \gamma \frac{\sqrt{\overline{v_{n(\mathrm{oc})}^2}}}{\sqrt{\overline{i_{n(\mathrm{sc})}^2}}} = R \frac{1 + (\omega L/R)^2}{1 + j\omega \left(RC - L/R + \omega^2 L^2 C/R\right)} \tag{3.25}$$

The correlation admittance is

$$Y_\gamma = \gamma^* \frac{\sqrt{\overline{i_{n(\mathrm{sc})}^2}}}{\sqrt{\overline{v_{n(\mathrm{oc})}^2}}} = \frac{1}{R} \frac{\left(1-\omega^2 LC\right)^2 + (\omega RC)^2}{1 - j\omega\left(RC - L/R + \omega^2 L^2 C/R\right)} \tag{3.26}$$

3.2 Noise Bandwidth

The bandwidth of a filter is commonly specified as the half-power or −3 dB bandwidth. Given a filter transfer function, the −3 dB bandwidth is simply the band of frequencies over which the filter gain is within 3 dB of its maximum gain. It can be analytically determined from the filter transfer function or measured with a sinusoidal signal source. The noise bandwidth of a filter differs from the −3 dB bandwidth. It is defined for a white noise signal rather than a sinusoidal signal. The noise bandwidth is an important concept in noise analyses. It is used to relate the mean-square noise voltage at the output of a filter to the spectral density of white noise at its input.

3.2.1 Definition of Noise Bandwidth

The noise bandwidth of a filter can be defined for low-pass and band-pass filters. We first do this for a low-pass filter. Let the filter have the voltage gain transfer function $A(s)$. Let the maximum value of $|A(j2\pi f)|$ be denoted by A_0. If white noise voltage having the constant spectral density $S_v(f) = K_v$ is applied to the input of the filter, the mean-square output voltage is given by

$$\overline{v_{o1}^2} = \int_0^\infty S_v(f) \left|A(j2\pi f)\right|^2 df = K_v \int_0^\infty \left|A(j2\pi f)\right|^2 df \tag{3.27}$$

Now let the white noise be applied to the input of an ideal low-pass filter having a gain of A_0 for $0 \le f \le B_n$ and a gain of 0 elsewhere. The mean-square output voltage is given by

$$\overline{v_{o2}^2} = \int_0^{B_n} S_v(f)\, A_0^2 df = K_v A_0^2 B_n \tag{3.28}$$

If B_n is chosen so that $\overline{v_{o1}^2} = \overline{v_{o2}^2}$, it follows by ratios that

$$B_n = \frac{1}{A_0^2} \int_0^\infty \left|A(j2\pi f)\right|^2 df \tag{3.29}$$

We define B_n as the noise bandwidth of the filter. It is the bandwidth of an ideal filter which passes the same mean-square noise voltage, where the input signal is white noise and the gain of the ideal filter is equal to the maximum value of the filter gain. The noise bandwidth of a band-pass filter is defined in a similar way. Graphical interpretations are shown in Fig.3.3.

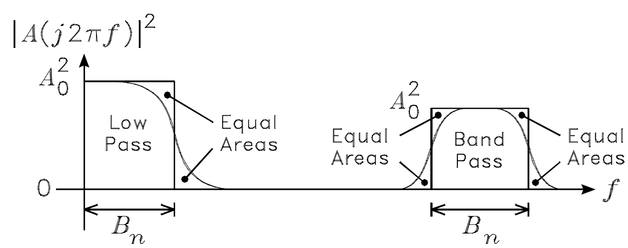

Figure 3.3: Graphical interpretations of filter noise bandwidths.

3.2.2 Noise Bandwidths of Low-Pass Filters

Two classes of low-pass filters are often used in measuring noise. One has n real poles, all with the same frequency. Its magnitude-squared transfer function is given by

$$|A(j2\pi f)|^2 = \frac{A_0^2}{\left[1 + (f/f_0)^2\right]^n} \tag{3.30}$$

The other is a n-pole Butterworth filter. Its magnitude-squared transfer function is given by

$$|A(j2\pi f)|^2 = \frac{A_0^2}{1 + (f/f_0)^{2n}} \tag{3.31}$$

For a given n, the Butterworth filter exhibits the flattest response before rolloff. However, because it has complex poles for $n \geq 2$, it exhibits ringing in its transient response. When this is undesirable, the real pole filter is used.

Table 3.1 gives the noise bandwidth for the two filters as a function of the number of poles n for $1 \leq n \leq 5$. For the real-pole filter, the noise bandwidth is given as a function of both the pole frequency f_0 and the

Table 3.1: Noise Bandwidths of Low-Pass Filters

Number of Poles	Slope dB/dec	Real Pole B_n		Butterworth B_n
1	20	$1.571f_0$	$1.571B_3$	$1.571B_3$
2	40	$0.785f_0$	$1.220B_3$	$1.111B_3$
3	60	$0.589f_0$	$1.155B_3$	$1.042B_3$
4	80	$0.491f_0$	$1.129B_3$	$1.026B_3$
5	100	$0.420f_0$	$1.114B_3$	$1.017B_3$

-3 dB bandwidth B_3. For the Butterworth filter, the noise bandwidth is given as a function of the -3 dB bandwidth B_3, which is equal to f_0. The table shows that the noise bandwidth approaches the -3 dB bandwidth as the number of poles is increased.

Figure 3.4 shows example plots of the dB gain versus frequency for three $4th$-order low-pass filters. The low-frequency gain is normalized to 0 dB for each. Curve a is for a Butterworth filter. Curve b is for a real pole filter with the same f_0 as the Butterworth. Curve c is for a real pole filter with f_0 increased so that it has the same -3 dB frequency as the Butterworth. From Table 3.1, it follows that the noise bandwidth for curve c is greater than that of the Butterworth by the factor $1.129/\left(0.491 \times 1.026\right) = 2.24$.

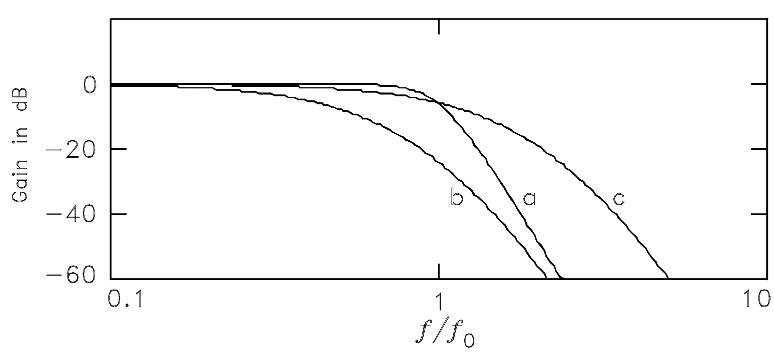

Figure 3.4: Example $4th$ order filter responses.

The unit step response of the three filters is shown in Fig. 3.5. Note the damped ringing for the Butterworth case. The peak overshoot is approximately 11%. All three responses exhibit an apparent delay before the voltage begins to rise. This is a characteristic of all low-pass filters of order greater than 1. Curve c exhibits the shortest rise time. This is because the filter exhibits the highest gain at high frequencies.

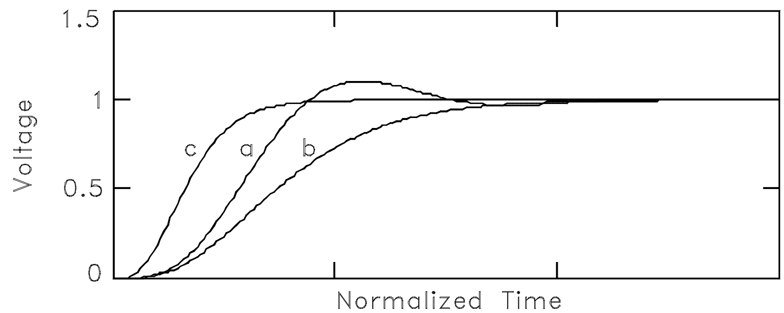

Figure 3.5: Unit step response for the three filters of Fig. 3.4.

3.2.3 Noise Bandwidth of Band-Pass Filters.

Band-pass filters are used in making spot noise measurements. The noise bandwidth of the filter must be small enough so that the spectral density of the input noise is approximately constant over the bandwidth. The spot noise voltage is obtained by dividing the rms noise output voltage of the filter by the square root of its noise bandwidth. Second-order bandpass filters are often used for these measurements. Such a filter has the voltage-gain transfer function

$$A(s) = A_0 \frac{(1/Q)(s/2\pi f_0)}{(s/2\pi f_0)^2 + (1/Q)(s/2\pi f_0) + 1} \tag{3.32}$$

where f_0 is the resonance frequency and Q is the quality factor. For $s = j2\pi f$, the maximum gain occurs at $f = f_0$ and is equal to A_0. The noise bandwidth is given by

$$B_n = \int_0^\infty \frac{(f/Qf_0)^2\, df}{\left[1 - (f/f_0)^2\right]^2 + (f/Qf_0)^2} = \frac{\pi f_0}{2Q} \tag{3.33}$$

Let B_3 be the -3 dB bandwidth of the filter. The quality factor is related to f_0 and B_3 by $Q = f_0/B_3$. It follows that an alternate expression for the noise bandwidth is

$$B_n = \frac{\pi}{2} B_3 \tag{3.34}$$

This expression is often used to approximate the noise bandwidth of band-pass filters of higher order than second.

A second-order band-pass filter having real poles can be realized by a first-order high-pass filter in cascade with a first-order low-pass filter. Such a filter is shown in Fig. 3.6. The op amp acts as a buffer to prevent loading effects. The transfer filter function is

$$\begin{aligned}\frac{V_o}{V_i} &= \frac{s/\omega_1}{1+s/\omega_1} \times \frac{1}{1+s/\omega_2} \\ &= \frac{\omega_2}{\omega_1+\omega_2} \frac{(1/\omega_1 + 1/\omega_2)\, s}{s^2/\omega_1\omega_2 + (1/\omega_1 + 1/\omega_2)\, s + 1}\end{aligned} \tag{3.35}$$

where the pole frequencies are given by

$$\omega_1 = 2\pi f_1 = \frac{1}{R_1 C_1} \tag{3.36}$$

$$\omega_2 = 2\pi f_2 = \frac{1}{R_2 C_2} \tag{3.37}$$

When Eq. (3.35) is compared to Eq. (3.32), it follows that

$$A_0 = \frac{\omega_2}{\omega_1+\omega_2} = \frac{f_2}{f_1+f_2} \tag{3.38}$$

$$\omega_0 = 2\pi f_0 = \sqrt{\omega_1\omega_2} = 2\pi\sqrt{f_1 f_2} \tag{3.39}$$

$$Q = \frac{\omega_0}{\omega_1+\omega_2} = \frac{f_0}{f_1+f_2} \tag{3.40}$$

Note that the pole frequencies f_1 and f_2 are not the -3 dB frequencies of the filter. The -3 dB frequencies are obtained by setting $|A(j2\pi f)|^2 = A_0^2/2$ and solving for f. The two solutions are

$$f_{b,a} = f_0 \left(\frac{1}{2Q} \pm \sqrt{\frac{1}{4Q^2} + 1} \right) \tag{3.41}$$

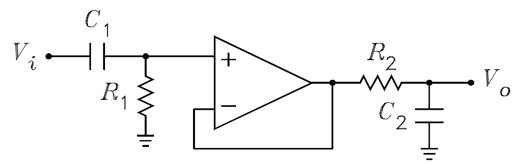

Figure 3.6: Band-pass filter consisting of a high-pass filter followed by a low-pass filter.

where the + sign is used for f_b and the − sign for f_a. The −3 dB bandwidth B_3 and the noise bandwidth B_n are given by

$$B_3 = \frac{f_0}{Q} = f_1 + f_2 = f_b - f_a \tag{3.42}$$

$$B_n = \frac{\pi f_0}{2Q} = \frac{\pi}{2}(f_1 + f_2) = \frac{\pi}{2}(f_b - f_a) \tag{3.43}$$

The Bode magnitude plot for the filter is shown in Fig. 3.7. The asymptotic gain is unity in the midband region. The maximum value of the actual gain is less than unity, i.e. $A_0 < 1$.

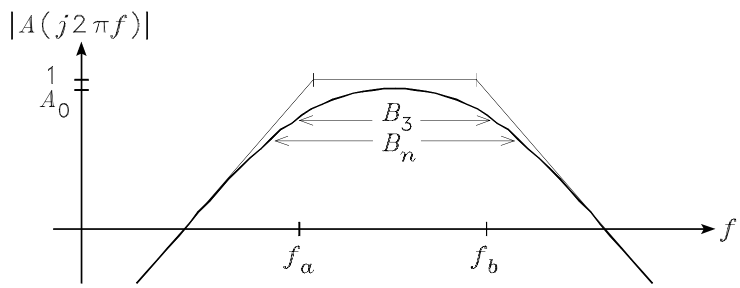

Figure 3.7: Bode magnitude plot for the 2*nd* order band-pass filter.

Example 3 *A parallel RLC circuit is shown in Fig. 3.2(c) in which the resistor is model by its Norton noise model. Calculate the mean-square voltage across the circuit.*

Solution. The impedance of the circuit can be written

$$Z(s) = \left[\frac{1}{R} + \frac{1}{Ls} + Cs\right]^{-1} = R\frac{s/\omega_0}{(s/\omega_0)^2 + (1/Q)(s/\omega_0) + 1} \tag{3.44}$$

where

$$\omega_0 = 2\pi f_0 = \frac{1}{\sqrt{LC}} \tag{3.45}$$

$$Q = \omega_0 RC = 2\pi f_0 RC \tag{3.46}$$

Let $S_i(f) = 4kT/R$ be the spectral density of the thermal noise current. The mean-square noise voltage across the circuit is given by

$$\overline{v_n^2} = \int_0^\infty S_i(f) \left|Z(j2\pi f)\right|^2 df = \frac{4kT}{R} \int_0^\infty \left|Z(j2\pi f)\right|^2 df \tag{3.47}$$

Because $Z(s)$ is of the same form as Eq. (3.32) with $A_0 = R$, the integral in Eq. (3.47) is equal to R^2 multiplied by the noise bandwidth of a second-order filter. Thus the mean-square noise voltage is given by

$$\overline{v_n^2} = \frac{4kT}{R} R^2 \frac{\pi f_0}{2Q} = \frac{kT}{C} \tag{3.48}$$

This is the same as for the parallel RC circuit of Section 3.1.4. It follows that the addition of a parallel inductor does not change the mean-square voltage across a parallel RC circuit.

3.2.4 Measuring Noise Bandwidth of a Filter

The noise bandwidth of any filter can be measured if a white noise source and another filter with a known noise bandwidth are available. The maximum gain A_0 of each filter must be known. Consider the test setup shown in Fig. 3.8. Let the spectral density of the source be denoted by $S_v(f) = S_0$. With both filters driven simultaneously from the source, the mean-square output voltages can be written

$$\overline{v_{o1}^2} = S_0 A_{01}^2 B_{n1} \tag{3.49}$$

$$\overline{v_{o2}^2} = S_0 A_{02}^2 B_{n2} \tag{3.50}$$

It follows by ratios that

$$B_{n2} = B_{n1} \times \frac{A_{01}^2}{A_{02}^2} \times \frac{\overline{v_{o2}^2}}{\overline{v_{o1}^2}} \tag{3.51}$$

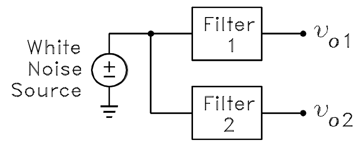

Figure 3.8: Illustration for measuring the noise bandwidth of a filter.

3.3 Measuring Noise

Noise is usually measured at an amplifier output where the voltage is the largest and easiest to measure. The output noise is referred to the input by dividing by the gain. A filter with a known noise bandwidth should precede the voltmeter to limit the bandwidth. The measuring voltmeter should have a bandwidth that is at least 10 times the filter bandwidth. The voltmeter crest factor is the ratio of the peak input voltage to the full scale rms meter reading at which the internal meter circuits overload. For a sine-wave, the minimum voltmeter crest factor is $\sqrt{2}$. For noise measurements, a higher crest factor is required. For Gaussian noise, a crest factor of 3 gives an error of 0.27%. A crest factor of 4 gives an error less than 0.0063%. To minimize errors caused by overload on noise peaks, measurements should be made on the lower one-third to one-half of the voltmeter scale.

A true rms voltmeter is preferred for noise measurements. Fig. 3.9 shows the block diagram of such a meter. The input signal v_{in} is applied to a calibrated attenuator which sets the voltage range of the meter. The attenuator output voltage is labeled v. This voltage is squared by an analog multiplier circuit. The average value of the squared voltage is taken. A low-pass filter performs this operation. The cutoff frequency of the filter must be much lower than the lowest frequency harmonic in the squared voltage waveform. The square root of the output of the averaging circuit is a dc voltage that is equal to the rms value of the ac voltage output of the attenuator. The operations are summarized by the equation

$$v_{\text{rms}} = \sqrt{\langle v^2(t) \rangle} \tag{3.52}$$

where the symbols $\langle \cdot \rangle$ denote a time average. This voltage is equal to the rms value of v_{in} to within a constant that is set by the input attenuator.

For a sine wave input signal, the internal signals are shown in the

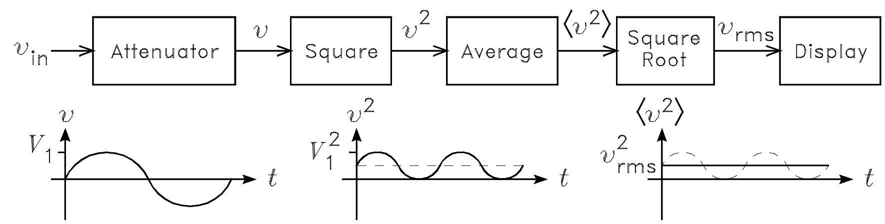

Figure 3.9: Diagram of a true rms voltmeter showing internal voltage waveforms.

figure. Note that the squared voltage has a frequency that is twice the input frequency and it always positive. For a sine wave, the average value of the square of the voltage is one-half the peak value. The average value is shown as a dashed line through the squared waveform. The averaging circuit passes the dc value of the squared waveform. The display is typically an analog meter or a digital readout.

An average responding meter is one which responds to the average rectified value of the input voltage but has a scale calibrated to read rms. The block diagram of such a meter is shown in Fig. 3.10. The circuit consists of an input attenuator which sets the voltmeter range, a full-wave rectifier which takes the absolute value of the signal, a low-pass filter averaging circuit which passes the dc value of the rectified voltage, a scale constant k_m, and a display. The scale constant is chosen so that the display reads the rms value of the voltage when the input signal is a sine wave.

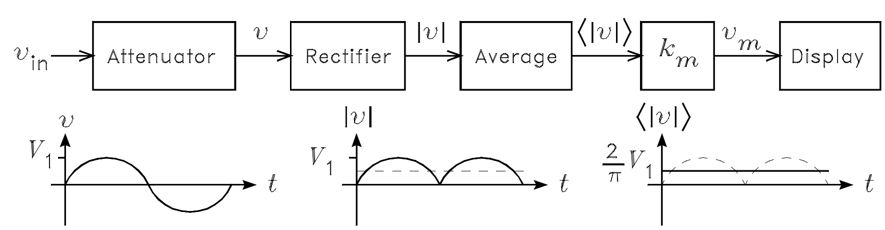

Figure 3.10: Diagram of an average responding voltmeter showing internal waveforms.

Let the attenuator output voltage be given by $v\,(t) = V_1 \sin{(2\pi f t)}$, where V_1 is a positive constant and f is the frequency. The period of the voltage is $T = 1/f$. The average value of $|v(t)|$ over one period is the same as it is over one-half a period. This is obtained as follows:

$$\langle |v(t)| \rangle = \frac{2}{T} \int_0^{T/2} |v(t)| \, dt = \frac{2V_1}{T} \int_0^{T/2} \sin(2\pi f t) \, dt = \frac{2}{\pi} V_1 \tag{3.53}$$

The rms value of $v(t)$ is given by

$$v_{\text{rms}} = \sqrt{\frac{2}{T} \int_0^{T/2} v^2(t) \, dt} = \sqrt{\frac{2V_1^2}{T} \int_0^{T/2} \sin^2(2\pi f t) \, dt} = \frac{V_1}{\sqrt{2}} \tag{3.54}$$

For the display to read the correct voltage, the scale constant k_m must satisfy

$$v_{\text{rms}} = k_m \langle |v(t)| \rangle \tag{3.55}$$

Thus k_m is given by

$$k_m = \frac{v_{\text{rms}}}{\langle |v(t)| \rangle} = \frac{V_1/\sqrt{2}}{2V_1/\pi} = \frac{\pi}{2\sqrt{2}} = 1.11 \tag{3.56}$$

Now let a noise voltage be applied to the meter. We assume that the noise has an average or dc value of zero and that its amplitude distribution can be modeled by a normal or Gaussian probability density function. The meter reading is given by

$$v_m = \frac{\pi}{2\sqrt{2}} \langle |v(t)| \rangle = \frac{\pi}{2\sqrt{2}} \int_{-\infty}^{\infty} |v| \, p(v) \, dv \tag{3.57}$$

where $p(v)$ is the probability density function given by

$$p(v) = \frac{1}{\sqrt{2\pi} v_{\text{rms}}} \exp\left(\frac{-v^2}{2v_{\text{rms}}^2} \right) \tag{3.58}$$

Because $|v| \, p(v)$ is an even function, the integral in Eq. (3.57) is twice the integral over positive values of v and is given by

$$\begin{aligned} v_m &= k_m \times \frac{2}{\sqrt{2\pi} v_{\text{rms}}} \int_0^{\infty} v \exp\left(\frac{-v^2}{2v_{\text{rms}}^2} \right) dv \\ &= \frac{\pi}{2\sqrt{2}} \times \frac{2}{\sqrt{2\pi} v_{\text{rms}}} \times v_{\text{rms}}^2 = \frac{\sqrt{\pi}}{2} v_{\text{rms}} \end{aligned} \tag{3.59}$$

It follows that the v_{rms} is given by

$$v_{\text{rms}} = \frac{2}{\sqrt{\pi}} v_m = 1.128 v_m \tag{3.60}$$

Thus the meter reading must be multiplied by 1.128 to obtain the correct reading with a Gaussian noise voltage. This is a 2.416 dB increase.

Chapter 4

Amplifier Noise Models

4.1 The $v_n - i_n$ Amplifier Noise Model

A general noise model of an amplifier can be obtained by reflecting all internal noise sources to the input. In order for the reflected sources to be independent of the source impedance, two noise sources are required – a series voltage source and a shunt current source. In general, these sources are correlated. The amplifier noise model is described in this section. The equivalent noise input voltage is derived for the case where the source is represented by a Thévenin equivalent. In addition, the equivalent noise input current is derived for the case where the source is represented by a Norton equivalent. A more general phasor analysis is used.

4.1.1 Thévenin Source

Equivalent Noise Input Voltage

Figure 4.1 shows the amplifier noise model with a Thévenin input source, where V_s is the source voltage, $Z_s = R_s + jX_s$ is the source impedance, V_{ts} is the thermal noise voltage generated by the source, and V_n and I_n are the noise sources representing the noise generated by the amplifier. The output voltage is given by

$$V_o = AV_i \frac{Z_L}{Z_o + Z_L} = \frac{AZ_i}{Z_s + Z_i} \frac{Z_L}{Z_o + Z_L} \left[V_s + (V_{ts} + V_n + I_n Z_s) \right] \quad (4.1)$$

where A is the voltage gain and Z_i is the input impedance. The equivalent noise input voltage V_{ni} is defined as the voltage in series with V_s

that generates the same noise voltage at the output as all noise sources in the circuit. It consists of the terms in parenthesis in Eq. (4.1) and is given by

$$V_{ni} = V_{ts} + V_n + I_n Z_s \tag{4.2}$$

Note that this is independent of both A and Z_i. It is simply the noise voltage across Z_i considering Z_i to be an open circuit.

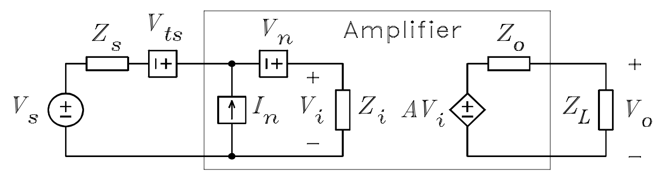

Figure 4.1: $v_n - i_n$ amplifier model with Thevénin source.

The mean-square value of V_{ni} is solved for as follows:

$$\begin{aligned}
\overline{v_{ni}^2} &= \overline{(V_{ts} + V_n + I_n Z_s)(V_{ts}^* + V_n^* + I_n^* Z_s^*)} \\
&= \overline{V_{ts}V_{ts}^*} + \overline{V_n V_n^*} + 2\,\mathrm{Re}\left[\left(\overline{V_n I_n^*}\right) Z_s^*\right] + \overline{(I_n Z_s)(I_n^* Z_s^*)} \\
&= 4kTR_s\Delta f + \overline{v_n^2} + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\,(\gamma Z_s^*) + \overline{i_n^2}\,|Z_s|^2 \\
&= 4kTR_s\Delta f + \overline{v_n^2} + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,(\gamma_r R_s + \gamma_i X_s) \\
&\quad + \overline{i_n^2}\left(R_s^2 + X_s^2\right)
\end{aligned} \tag{4.3}$$

where $\gamma = \gamma_r + j\gamma_i$ is the correlation coefficient between V_n and I_n and it is assumed that V_{ts} is independent of both V_n and I_n. For $|Z_s|$ very small, $\overline{v_{ni}^2} \simeq \overline{v_n^2}$ and the correlation coefficient is not important. Similarly, for $|Z_s|$ very large, $\overline{v_{ni}^2} \simeq \overline{i_n^2}\,|Z_s|^2$ and the correlation coefficient is again not important.

Effect of Series and Shunt Impedances at the Input

Figure 4.2(a) shows the input circuit of an amplifier with an impedance Z_1 added in series with a Thévenin source. The noise source V_{t1} models the thermal noise generated by Z_1. As is shown above, the equivalent noise voltage in series with the source can be solved for by first solving for the open-circuit input voltage, i.e. the input voltage considering Z_i

to be an open circuit. It is given by

$$\begin{aligned} V_{i(oc)} &= V_s + V_{ts} + V_{t1} + V_n + I_n (Z_s + Z_1) \\ &= V_s + V'_{ni} \end{aligned} \tag{4.4}$$

where V'_{ni} is the equivalent noise voltage in series with the source. It is given by

$$\begin{aligned} V'_{ni} &= V_{ts} + V_{t1} + V_n + I_n (Z_s + Z_1) \\ &= V_{ts} + V'_n + I'_n Z_s \end{aligned} \tag{4.5}$$

where V'_n and I'_n are the new values of V_n and I_n on the source side of Z_1. It follows from this equation that

$$V'_n = V_{t1} + V_n + I_n Z_1 \tag{4.6}$$

$$I'_n = I_n \tag{4.7}$$

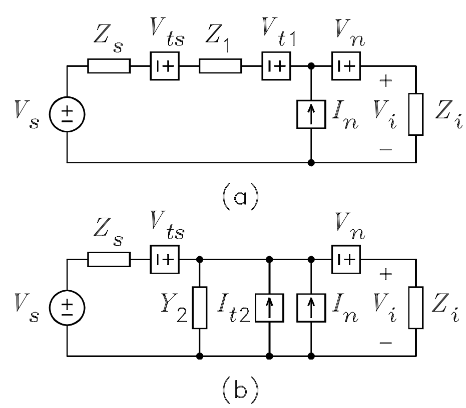

Figure 4.2: Amplifier input circuit with Thévenin source. (a) Series impedance added at input. (b) Shunt admittance added at input.

It follows that the addition of a series impedance at the input of an amplifier increases the V_n noise but does not change the I_n noise. If Z_1 is not to increase the noise, $|Z_1|$ should be as small as possible. If Z_1 is lossless, it generates no noise itself so that $V_{t1} = 0$. The mean-square values and the correlation coefficient for V'_n and I'_n are given by

$$\overline{v'^2_n} = \overline{v^2_{t1}} + \overline{v^2_n} + 2\sqrt{\overline{v^2_n}}\sqrt{\overline{i^2_n}}\,\mathrm{Re}\,(\gamma Z_1^*) + \overline{i^2_n}\,|Z_1|^2 \tag{4.8}$$

$$\overline{i_n'^2} = \overline{i_n^2} \tag{4.9}$$

$$\gamma' = \frac{\gamma\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}} + \overline{i_n^2} Z_1}{\sqrt{\overline{v_n'^2}}\sqrt{\overline{i_n'^2}}} \tag{4.10}$$

The mean-square equivalent noise input voltage on the source side of Z_1 is given by

$$\begin{aligned}\overline{v_{ni}'^2} &= 4kT\,\mathrm{Re}\,(Z_s + Z_1)\,\Delta f + \overline{v_n^2} \\ &\quad + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\left[\gamma\left(Z_s^* + Z_1^*\right)\right] + \overline{i_n^2}\left|Z_s + Z_1\right|^2 \end{aligned} \tag{4.11}$$

Figure 4.2(b) shows the input circuit of an amplifier with a admittance Y_2 added in parallel with the source. The noise source I_{t1} models the thermal noise generated by Y_2. The open-circuit input voltage is given by

$$\begin{aligned} V_{i(oc)} &= (V_s + V_{ts})\,\frac{1/Y_2}{Z_s + 1/Y_2} + V_n + (I_{t1} + I_n)\,Z_s \| (1/Y_2) \\ &= \left(V_s + V_{ni}''\right)\frac{1/Y_2}{Z_s + 1/Y_2} \end{aligned} \tag{4.12}$$

where V_{ni}'' is the equivalent noise voltage in series with the source. It is given by

$$\begin{aligned} V_{ni}'' &= V_{ts} + V_n\,(1 + Z_s Y_2) + (I_{t1} + I_n)\,Z_s \\ &= V_{ts} + V_n'' + I_n'' Z_s \end{aligned} \tag{4.13}$$

where V_n'' and I_n'' are the new values of V_n and I_n on the source side of Y_2. It follows from this equation that

$$V_n'' = V_n \tag{4.14}$$

$$I_n'' = I_{t1} + V_n Y_2 + I_n \tag{4.15}$$

It follows that the addition of a parallel impedance at the input of an amplifier increases the I_n noise but does not change the V_n noise. If Y_2 is not to increase the noise, $|Y_2|$ should be as small as possible. If Y_2 is lossless, it generates no noise itself so that $I_{t1} = 0$. The mean-square values and the correlation coefficient for V_n'' and I_n'' are given by

$$\overline{v_n''^2} = \overline{v_n^2} \tag{4.16}$$

$$\overline{i_n''^2} = \overline{i_{t2}^2} + \overline{v_n^2}\,|Y_2|^2 + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\,(\gamma Y_2) + \overline{i_n^2} \tag{4.17}$$

$$\gamma'' = \frac{\overline{v_n^2}Y_2^* + \gamma\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}}{\sqrt{\overline{v_n''^2}}\sqrt{\overline{i_n''^2}}} \tag{4.18}$$

The mean-square equivalent noise input voltage on the source side of Y_2 is given by

$$\begin{aligned}\overline{v_{ni}''^2} &= \overline{v_{ts}^2} + \overline{v_n^2}\,|1 + Z_sY_2|^2 + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\,[\gamma Z_s^*\,(1 + Z_sY_2)] \\ &\quad + \left(\overline{i_{t1}^2} + \overline{i_n^2}\right)|Z_s|^2 \end{aligned} \tag{4.19}$$

Dc bias networks and rf matching networks usually consist of series and parallel elements at the input to an amplifier. The effect of these elements on the amplifier noise can be analyzed by transforming the V_n and I_n sources from the amplifier input back to the source by use of the above relations. This is illustrated in the following example.

Example 1 *Figure 4.3shows the input circuit of an amplifier. It is given that $R_s = 75\ \Omega$, $R_1 = 1\ k\Omega$, $C = 10\ nF$, $R_2 = 100\ \Omega$, $\sqrt{\overline{v_n^2}} = 2\ nV$, $\sqrt{\overline{i_n^2}} = 1.5\ pA$, $\gamma = 0.2{+}j0.1$. The noise specifications are for a frequency $f = 100kHz$ and a bandwidth $\Delta f = 1\ Hz$. Calculate $\sqrt{\overline{v_{ni}^2}}$ in series with the source.*

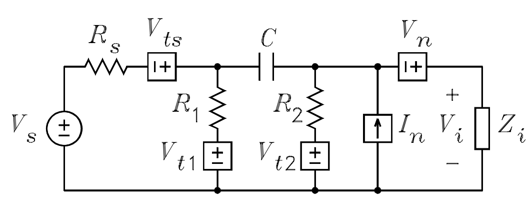

Figure 4.3: Amplifier input circuit.

Solution. To the left of R_2

$$\sqrt{\overline{v_{na}^2}} = \sqrt{\overline{v_n^2}} = 2\ \text{nV}$$

$$\sqrt{\overline{i_{na}^2}} = \left(\frac{4kT\Delta f}{R_2} + \frac{\overline{v_n^2}}{R_2^2} + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\left(\frac{\gamma}{R_2}\right) + \overline{i_n^2}\right)^{1/2} = 24 \text{ pA}$$

$$\gamma_a = \frac{\overline{v_n^2}/R_2 + \gamma\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}}{\sqrt{\overline{v_{na}^2}}\sqrt{\overline{i_{na}^2}}} = 0.847 + j6.26 \times 10^{-3}$$

The capacitor impedance is $Z_C = 1/j2\pi f = -j159\ \Omega$. To the left of C

$$\sqrt{\overline{v_{nb}^2}} = \left(\overline{v_{na}^2} + 2\sqrt{\overline{v_{na}^2}}\sqrt{\overline{i_{na}^2}}\,\mathrm{Re}\left(\gamma_a Z_C^*\right) + \overline{i_n^2}\,|Z_C|^2\right)^{1/2} = 4.3 \text{ nV}$$

$$\sqrt{\overline{i_{nb}^2}} = \sqrt{\overline{i_{na}^2}} = 24 \text{ pA}$$

$$\gamma_b = \frac{\gamma_a\sqrt{\overline{v_{na}^2}}\sqrt{\overline{i_{na}^2}} + \overline{i_{na}^2}Z_C}{\sqrt{\overline{v_{nb}^2}}\sqrt{\overline{i_{nb}^2}}} = 0.394 - j0.885$$

To the left of R_1

$$\sqrt{\overline{v_{nc}^2}} = \sqrt{\overline{v_{nb}^2}} = 4.3 \text{ nV}$$

$$\sqrt{\overline{i_{nc}^2}} = \left(\frac{\overline{v_n^2}}{R_1^2} + \frac{4kT\Delta f}{R_1} + \overline{i_{nb}^2}\right)^{1/2} = 26.3 \text{ pA}$$

$$\gamma_c = \frac{\overline{v_{nc}^2}/R_1 + \gamma_b\sqrt{\overline{v_{nb}^2}}\sqrt{\overline{i_{nb}^2}}}{\sqrt{\overline{v_{nc}^2}}\sqrt{\overline{i_{nc}^2}}} = 0.523 - j0.807$$

The equivalent noise voltage in series with the source is

$$\begin{aligned}\sqrt{\overline{v_{ni}^2}} &= \left(4kTR_s\Delta f + \overline{v_{nc}^2} + 2\sqrt{\overline{v_{nc}^2}}\sqrt{\overline{i_{nc}^2}}\,\mathrm{Re}\left(\gamma_c R_s\right) + \overline{i_{nc}^2}R_s^2\right)^{1/2} \\ &= 5.69 \text{ nV}\end{aligned}$$

4.1.2 Norton Source

Figure 4.4 shows the amplifier model with a Norton input source, where I_s is the source current, $Y_s = G_s + jB_s$ is the source admittance, and I_{ts} is the thermal noise current generated by the source. The output voltage is given by

$$V_o = AV_i\frac{Z_L}{Z_o + Z_L} = \frac{A}{Y_s + Y_i}\,\frac{Z_L}{Z_o + Z_L}\left[I_s + (I_{ts} + V_nY_s + I_n)\right] \qquad (4.20)$$

where A is the voltage gain and Y_i is the input admittance. The equivalent noise input current I_{ni} is defined as the current in parallel with I_s that generates the same noise voltage at the output as all noise sources in the circuit. Its value is given by

$$I_{ni} = I_{ts} + V_n Y_s + I_n \tag{4.21}$$

Like V_{ni}, this is independent of both A and Z_i. It is simply the current through Z_i considering Z_i to be a short circuit.

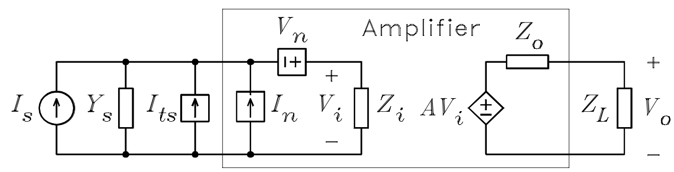

Figure 4.4: $v_n - i_n$ amplifier model with Norton source.

The mean-square value of I_{ni} is solved for as follows:

$$\begin{aligned}
\overline{i_{ni}^2} &= \overline{(I_{ts} + V_n Y_s + I_n)\left(I_{ts}^* + V_n^* Y_s^* + I_n^*\right)} \\
&= \overline{I_{ts} I_{ts}^*} + \overline{(V_n Y_s)\left(V_n^* Y_s^*\right)} + 2\,\mathrm{Re}\left[\left(\overline{V_n I_n^*}\right) Y_s\right] + \overline{I_n I_n^*} \\
&= 4kTG_s\Delta f + \overline{v_n^2}\,|Y_s|^2 + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\,(\gamma Y_s) + \overline{i_n^2} \\
&= 4kTG_s\Delta f + \overline{v_n^2}\left(G_s^2 + B_s^2\right) \\
&\quad + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,(\gamma_r G_s - \gamma_i B_s) + \overline{i_n^2}
\end{aligned} \tag{4.22}$$

where $\gamma = \gamma_r + j\gamma_i$ is the complex correlation coefficient between V_n and I_n.

Effect of Series and Shunt Impedances at the Input

Figure 4.2(a) shows the input circuit of an amplifier with an impedance Z_1 added in series with a Norton source. The noise source V_{t1} models the thermal noise generated by Z_1. As is shown above, the equivalent noise current in parallel with the source can be solved for by first solving for the short-circuit input current, i.e. the input voltage considering Z_i

to be a short circuit. It is given by

$$\begin{aligned} I_{i(sc)} &= \frac{1/Y_s}{1/Y_s + Z_1}(I_s + I_{ts}) + \frac{V_{t1} + V_n}{1/Y_s + Z_1} + I_n \\ &= \frac{1/Y_s}{1/Y_s + Z_1}\left(I_s + I'_{ni}\right) \end{aligned} \tag{4.23}$$

where I'_{ni} is the equivalent noise current in parallel with the source. It is given by

$$\begin{aligned} I'_{ni} &= I_{ts} + (V_{t1} + V_n) Y_s + I_n (1 + Y_s Z_1) \\ &= I_{ts} + V'_n Y_s + I'_n \end{aligned} \tag{4.24}$$

where V'_n and I'_n are the new values of V_n and I_n on the source side of Z_1. It follows from this equation that

$$V'_n = V_{t1} + V_n + I_n Z_1 \tag{4.25}$$

$$I'_n = I_n \tag{4.26}$$

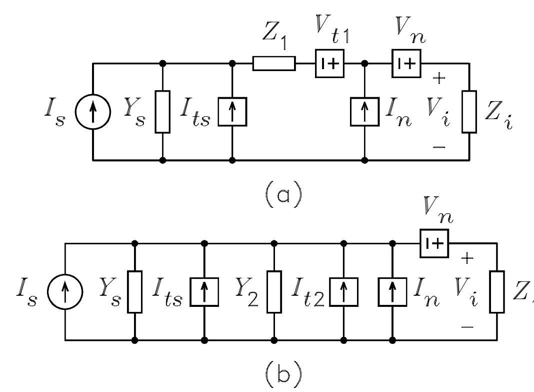

Figure 4.5: Amplifier input circuit with Norton source. (a) Series impedance added at input. (b) Shunt admittance added at input.

It follows that the addition of a series impedance at the input of the amplifier increases the V_n noise but does not change the I_n noise. If Z_1 is not to increase the noise, $|Z_1|$ should be as small as possible. If Z_1 is lossless, it generates no noise itself so that $V_{t1} = 0$. The mean-square values and the correlation coefficient for V'_n and I'_n are given by Eqs.

(4.8) through (4.10).The mean-square equivalent noise input current on the source side of Z_1 is given by

$$\begin{aligned}\overline{i_{ni}'^2} &= 4kT\,\mathrm{Re}\,(Y_s)\,\Delta f + \left(\overline{v_{t1}^2} + \overline{v_n^2}\right)|Y_s|^2 \\ &\quad + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\left[\gamma Y_s\left(1 + Y_s^* Z_1^*\right)\right] + \overline{i_n^2}\,|1 + Y_s Z_1|^2 \qquad (4.27)\end{aligned}$$

Figure 4.2(b) shows the input circuit of an amplifier with a admittance Y_2 added in parallel with the source. The noise source I_{t1} models the thermal noise generated by Y_2. The short-circuit input current is given by

$$\begin{aligned}I_{i(sc)} &= I_s + I_{ts} + I_{t1} + I_n + V_n Y_2 \\ &= I_s + I_{ni}'' \qquad (4.28)\end{aligned}$$

where I_{ni}'' is the equivalent noise current in parallel with the source. It is given by

$$\begin{aligned}I_{ni}'' &= I_{ts} + I_{t1} + I_n + V_n\left(Y_s + Y_2\right) \\ &= I_{ts} + I_n'' + V_n'' Y_s \qquad (4.29)\end{aligned}$$

where V_n'' and I_n'' are the new values of V_n and I_n on the source side of Y_2. It follows from this equation that

$$V_n'' = V_n \qquad (4.30)$$

$$I_n'' = I_{t2} + I_n + V_n Y_2 \qquad (4.31)$$

It follows that the addition of a parallel impedance at the input of an amplifier increases the I_n noise but does not change the V_n noise. If Y_2 is not to increase the noise, $|Y_2|$ should be as small as possible. If Y_2 is lossless, it generates no noise itself so that $I_{t2} = 0$. The mean-square values and the correlation coefficient for V_n'' and I_n'' are given by Eqs. (4.16) through (4.18). The mean-square equivalent noise input current on the source side of Y_2 is given by

$$\begin{aligned}\overline{i_{ni}''^2} &= \overline{v_{ts}^2}\,|Y_s|^2 + \overline{v_n^2}\,|Y + Y_2|^2 \\ &\quad + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\left[\gamma\left(Y_s + Y_2\right)\right] + \overline{i_{t1}^2} + \overline{i_n^2} \qquad (4.32)\end{aligned}$$

4.1.3 Measuring $\overline{v_n^2}$ and $\overline{i_n^2}$

Let the amplifier be driven by a source with a resistive output impedance, i.e. $Z_s = R_s$. With $V_s = 0$ in Eq. (4.1), the mean-square noise voltage at the amplifier output in Fig. 4.1(a) is given by

$$\begin{aligned}\overline{v_{no}^2} = &\left|\frac{AZ_i}{R_s+Z_i}\right|^2 \Big[4kTR_s\Delta f + \overline{v_n^2} \\ &+2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}R_s \operatorname{Re}(\gamma) + \overline{i_n^2}\left|R_s\right|^2\Big]\end{aligned} \tag{4.33}$$

If the source is replaced by a short circuit, i.e. $R_s = 0$, this equation can be solved for $\overline{v_n^2}$ to obtain

$$\overline{v_n^2} = \frac{\overline{v_{no}^2}}{|A|^2} \tag{4.34}$$

The value of A is measured at a frequency where the amplifier gain is a maximum. The gain is measured as the ratio of the amplifier output voltage to its input voltage and is independent of R_s.

Let a resistor R_1 be added in series with the source. If R_1 is very large, the last term in Eq. (4.33) dominates and $\overline{v_{no}^2}$ can be written

$$\overline{v_{no}^2} \simeq \left|\frac{AZ_i}{R_s+R_1+Z_i}\right|^2 \overline{i_n^2}\left(R_s+R_1\right)^2 \tag{4.35}$$

To solve for $\overline{i_n^2}$, this equation implies that Z_i must be known. This can be circumvented by a simple procedure. Drive the amplifier with the source and the added resistor. Adjust the source voltage to obtain a convenient voltage at the amplifier output. Denote this by V_{o1}. The frequency should be the same as that used to determine $\overline{v_n^2}$. Disconnect the source from the input and measure its open-circuit source voltage. Denote this by V_{s1}. To minimize voltmeter loading, this should be measured with the added resistor disconnected. It follows that

$$\left|\frac{V_{o1}}{V_{s1}}\right| = \left|\frac{AZ_i}{R_s+R_1+Z_i}\right| \tag{4.36}$$

Replace the source and added resistor at the amplifier input with a single resistor having the value R_s+R_1. The mean-square output voltage is given by

$$\overline{v_{no}^2} = \left|\frac{AZ_i}{R_s+R_1+Z_i}\right|^2 \overline{i_n^2}R_s^2 = \left|\frac{V_{o1}}{V_{s1}}\right|^2 \overline{i_n^2}\left(R_s+R_1\right)^2 \tag{4.37}$$

This can be solved for $\overline{i_n^2}$ to obtain

$$\overline{i_n^2} = \left|\frac{V_{s1}}{V_{o1}}\right|^2 \frac{\overline{v_{no}^2}}{(R_s + R_1)^2} \tag{4.38}$$

4.2 Noise in Multistage Amplifiers

Figure 4.6 shows the first two stages of a multistage amplifier having N stages. The input circuit of each stage is modeled by its input impedance. Each output circuit is modeled by a Norton equivalent circuit consisting of a parallel current source and impedance. The equivalent noise input voltage for each stage is shown as a series voltage source preceding that stage. For the first stage, it is given by

$$V_{ni1} = V_{ts} + V_{n1} + I_{n1} Z_s \tag{4.39}$$

where V_{ts} is the thermal noise generated by $R_s = \mathrm{Re}\,(Z_s)$. For the succeeding stages it is given by

$$V_{nij} = V_{nj} + I_{nj} Z_{o(j-1)} \tag{4.40}$$

Note that each Z_i and Z_o is noiseless in the amplifier noise model.

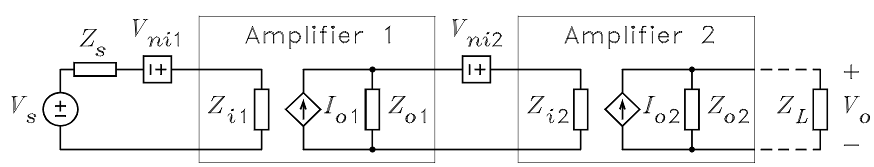

Figure 4.6: Multi-stage amplifier.

The short-circuit output current from the jth stage can be written

$$I_{oj} = g_{mj} V_{ij} = \frac{g_{mj} Z_{ij}}{Z_{o(j-1)} + Z_{ij}} V_{ij(\mathrm{oc})} = G_{mj} V_{ij(\mathrm{oc})} \tag{4.41}$$

where g_{mj} is the transconductance of the jth stage, V_{ij} is the voltage across Z_{ij}, $Z_{o(j-1)} = Z_s$ for $j = 1$, and $V_{ij(\mathrm{oc})}$ is the open circuit input voltage of the jth stage, i.e. the input voltage with Z_{ij} replaced with an open circuit. This equation defines the transconductance G_{mj} given by

$$G_{mj} = \frac{g_{mj} Z_{ij}}{Z_{o(j-1)} + Z_{ij}} \tag{4.42}$$

It represents the ratio of the short-circuit output current to the open-circuit input voltage of the jth stage.

The open-circuit input voltage to the $(j+1)$st stage is given by

$$V_{i(j+1)(\mathrm{oc})} = I_{oj} Z_{oj} = G_{mj} V_{ij(\mathrm{oc})} Z_{oj} \tag{4.43}$$

But the open-circuit input voltage to the $(j+1)$st stage is equal to the open-circuit output voltage of the jth stage, i.e. $V_{i(j+1)(\mathrm{oc})} = V_{oj(\mathrm{oc})}$. Also, the open-circuit input voltage to the jth stage is the open-circuit output voltage of the $(j-1)$st stage, i.e. $V_{ij(\mathrm{oc})} = V_{o(j-1)(\mathrm{oc})}$. Thus we can write

$$\frac{V_{i(j+1)(\mathrm{oc})}}{V_{ij(\mathrm{oc})}} = \frac{V_{oj(\mathrm{oc})}}{V_{o(j-1)(\mathrm{oc})}} = G_{mj} Z_{oj} \tag{4.44}$$

Let $A_v = V_o/V_s$ be the overall voltage gain of the circuit. To calculate A_v, we set all noise sources to zero. For $j=1$, $V_{o(j-1)(\mathrm{oc})} = V_s$. Thus we can write

$$\begin{aligned} A_v &= \frac{V_{o1(\mathrm{oc})}}{V_s} \times \frac{V_{o2(\mathrm{oc})}}{V_{o1(\mathrm{oc})}} \times \cdots \times \frac{V_{oN(\mathrm{oc})}}{V_{o(N-1)(\mathrm{oc})}} \times \frac{V_o}{V_{oN(\mathrm{oc})}} \\ &= G_{m1} Z_{o1} \times G_{m2} Z_{o2} \times \cdots \times G_{mN} Z_{oN} \times \frac{Z_L}{Z_{oN} + Z_L} \\ &= G_{m1} Z_{o1} \times G_{m2} Z_{o2} \times \cdots \times G_{mN} \left(Z_{oN} \| Z_L \right) \end{aligned} \tag{4.45}$$

With the noise sources not set to zero, the open circuit input voltage to the $(j+1)$st stage is given by

$$V_{i(j+1)(\mathrm{oc})} = I_{oj} Z_{oj} + V_{ni(j+1)} = G_{mj} V_{ij(\mathrm{oc})} Z_{oj} + V_{ni(j+1)} \tag{4.46}$$

But $V_{i(j+1)(\mathrm{oc})} = V_{oj(\mathrm{oc})}$ and $V_{ij(\mathrm{oc})} = V_{o(j-1)(\mathrm{oc})}$. Thus we can write

$$V_{oj(\mathrm{oc})} = G_{mj} V_{o(j-1)(\mathrm{oc})} Z_{oj} + V_{ni(j+1)} \tag{4.47}$$

The output voltage is given by

Don't erase the spaces. They have a width of -0.13 inches. The vertical space in the matrices has a height of 0.15 inches. This can be changed to adjust the height of the brackets.

$$\begin{aligned} V_o &= G_{mN} V_{o(N-1)(\mathrm{oc})} \left(Z_{oN} \| Z_L \right) \\ &= G_{mN} \left[G_{m(N-1)} V_{o(N-2)(\mathrm{oc})} Z_{o(N-1)} + V_{niN} \right] \left(Z_{oN} \| Z_L \right) \\ &= \cdots \\ &= A_v \left[V_s + V_{ni1} + \frac{V_{ni2}}{G_{m1} Z_{o1}} + \cdots \right. \\ &\quad \left. + \frac{V_{niN}}{G_{m1} Z_{o1} G_{m2} Z_{o2} \cdots G_{m(N-1)} Z_{o(N-1)}} \right] \end{aligned} \tag{4.48}$$

It follows from this equation that the equivalent noise input voltage in series with V_s is given by

$$\begin{aligned}
V_{ni} &= V_{ni1} + \frac{V_{ni2}}{G_{m1}Z_{o1}} + \cdots + \frac{V_{niN}}{G_{m1}Z_{o1}G_{m2}Z_{o2}\cdots G_{m(N-1)}Z_{o(N-1)}} \\
&= V_{ts} + V_{n1} + I_{n1}Z_s + \frac{V_{n2} + I_{n2}Z_{o1}}{G_{m1}Z_{o1}} + \cdots \\
&\quad + \frac{V_{nN} + I_{nN}Z_{o(N-1)}}{G_{m1}Z_{o1}G_{m2}Z_{o2}\cdots G_{m(N-1)}Z_{o(N-1)}} \\
&= V_{ts} + V_{n1} + \frac{V_{n2}}{G_{m1}Z_{o1}} + \frac{V_{n3}}{G_{m1}Z_{o1}G_{m2}Z_{o2}} \\
&\quad + \cdots + \frac{V_{nN}}{G_{m1}Z_{o1}G_{m2}Z_{o2}\cdots G_{m(N-1)}Z_{o(N-1)}} \\
&\quad + I_{n1}Z_s + \frac{I_{n2}}{G_{m1}} + \frac{I_{n3}}{G_{m1}Z_{o1}G_{m2}} \\
&\quad + \cdots + \frac{I_{nN}}{G_{m1}Z_{o1}G_{m2}Z_{o2}\cdots G_{m(N-1)}}
\end{aligned} \tag{4.49}$$

It can be seen that the v_n noise of any stage following the first stage is divided by the open-circuit voltage gain of the first stage. If this gain is sufficiently high, the v_n noise of all stages after the first stage can be neglected. This also minimizes the i_n noise of all stages after the second stage. The i_n noise of the second stage is divided by G_{m1}. Unless G_{m1} is large, the only way to minimize the i_{n2} term is to use a second stage which exhibits a low i_n noise. For a single bipolar transistor, $G_m \leq g_m = I_C/V_T$, where I_C is the collector bias current and V_T is the thermal voltage. For $I_C = 1$ mA and $V_T = 25$ mV, $g_m = 0.04$. For field effect devices, the g_m is usually lower. Therefore, minimization of the i_{n2} noise can be difficult to achieve by maximizing G_{m1}.

Example 2 *The figure shows a two-stage amplifier. It is given that* $R_S = 1\ k\Omega$, $R_L = 10\ k\Omega$. *For each stage,* $R_i = 20\ k\Omega$, $R_o = 100\ k\Omega$, $i_o = g_m v_i$, $g_m = 1/25$, $\sqrt{\overline{v_n^2}} = 10\ nV$, *and* $\sqrt{\overline{i_{n1}^2}} = 1\ pA$, *where* $\Delta f = 1$ *Hz. The noise sources are uncorrelated. The noise generated by* R_L *can be neglected. (a) Solve for* $G_m = i_{o(sc)}/v_{i(oc)}$ *for each stage. (b) Solve for the voltage gain* $A = v_o/v_s$. *(c) Solve for the components of* $\sqrt{\overline{v_{ni}^2}}$ *due to* v_{ts}, v_{n1}, v_{n2}, i_{n1}, *and* i_{n2}. *(d) Solve for* $\sqrt{\overline{v_{ni}^2}}$ *due to all noise sources. (e) Which noise component dominates?*

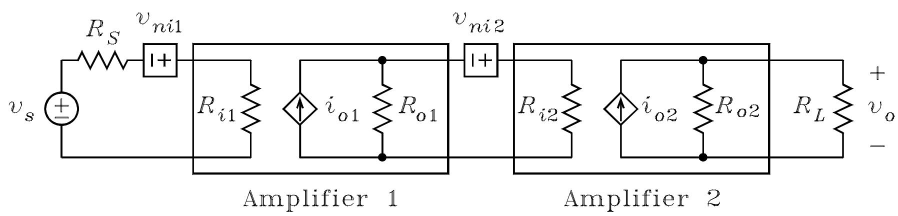

Figure 4.7: Two stage amplifier.

Solution. (a) $G_{m1} = g_{m1}R_{i1}/(R_S + R_{i1}) = 1/25$, $G_{m2} = g_{m2}R_{i2} \div (R_{o2} + R_{i2}) = 1/50$, (b) $A = G_{m1}R_{o1}G_{m2}R_{o2}\|R_L = 7.27 \times 10^5$, (c) $\sqrt{\overline{v_{ts}^2}} = 1.27$ nV, $\sqrt{\overline{v_{n1}^2}} = 10$ nV, $\sqrt{\overline{v_{n2}^2}}/G_{m1}R_{o1} = 2.5$ pV, $\sqrt{\overline{i_{n1}^2}}R_S = 100$ pV, $\sqrt{\overline{i_{n2}^2}}/G_{m1} = 25$ pV, $\sqrt{\overline{v_{ni}^2}} = 10.1$ nV. (d) The first stage $\sqrt{\overline{v_{n1}^2}}$ noise dominates.

4.3 Operational Amplifier Noise

Op-amp noise models are variations of the $v_n - i_n$ amplifier noise model. The general noise model puts a noise voltage source and a noise current source at each input to the op amp. Thus four noise sources are required. In general, the sources are correlated. However, the correlation between the two sources on one input and the two sources on the other input would be expected to be weak. The general noise model is given in Fig. 4.8(a).

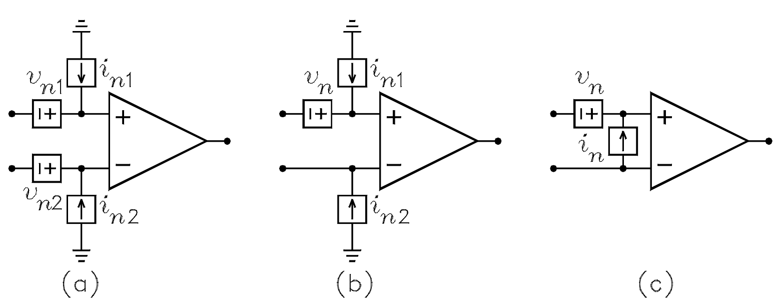

Figure 4.8: Op-amp noise models.

If it is assumed that the op amp responds only to the differential input voltage, the two noise voltage sources in the general model can be replaced by a single noise voltage source at either input. Fig. 4.8(b) shows the modified model, where the source is put at the non-inverting input and $v_n = v_{n1} - v_{n2}$. In general, v_n in this circuit is correlated to both i_{n1} and i_{n2}. Its mean-square value is $\overline{v_n^2} = \overline{v_{n1}^2} + \overline{v_{n2}^2}$, where correlation between v_{n1} and v_{n2} has been neglected.

An even simpler noise model can be obtained if the two noise current sources are replaced by a single differential noise current source as shown in Fig. 4.8(c). This model is not as accurate as the other two. In making calculations that use specified noise data on op-amps, it is important to use the noise model for which the data apply. The following two examples illustrate op amp noise calculations. In each case, the op amp is modeled by the circuit in Fig. 4.8(b).

Example 3 *Solve for $\overline{v_{ni}^2}$ for the op amp circuit in Fig. 4.9(a).*

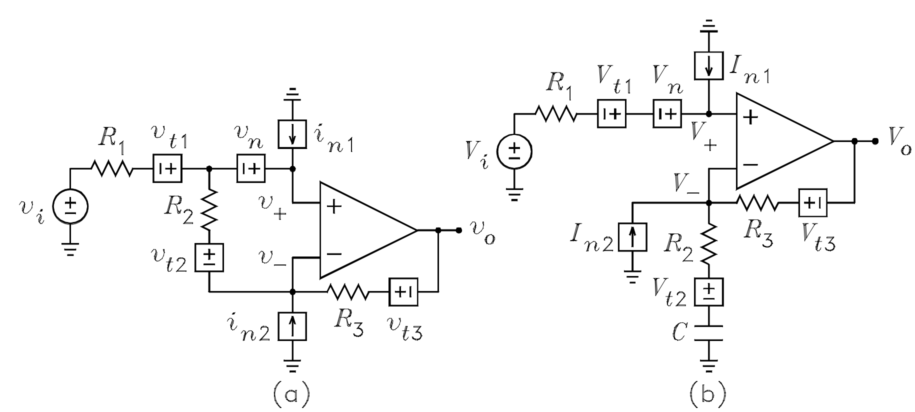

Figure 4.9: Example op amp circuits to illustrate noise calculations.

Solution. Because R_2 connects between the two op-amp inputs, the voltages at the two inputs due to v_{t1}, v_{t2}, and v_{t3} are correlated. This makes it impossible to calculate the thermal noise voltage at each input in terms of the equivalent resistance seen looking out of the input. The voltage at each input must be solved for in terms of each source. We do this by superposition. We have

$$\begin{aligned} v_+ &= (v_i + v_{t1}) \frac{R_2 + R_3}{R_1 + R_2 + R_3} + (v_{t2} + v_{t3} + v_o) \frac{R_1}{R_1 + R_2 + R_3} + v_n \\ &\quad + i_{n1} R_1 \| (R_2 + R_3) + i_{n2} \frac{R_1 R_3}{R_1 + R_2 + R_3} \end{aligned} \tag{4.50}$$

$$\begin{aligned} v_- &= (v_i + v_{t1} - v_{t2}) \frac{R_3}{R_1 + R_2 + R_3} + (v_{t3} + v_o) \frac{R_1 + R_2}{R_1 + R_2 + R_3} \\ &\quad + i_{n1} \frac{R_1 R_3}{R_1 + R_2 + R_3} + i_{n2} (R_1 + R_2) \| R_2 \end{aligned} \tag{4.51}$$

If we consider the op amp to be ideal except for its noise, its output voltage can be solved for by setting $v_+ = v_-$. This yields

$$\begin{aligned} v_o &= v_i + v_{t1} + v_{t2} \frac{R_1 + R_3}{R_2} - v_{t3} \\ &\quad + v_n \left(1 + \frac{R_1 + R_3}{R_2}\right) + i_{n1} R_1 - i_{n2} R_3 \end{aligned} \tag{4.52}$$

Because the coefficient of v_i is unity in this equation, it follows that v_{ni} is the sum of the noise terms. Its mean-square value is given by

$$\begin{aligned} \overline{v_{ni}^2} &= 4kT \left[R_1 + \frac{(R_1 + R_3)^2}{R_2} + R_3 \right] \Delta f + \overline{v_n^2} \left(1 + \frac{R_1 + R_3}{R_2}\right)^2 \\ &\quad + 2\rho_1 \sqrt{\overline{v_n^2}} \sqrt{\overline{i_{n1}^2}} \left(1 + \frac{R_1 + R_3}{R_2}\right) R_1 \\ &\quad - 2\rho_2 \sqrt{\overline{v_n^2}} \sqrt{\overline{i_{n2}^2}} \left(1 + \frac{R_1 + R_3}{R_2}\right) R_3 + \overline{i_{n1}^2} R_1^2 + \overline{i_{n2}^2} R_3^2 \end{aligned} \tag{4.53}$$

where ρ_1 and ρ_2, respectively, are the correlation coefficients between v_n and i_{n1} and between v_n and i_{n2}.

It is not common to see a resistor shunting the two inputs of an op amp as R_2 does in the circuit of Fig. 4.9(a). If the op amp is considered to be ideal and noiseless, this resistor has no effect on the circuit. When noise is included, Eq. (4.53) shows that the noise increases as R_2 is made smaller. An input compensation network consisting of a resistor or a resistor in series with a capacitor is sometimes connected in shunt between the two inputs of an op amp to prevent oscillations. This network has the effect of reducing the loop gain of the circuit at high frequencies. Although input compensation can be effective in eliminating oscillation problems when other methods fail, Eq. (4.53) shows that it should be used cautiously when noise is a consideration.

Example 4 *Solve for $\overline{v_{ni}^2}$ for the op amp circuit in Fig. 4.9(b).*

Solution. The circuit has a capacitor, thus a phasor analysis must be used. Unlike the circuit of Fig. 4.9(a), there is no impedance element connecting the two op amp inputs. This makes it possible to solve for the thermal noise voltage at each input from the Thévenin impedance looking out of that input. Using superposition, we can write

$$V_+ = V_i + V_{t1} + V_n + I_{n1} R_1 \tag{4.54}$$

$$V_- = V_o \frac{R_2 + 1/j\omega C}{R_2 + R_3 + 1/j\omega C} + I_{n2} Z_- + V_{t-} \tag{4.55}$$

where V_{t-} is the thermal noise generated by the impedance Z_- given by

$$Z_- = \left(R_2 + \frac{1}{j\omega C} \right) \| R_3 = \frac{R_3 \left(1 + j\omega R_2 C\right)}{1 + j\omega \left(R_2 + R_3\right) C}$$

The output voltage can be solved for by setting $V_+ = V_-$. It is given by

$$\begin{aligned} V_o \; = \; & \frac{1 + j\omega \left(R_2 + R_3\right) C}{1 + j\omega R_2 C} \Big[V_i + V_{t1} + V_n \\ & + I_{n1} R_1 - I_{n2} Z_- - V_{t-} \Big] \end{aligned} \tag{4.56}$$

The equivalent noise input voltage is given by the sum of terms in the brackets with the exception of the V_i term. Its mean-square value is

$$\begin{aligned} \overline{v_{ni}^2} \; = \; & 4kT \left\{ R_1 + R_3 \frac{1 + \omega^2 R_2 \left(R_2 + R_3\right) C^2}{1 + \left[\omega \left(R_2 + R_3\right) C\right]^2} \right\} \Delta f \\ & + \overline{v_n^2} + 2\sqrt{\overline{v_n^2}} \sqrt{\overline{i_{n1}^2}} R_1 \operatorname{Re}\left(\gamma_1\right) \\ & - 2\sqrt{\overline{v_n^2}} \sqrt{\overline{i_{n1}^2}} \operatorname{Re}\left[\gamma_2 \frac{R_3 \left(1 - j\omega R_2 C\right)}{1 - j\omega \left(R_2 + R_3\right) C} \right] \\ & + \overline{i_{n1}^2} R_1^2 + \overline{i_{n2}^2} \frac{R_3^2 \left[1 + \left(\omega R_2 C\right)^2\right]}{1 + \left[\omega \left(R_2 + R_3\right) C\right]^2} \end{aligned} \tag{4.57}$$

where $\overline{v_{t-}^2} = 4kT \operatorname{Re}\left(Z_-\right)$ has been used and γ_1 and γ_2, respectively, are the complex correlation coefficients between V_n and I_{n1} and between V_n and I_{n2}.

4.4 Noise Reduction with Parallel Devices

A method which can be used to reduce the noise generated in an amplifier input stage is to realize that stage with several active devices in parallel, e.g. parallel BJTs or parallel FETs. Fig. 4.10 shows the diagram of an amplifier input stage having N identical devices in parallel. For simplicity, only the first two are shown. The noise source V_{ts} models the thermal noise generated by the source resistance $R_s = \mathrm{Re}(Z_s)$. Each amplifier stage is modeled by the $v_n - i_n$ amplifier noise model having an input impedance Z_i. The output circuit is modeled by a Norton equivalent circuit consisting of a parallel current source and impedance.

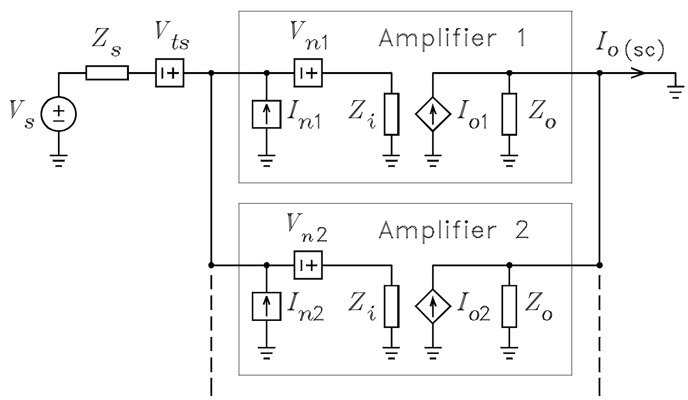

Figure 4.10: Parallel amplifiers.

To solve for the equivalent noise input voltage, we solve for the Thévenin equivalent circuit seen looking out of the parallel connected inputs. This circuit must contain all noise sources. As a first step in doing this, the noise current sources I_{nj} can all be combined into a single current source with a value equal to the sum of the N currents. The voltage in series with V_s which generates the same noise as this current is then $Z_s \sum I_{nj}$. The next step is to replace the noise voltage sources V_{nj} with a single noise voltage in series with V_s. This source is equal to the common-mode value of the V_{nj} sources, i.e. it has the value $(1/N) \sum V_{nj}$. The final circuit is shown in Fig. 4.11 where the N amplifier stages have been combined into a single equivalent amplifier.

Because the latter step is not obvious, a simple proof is presented for the case $N = 3$. It is straightforward to extend the proof for other

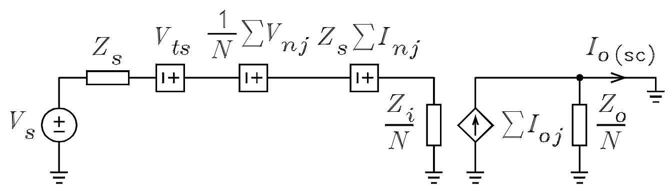

Figure 4.11: Equivalent circuit.

values of N. The three noise sources V_{n1}, V_{n2}, and V_{n3} can be written

$$V_{n1} = V_a + V_b + V_c \tag{4.58}$$

$$V_{n2} = V_a - V_b + V_d \tag{4.59}$$

$$V_{n3} = V_a - V_c - V_d \tag{4.60}$$

where V_a, V_b, V_c, and V_d are given by

$$V_a = \frac{1}{3}\left(V_{n1} + V_{n2} + V_{n3}\right) \tag{4.61}$$

$$V_b = \frac{1}{3}\left(V_{n1} - V_{n2}\right) \tag{4.62}$$

$$V_c = \frac{1}{3}\left(V_{n1} - V_{n3}\right) \tag{4.63}$$

$$V_d = \frac{1}{3}\left(V_{n2} - V_{n3}\right) \tag{4.64}$$

Because of linearity, superposition of V_a, V_b, V_c, and V_d can be used to obtain the total response of the circuit. When V_a alone is active, the three sources are equal and the response of the circuit is the same as for a single source V_a in series with V_s. When V_b alone is active, amplifiers 1 and 2 are driven differentially and the output currents cancel. The same happens between amplifiers 1 and 3 when V_c is active and between amplifiers 2 and 3 when V_d is active. This completes the proof.

It follows from Fig. 4.11 that the equivalent noise input voltage is given by

$$V_{ni} = V_{ts} + \frac{1}{N}\sum_{j=1}^{N} V_{nj} + Z_s \sum_{j=1}^{N} I_{nj} \tag{4.65}$$

This has the mean-square value

$$\overline{v_{ni}^2} = 4kTR_s\Delta f + \frac{\overline{v_n^2}}{N} + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\left(\gamma Z_s^*\right) + N\overline{i_n^2}\left|Z_s\right|^2 \qquad (4.66)$$

where γ is the correlation coefficient between V_n and I_n for any one of the N identical stages.

If $Z_s = 0$, Eq. (4.66) reduces to $\overline{v_{ni}^2} = \overline{v_n^2}/N$. In this case, the noise can theoretically be reduced to any desired level if N is made large enough. For $Z_s \neq 0$, Eq. (4.66) predicts that $\overline{v_{ni}^2} \to \infty$ for $N \to 0$ or $N \to \infty$. Thus there is a value of N that minimizes the noise. It is solved for by setting $d\overline{v_{ni}^2}/dN = 0$ and solving for N. It is given by

$$N = \frac{1}{|Z_s|}\sqrt{\frac{\overline{v_n^2}}{\overline{i_n^2}}} \qquad (4.67)$$

This expression shows that N decreases as $|Z_s|$ increases. It follows that the noise cannot be reduced by paralleling input devices if the source impedance is large.

4.5 Noise Reduction with an Input Transformer

A transformer at the input of an amplifier may improve its noise performance. Fig. 4.12(a) shows a signal source connected to an amplifier through a transformer with a turns ratio $1 : n$. Resistors R_1 and R_2, respectively, represent the primary and the secondary winding resistances. Fig. 4.12(b) shows the equivalent circuit seen by the amplifier input with all noise sources shown. The source V_{t1} represents the thermal noise generated by the effective source resistance $n^2\left(R_s + R_1\right) + R_2$, where $R_s = \mathrm{Re}\left(Z_s\right)$. The open-circuit input voltage is given by

$$V_{i(oc)} = nV_s + V_{t1} + V_n + I_n\left[n^2\left(Z_s + R_1\right) + R_2\right] \qquad (4.68)$$

The equivalent noise input voltage referred to the source is obtained by factoring the turns ratio n from the expression for $V_{i(oc)}$ and retaining all terms except the V_s term. It is given by

$$V_{i(oc)} = \frac{V_{t1} + V_n}{n} + I_n\left[n\left(Z_s + R_1\right) + \frac{R_2}{n}\right] \qquad (4.69)$$

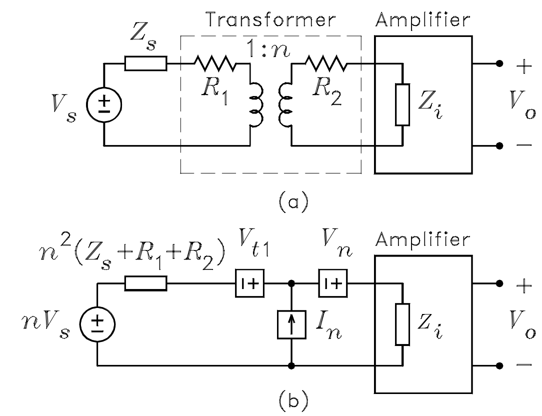

Figure 4.12: Transformer coupled amplifier.

This expression obtained can be converted into a mean-square sum to obtain

$$\begin{aligned}\overline{v_{ni}^2} &= 4kT\left(R_s + R_1 + \frac{R_2}{n^2}\right)\Delta f + \frac{\overline{v_n^2}}{n^2} \\ &+2\,\mathrm{Re}\left[\gamma\left(Z_s^* + R_1 + \frac{R_2}{n^2}\right)\right]\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}} \\ &+n^2\overline{i_n^2}\left|Z_s + R_1 + \frac{R_2}{n^2}\right|^2 \qquad (4.70)\end{aligned}$$

where γ is the correlation coefficient between V_n and I_n.

Because the series resistance of a transformer winding is proportional to the number of turns in the winding, it follows that $R_2/R_1 \propto n$. This makes it difficult to determine the value of n which minimizes $\overline{v_{ni}^2}$. In the case that $|Z_s| >> R_1 + R_2/n^2$, the expression for $\overline{v_{nis}^2}$ is given approximately by

$$\overline{v_{ni}^2} \simeq 4kTR_s\Delta f + \frac{\overline{v_n^2}}{n^2} + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\left(\gamma Z_s^*\right) + n^2\overline{i_n^2}\,|Z_s|^2 \qquad (4.71)$$

This is minimized when n^2 is given by

$$n^2 = \frac{1}{|Z_s|}\sqrt{\frac{\overline{v_n^2}}{\overline{i_n^2}}} \qquad (4.72)$$

In this case, the magnitude of the effective source impedance seen by the amplifier is $n^2 |Z_s| = \sqrt{\overline{v_n^2}}/\sqrt{\overline{i_n^2}}$.

If the source resistance is small, the transformer winding resistance can be a significant contributor to the thermal noise at the amplifier input. For this reason, a transformer can result in a decreased SNR compared to the case without the transformer.

Chapter 5

RF Amplifier Models

The rf (radio frequency) band bridges the range of frequencies from as low as 9 kHz to frequencies of 10 GHz and higher. The lowest frequencies are used for radio navigation and cw (continuous wave) systems such as beacons. On the low end of the broadcast band, the North American am broadcast band occupies the spectrum from 525 kHz to 1605 kHz. On the upper end, airborne Doppler radar systems operate as high as 8.8 GHz.

5.1 Transmission Line Fundamentals

A transmission line is a waveguide system in which the propagating wave is a TEM or transverse electromagnetic wave. To support such a wave, the line must consist of at least two conductors. Coaxial transmission lines are used at frequencies up to approximately 10 GHz. At the higher frequencies, small-diameter lines are required. The smaller lines exhibit higher losses and reduced power handling capability. These problems are solved by replacing the transmission lines with waveguides at the higher frequencies. Although the waves in a waveguide are not TEM waves, transmission line models of waveguides are commonly used to analyze waveguide systems.

5.1.1 Forward Wave

A transmission line is a one dimensional system which can support waves propagating in only two directions. These are the forward direction and the reverse direction. Although the waves consist of electric and magnetic fields between the line conductors, they can be modeled by voltage and current waves. Fig. 5.1(a) illustrates a transmission line section with a wave propagating in the forward or $+z$ direction. We assume a sinusoidal excitation so that the voltage and current can be represented by phasors. Let V_0^+ be the phasor amplitude of the forward wave. The phasor voltage and current on the line are given by

$$V^+(z) = V_0^+ e^{-j\beta z} \tag{5.1}$$

$$I^+(z) = \frac{V_0^+}{Z_c} e^{-j\beta z} = V_0^+ Y_c e^{-j\beta z} \tag{5.2}$$

where β is the propagation constant, Z_c is the characteristic impedance, and Y_c is the characteristic admittance. For a lossless line, these are given by

$$\beta = \omega\sqrt{LC} \tag{5.3}$$

$$Z_c = \frac{1}{Y_c} = \sqrt{\frac{L}{C}} \tag{5.4}$$

where $\omega = 2\pi f$ is the radian frequency, L is the line inductance per meter (H/m), and C is the line capacitance per meter (F/m). For a lossless line, β, Z_c, and Y_c are real. For a lossy line, they are complex. The analysis presented here assumes a lossless line in all cases.

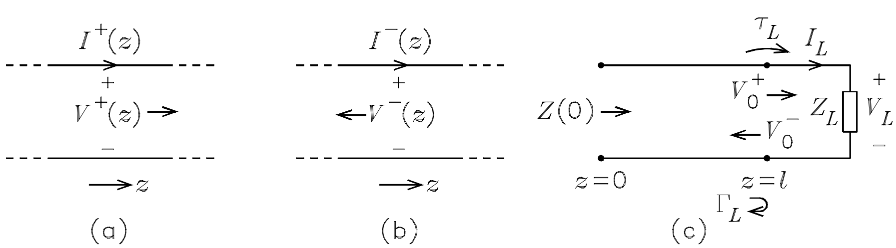

Figure 5.1: (a) Transmission line with a $+z$ wave. (b) Transmission with a $-z$ wave. (c) Line of length ℓ.

The time domain waves are obtained by multiplying the phasor representations by $\exp(j\omega t)$ and taking the real part. If V_0^+ is real, the time domain voltage is given by

$$v^+(t) = V_0^+ \cos(\omega t - \beta z) \tag{5.5}$$

A point on the wave is defined by the condition $\omega t - \beta z = \theta_0$, where θ_0 is a constant. It follows from this condition that $\omega - \beta dz/dt = 0$. The velocity of the wave is given by

$$v_p = \frac{dz}{dt} = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \tag{5.6}$$

This velocity is called the phase velocity. The wavelength λ is the distance the wave propagates in one period. It is given by

$$\lambda = v_p T = \frac{v_p}{f} = \frac{2\pi}{\beta} \tag{5.7}$$

The average power propagating in the forward wave is given by

$$P^+ = \mathrm{Re}\,\overline{V^+(z)\,I^+(z)^*} = \mathrm{Re}\left(\frac{\overline{V_0^+ V_0^{+*}}}{Z_c}\right) = \frac{\overline{v_0^{+2}}}{Z_c} \tag{5.8}$$

where $\overline{v_0^{+2}}$ is the mean-square value of V_0^+. Note that P^+ is independent of z because of the assumption of a lossless line.

5.1.2 Reverse Wave

Figure 5.1(b) illustrates a line with a wave propagating in the reverse or $-z$ direction. Note that the reference directions for the voltage and current are taken to be the same as those for the forward wave. The phasor voltage, phasor current, and average power flow are given by

$$V^-(z) = V_0^- e^{+j\beta z} \tag{5.9}$$

$$I^-(z) = -\frac{V_0^-}{Z_c} e^{+j\beta z} = -Y_c V_0^- e^{+j\beta z} \tag{5.10}$$

$$P^- = \mathrm{Re}\,\overline{V^-(z)\,[-I^-(z)]^*} = \mathrm{Re}\left(\frac{\overline{V_0^- V_0^{-*}}}{Z_c}\right) = \frac{\overline{v_0^{-2}}}{Z_c} \tag{5.11}$$

where $\overline{v_0^2}$ is the mean-square value of V_0^-. At any point on the line, the time domain solutions for the voltage and current vary sinusoidally with time with a mean value of zero. The negative sign in the equation for $I^-(z)$ means that the direction of instantaneous current flow is to the left in Fig. 5.1(b) when the instantaneous voltage is positive. Similarly, the direction of current flow is to the right when the voltage is negative. The use of $-I^-(z)$ in the equation for P^- is necessary to make the P^- positive when the source is to the right of the observation point.

5.1.3 Reflection Coefficient

Figure 5.1(c) illustrates a line of length ℓ connected to a load impedance Z_L. At $z = \ell$, denote the phasor amplitudes of the forward and reverse voltage waves by V_0^+ and V_0^-, respectively. At the load, we can write

$$V_L = V_0^+ + V_0^- = I_L Z_L \tag{5.12}$$

$$I_L = \frac{V_0^+}{Z_c} - \frac{V_0^-}{Z_c} \tag{5.13}$$

These equations can be solved for the load reflection coefficient Γ_L given by

$$\Gamma_L = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_c}{Z_L + Z_c} = \frac{\widehat{Z}_L - 1}{\widehat{Z}_L + 1} \tag{5.14}$$

where $\widehat{Z}_L = Z_L/Z_c$ is the normalized load impedance. We conclude from this equation that $V_0^- = 0$ if $Z_L = Z_c$. Thus the characteristic impedance of a line is equal to the value of the load impedance which causes no reflections. In this case, the load is said to be matched to the line and the forward power propagating on the line is absorbed by the load with no reflected power. Alternately, Γ_L can be expressed as a function of the characteristic admittance Y_c and the load admittance $Y_L = 1/Z_L$. It is given by

$$\Gamma_L = \frac{V_0^-}{V_0^+} = \frac{Y_c - Y_L}{Y_c + Y_L} = \frac{1 - \widehat{Y}_L}{1 + \widehat{Y}_L} \tag{5.15}$$

where $\widehat{Y}_L = Y_L/Y_c$ is the normalized load admittance.

For any point $0 \le z \le \ell$ on the line in Fig. 5.1(c), the total voltage is given by

$$V(z) = V_0^+ e^{-j\beta(z-\ell)} + V_0^- e^{+j\beta(z-\ell)} \tag{5.16}$$

The reflection coefficient at that point is the ratio of the reverse voltage to the forward voltage. It is given by

$$\Gamma(z) = \frac{V_0^- e^{+j\beta(z-\ell)}}{V_0^+ e^{-j\beta(z-\ell)}} = \Gamma_L e^{+j2\beta(z-\ell)} \tag{5.17}$$

It follows from the expressions that $|\Gamma(z)| \le 1$. Thus a plot of $\Gamma(z)$ on the complex plane is a point that lies in or on a circle having a radius $\rho = 1$. If z is increased, i.e. one moves closer to the load end, $\Gamma(z)$ rotates clockwise on a circle of radius $\rho = |\Gamma_L|$. If z is decreased, i.e. one moves away from the load end, $\Gamma(z)$ rotates counterclockwise on the same circle. If z is changed by $\lambda/2$, $2\beta z$ changes by 2π. Thus $\Gamma(z)$ rotates 360^{o} each time z changes by $\lambda/2$.

5.1.4 Transmission Coefficient

The total voltage across the load in Fig. 5.1(c) is

$$V_L = V_0^+ + V_0^- = V_0^+ (1 + \Gamma_L) \tag{5.18}$$

The load transmission coefficient τ_L is defined by

$$\tau_L = \frac{V_L}{V_0^+} = 1 + \Gamma_L = \frac{2Z_L}{Z_c + Z_L} \tag{5.19}$$

Note that $\tau_L = 1$ for $Z_L = Z_c$, $\tau_L = 0$ for $Z_L = 0$, and $\tau_L = 2$ for $Z_L \to \infty$. In the latter case, the reflected wave is equal to the incident wave so that the sum at $z = \ell$ has an amplitude that is twice that of the incident wave.

5.1.5 Power Delivered to a Load

The power delivered to the load in Fig. 5.1(c) is given by

$$\begin{aligned} P_L &= \operatorname{Re} \overline{V_L I_L^*} = \operatorname{Re} \overline{\left[V_0^+ + V_0^-\right] \left[\left(V_0^+ - V_0^-\right)/Z_c\right]^*} \\ &= \frac{1}{Z_c} \operatorname{Re} \overline{V_0^+ (1 + \Gamma_L) V_0^{+*} \left(1 - \Gamma_L^*\right)} = \frac{\overline{v_0^{+2}}}{Z_c} \left(1 - |\Gamma_L|^2\right) \end{aligned} \tag{5.20}$$

This expression is of the form $P_L = P^+ - P^-$, where $P^+ = \overline{v_0^{+2}}/Z_c$ is the forward power and $P^- = |\Gamma_L|^2 P^+$ is the reverse power. Alternate expressions for the load power are

$$\begin{aligned} P_L &= \operatorname{Re} \overline{V_L \left(\frac{V_L}{Z_L}\right)^*} = \operatorname{Re}\left(\frac{\overline{V_L V_L^*}}{Z_L^*}\right) \\ &= \overline{v_L^2} \operatorname{Re}(Y_L^*) = \overline{v_{0+}^2} \, |1 + \Gamma_L|^2 \operatorname{Re}(Y_L^*) \end{aligned} \tag{5.21}$$

and

$$\begin{aligned} P_L &= \operatorname{Re} \overline{(I_L Z_L) I_L^*} = \operatorname{Re}\left(\overline{I_L I_L^*} Z_L\right) \\ &= \overline{i_L^2} \operatorname{Re}(Z_L) = \frac{\overline{v_{0+}^2}}{Z_c^2} |1 + \Gamma_L|^2 \operatorname{Re}(Z_L) \end{aligned} \tag{5.22}$$

where $Y_L^* = 1/Z_L^*$.

5.1.6 Line Impedance

For $0 \le z \le \ell$, the voltage in Fig. 5.1(c) is given by Eq. (5.16). The current is given by

$$I(z) = \frac{V_0^+}{Z_c} e^{-j\beta(z-\ell)} - \frac{V_0^-}{Z_c} e^{+j\beta(z-\ell)} \tag{5.23}$$

The impedance seen looking into the line at z is $Z(z) = V(z)/I(z)$. It is given by

$$\begin{aligned} Z(z) &= Z_c \frac{V_0^+ e^{-j\beta(z-\ell)} + V_0^- e^{+j\beta(z-\ell)}}{V_0^+ e^{-j\beta(z-\ell)} - V_0^- e^{+j\beta(z-\ell)}} \\ &= Z_c \frac{1 + \Gamma_L e^{+j2\beta(z-\ell)}}{1 - \Gamma_L e^{+j2\beta(z-\ell)}} = Z_c \frac{\widehat{Z}_L + j\tan[\beta(\ell - z)]}{1 + j\widehat{Z}_L \tan[\beta(\ell - z)]} \end{aligned} \tag{5.24}$$

The admittance is given by

$$Y(z) = Y_c \frac{1 - \Gamma_L e^{+j2\beta(z-\ell)}}{1 + \Gamma_L e^{+j2\beta(z-\ell)}} = Y_c \frac{\widehat{Y}_L + j\tan[\beta(\ell - z)]}{1 + j\widehat{Y}_L \tan[\beta(\ell - z)]} \tag{5.25}$$

5.2 The Smith Chart

The Smith chart is a polar plot of reflection coefficient on which curves of constant normalized resistance and reactance are plotted. In Eq. (5.17), let $\Gamma_0 = \Gamma_L \exp(-j2\beta\ell)$. It follows that the reflection coefficient can be written

$$\Gamma(z) = \Gamma_0 e^{+j2\beta z} = |\Gamma_0| e^{+j(2\beta z + \varphi_0)} \tag{5.26}$$

where φ_0 is the angle of Γ_0. This equation defines the generalized reflection coefficient. It can be used to predict the reflection coefficient at any point on the line in terms of the reflection coefficient Γ_0 at $z = 0$. The point $z = 0$ is arbitrary and can be at any point on the line. For example, if $z = 0$ is taken to be a point between the source and the load, z is positive when moving toward the load and negative when moving away from the load. The angle of $\Gamma(z)$ varies as $2\beta z = 4\pi z/\lambda$ radians or $360\beta z/\pi = 720z/\lambda$ degrees. For positive z, $\Gamma(z)$ rotates counterclockwise on a circle of radius $\rho = |\Gamma_0|$. For negative z, $\Gamma(z)$ rotates clockwise on the same circle. If z is increased or decreased by $\lambda/4$, the angle changes by $\pm 180^{\circ}$ and $\Gamma(z \pm \lambda/4) = -\Gamma(z)$. If z is increased or decreased by $\lambda/2$, the angle changes by $\pm 360^{\circ}$ and $\Gamma(z \pm \lambda/2) = \Gamma(z)$.

At any point on the line, the normalized impedance and the reflection coefficient can be written in the rectangular forms

$$\widehat{Z}(z) = \frac{Z}{Z_c} = r + jx \tag{5.27}$$

$$\Gamma(z) = \frac{\widehat{Z}(z) - 1}{\widehat{Z}(z) + 1} = p + jq \tag{5.28}$$

Thus we can write

$$p + jq = \frac{r + jx - 1}{r + jx + 1} \tag{5.29}$$

When this equation is separated into its real and imaginary parts, the following equations can be obtained

$$\left(p - \frac{r}{1+r}\right)^2 + q^2 = \frac{1}{(1+r)^2} \tag{5.30}$$

$$(p-1)^2 + \left(q - \frac{1}{x}\right)^2 = \frac{1}{x^2} \tag{5.31}$$

These equations represent circles in the (p, q) plane whose centers and radii are functions of r and x. For constant r, $\Gamma(z)$ lies on a circle of radius $\rho = 1/(1+r)$ centered at the point $(r/(1+r), 0)$. For constant x, $\Gamma(z)$ lies on a circle of radius $\rho = 1/|x|$ centered at the point $(1, 1/x)$. The magnitude sign is necessary because x can be either positive or negative.

Figure 5.2 shows the Smith chart. The circles with centers on the horizontal axis are circles of constant r. On the circle labeled $r = 1$, the real part of the impedance is equal to Z_c. The circles with centers above the horizontal axis represent contours of constant $x > 0$. The circles with centers below the horizontal axis represent contours of constant $x < 0$. The horizontal axis represents $x = 0$. The left end of the horizontal axis represents $r = 0$. The right end of the axis represents $r = \infty$. The origin represents the point $\Gamma(z) = 0$. At this point, the line is matched and $\widehat{Z}(z) = 1$, i.e. $Z = Z_c$. Clockwise rotation corresponds to increasing z, i.e. moving toward the load. Counterclockwise rotation corresponds to decreasing z, i.e. moving toward the source.

The normalized impedance on the line is related to the reflection coefficient by

$$\widehat{Z}(z) = \frac{Z(z)}{Z_c} = \frac{1 + \Gamma(z)}{1 - \Gamma(z)} \tag{5.32}$$

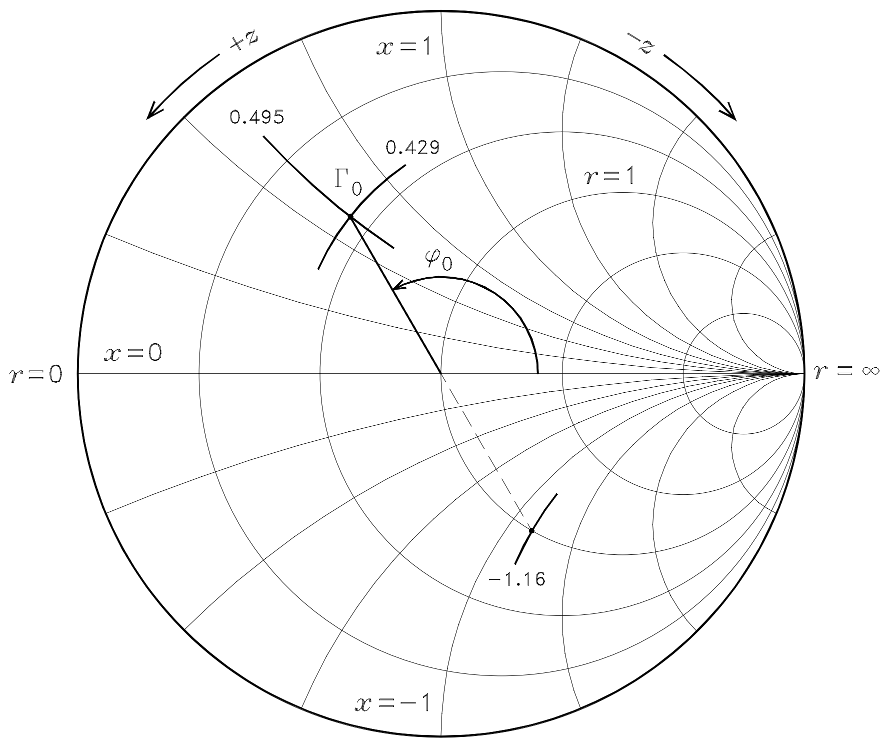

Figure 5.2: Impedance Smith chart. $+z$ direction moves from the source end toward the load end. $-z$ direction moves from the load end toward the source end.

For the point $\Gamma_0 = 0.5\angle 120^o$ that is labeled in Fig. 5.2, this equation gives $\widehat{Z} = 0.429 + j0.495$. Intersecting arcs of the circles corresponding to $r = 0.429$ and $x = 0.495$ are shown on the figure. If z is increased by $\pm\lambda/4$, Γ_0 rotates $\pm 180^o$ around the chart and the normalized impedance is given by

$$\widehat{Z}(z \pm \lambda/4) = \frac{1 - \Gamma(z)}{1 + \Gamma(z)} = \frac{1}{\widehat{Z}(z)} = \widehat{Y}(z) \tag{5.33}$$

Thus rotating 180^o in either direction converts the normalized impedance at z into the normalized admittance at z. This point is shown at the end of the dashed line in Fig. 5.2 where an arc of the circle corresponding to $x = -1.16$ is shown. For $\Gamma(z) = 0.5\angle 120^o$, we have $\widehat{Y}(z) = 1 - j1.16$. It follows that adding a parallel susceptance $B = 1.16Z_c$ across the line at the point z will result in $\Gamma(z) = 0$, which is the condition for a matched line. Note that rotating $\pm 180^o$ to convert $\widehat{Z}$ into $\widehat{Y}$ and vice versa does not correspond to a change in position z on the line.

The chart shown in Fig. 5.2 is an impedance chart. To obtain an admittance chart, we write the normalized admittance on the line

$$\widehat{Y}(z) = \frac{Y(z)}{Y_c} = \frac{Z_c}{Z(z)} = \frac{1 + \Gamma(z)}{1 - \Gamma(z)} = g + jb \tag{5.34}$$

With $\Gamma(z) = p + jq$, the equations of constant g and b on the (p, q) plane are found to be

$$\left(p + \frac{g}{1+g}\right)^2 + q^2 = \frac{1}{(1+g)^2} \tag{5.35}$$

$$(p+1)^2 + \left(q + \frac{1}{b}\right)^2 = \frac{1}{b^2} \tag{5.36}$$

Contours of constant g are circles with centers at $(-g/(1+g), 0)$ and radii $\rho = 1/(1+g)$. Contours of constant b are circles with centers at $(-1, -1/b)$ and radii $\rho = 1/|b|$.

The admittance chart is shown in Fig. 5.3. The circles with centers on the horizontal axis are circles of constant g. On the circle labeled $g = 1$, the real part of the admittance is equal to Y_c. The circles with centers above the horizontal axis represent contours of constant $b < 0$. The circles with centers below the horizontal axis represent contours of constant $b > 0$. The horizontal axis represents $b = 0$. The left end of the horizontal axis represents $g = \infty$. The right end of the axis represents $g = 0$. The origin represents the point $\Gamma(z) = 0$. At this point, the line is matched and $\widehat{Y}(z) = 1$.

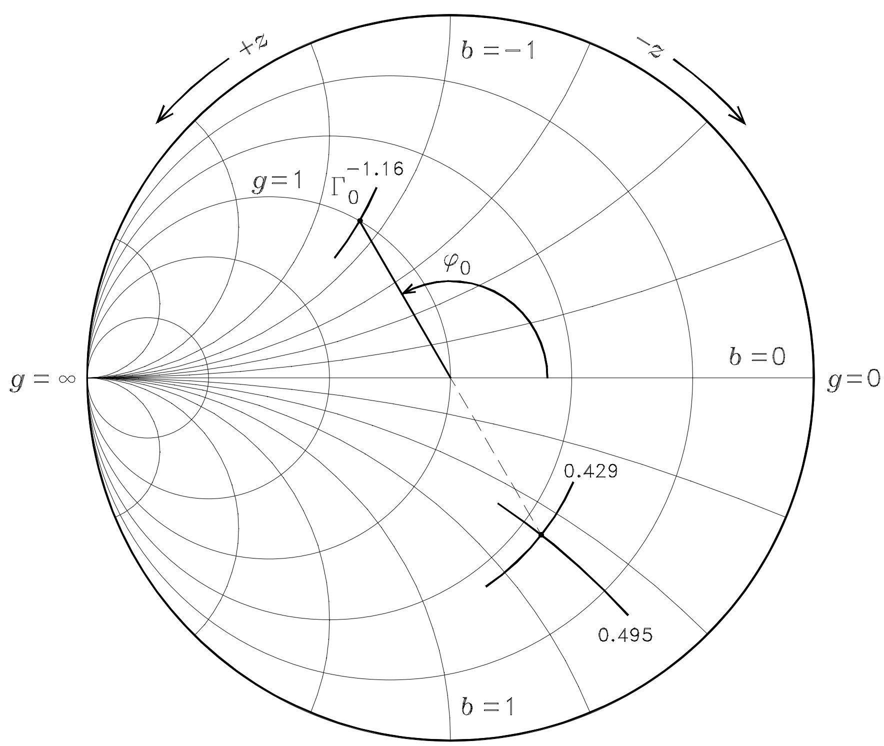

Figure 5.3: Admittance Smith chart. $+z$ direction moves from the source end toward the load end. $-z$ direction moves from the load end toward the source end.

The point $\Gamma_0 = 0.5\angle 120^{\mathrm{o}}$ is labeled in Fig. 5.3. Note that this is at the same position as it is in the impedance chart of Fig. 5.2. Thus the reflection coefficient is invariant to a change from an impedance chart to an admittance chart and vice versa. For $\Gamma(z) = \Gamma_0$, Eq. (5.34) gives $\widehat{Y}(z) = 1 - j1.16$. The dashed line in the figure shows the Γ_0 phasor rotated by 180^{o}. This converts the normalized admittance into the normalized impedance. It is given by $\widehat{Z}(z) = 0.429 + j0.495$.

5.3 The Signal Flow Graph

S-parameter analysis is facilitated by the use of signal flow graphs. A signal flow graph, or flow graph for short, is a graphical representation of a set of linear equations which can be used to write by inspection the solution to the set of equations. For example, consider the set of equations

$$x_2 = Ax_1 + Bx_2 + Cx_5 \tag{5.37}$$

$$x_3 = Dx_1 + Ex_2 \tag{5.38}$$

$$x_4 = Fx_3 + Gx_5 \tag{5.39}$$

$$x_5 = Hx_4 \tag{5.40}$$

$$x_6 = Ix_3 \tag{5.41}$$

where x_1 through x_6 are variables and A through I are constants. These equations can be represented graphically as shown in Fig. 5.4. The graph has a node for each variable with branches connecting the nodes labeled with the constants A through I. The node labeled x_1 is called a source node because it has only outgoing branches. The node labeled x_6 is called a sink node because it has only incoming branches. The path from x_1 to x_2 to x_3 to x_6 is called a forward path because it originates at a source node and terminates at a non-source node and along which no node is encountered twice. The path gain for this forward path is AEI. The path from x_2 to x_3 to x_4 to x_5 and back to x_2 is called a feedback path because it originates and terminates on the same node and along which no node is encountered more than once. The loop gain for this feedback path is $EFHC$.

Mason's formula can be used to calculate the transmission gain from a source node to any non-source node in a flow graph. The formula is

$$T = \frac{1}{\Delta}\sum_k P_k \Delta_k \tag{5.42}$$

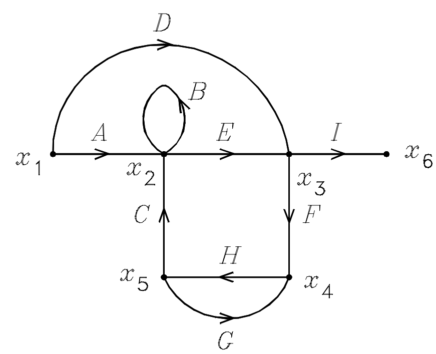

Figure 5.4: Example signal flow graph.

where P_k is the gain of the kth forward path, Δ is the graph determinant, and Δ_k is the determinant with the kth forward path erased. The determinant is given by

$$\begin{aligned}\Delta \;=\; & 1 - (\text{sum of all loop gains}) \\ & + \begin{pmatrix}\text{sum of the gain products of all possible} \\ \text{combinations of two non-touching loops}\end{pmatrix} \\ & - \begin{pmatrix}\text{sum of the gain products of all possible} \\ \text{combinations of three non-touching loops}\end{pmatrix} \\ & + \begin{pmatrix}\text{sum of the gain products of all possible} \\ \text{combinations of four non-touching loops}\end{pmatrix} \\ & - \cdots \end{aligned} \tag{5.43}$$

For the flow graph in Fig. 5.4, the objective is to solve for the gain from node x_1 to node x_6. There are two forward paths from x_1 to x_6 and three loops. Two of the loops do not touch each other. Thus the product of these two loop gains appears in the expression for Δ. The path gains and the determinant are given by

$$P_1 = AEI \tag{5.44}$$

$$P_2 = DI \tag{5.45}$$

$$\Delta = 1 - (B + CEFH + GH) + B \times GH \tag{5.46}$$

Path P_1 touches two loops while path P_2 touches one loop. The determinants with each path erased are given by

$$\Delta_1 = 1 - GH \tag{5.47}$$

$$\Delta_2 = 1 - (B + GH) + B \times GH \tag{5.48}$$

Thus the overall gain from x_1 to x_6 is given by

$$\frac{x_6}{x_1} = \frac{AEI \times (1 - GH) + DI \times [1 - (B + GH) + B \times GH]}{1 - (B + CEFH + GH) + B \times GH} \tag{5.49}$$

Example 1 *Figure 5.5 shows a transmission line of length ℓ connected between a source and a load. Draw the flow graph for the system and use it to solve for V_L/V_s and the load power P_L.*

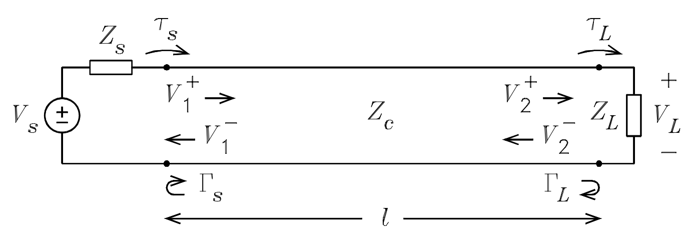

Figure 5.5: Transmission line connected between a source and load.

Solution. The source transmission coefficient τ_s, the source reflection coefficient Γ_s, the load transmission coefficient τ_L, and the load reflection coefficient Γ_L are given by

$$\tau_s = \frac{V_1^+}{V_s} = \frac{Z_c}{Z_s + Z_c} \tag{5.50}$$

$$\Gamma_s = \frac{V_1^+}{V_1^-} = \frac{Z_s - Z_c}{Z_s + Z_c} \tag{5.51}$$

$$\tau_L = \frac{V_o}{V_2^+} = \frac{2Z_L}{Z_c + Z_L} \tag{5.52}$$

$$\Gamma_L = \frac{V_2^-}{V_2^+} = \frac{Z_L - Z_c}{Z_L + Z_c} \tag{5.53}$$

In addition, we have the relations

$$\frac{V_2^+}{V_1^+} = \frac{V_1^-}{V_2^-} = e^{-j\beta\ell} \tag{5.54}$$

The flow graph is shown in Fig. 5.6. There is one loop. The determinant is

$$\Delta = 1 - \Gamma_s \Gamma_L e^{-j2\beta\ell} \tag{5.55}$$

It follows that

$$\frac{V_L}{V_s} = \frac{1}{\Delta}\tau_s \tau_L e^{-j\beta\ell} = \frac{\tau_s \tau_L e^{-j\beta\ell}}{1 - \Gamma_s \Gamma_L e^{-j2\beta\ell}} \tag{5.56}$$

By the third relation in Eq. (5.21), the load power is given by

$$P_L = \overline{v_L^2}\,\mathrm{Re}\,(Y_L^*) = \overline{v_s^2}\frac{|\tau_s \tau_L|^2}{|1 - \Gamma_s \Gamma_L e^{-j2\beta\ell}|^2}\,\mathrm{Re}\,(Y_L^*)$$

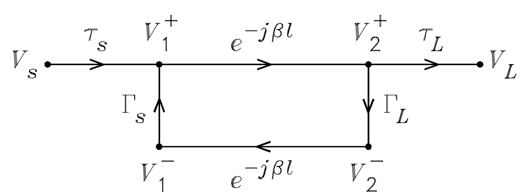

Figure 5.6: Signal-flow graph.

5.3.1 Thévenin and Norton Equivalents of Source and Line

The Thévenin equivalent circuit seen looking to the left of Z_L in Fig. 5.5 can be represented by an open-circuit voltage $V_{o(oc)}$ in series with an impedance Z_{out}. The Norton equivalent circuit is a short-circuit current $I_{o(sc)}$ in parallel with Z_{out}. To calculate $V_{o(oc)}$, we set $Z_L = \infty$. In this case, $\tau_L = 2$ and $\Gamma_L = 1$. It follows from Eqs. (5.55) and (5.56) that

$$V_{o(oc)} = V_s \frac{2\tau_s e^{-j\beta\ell}}{1 - \Gamma_s e^{-j2\beta\ell}} \tag{5.57}$$

Although Eqs. (5.23) or (5.24) can be used to solve for Z_{out}, it can also be solved for from the flow graph. The impedance is given by

$Z_{\text{out}} = V_{o(oc)}/I_{o(sc)}$, where $I_{o(sc)}$ is the current through Z_L with $Z_L = 0$. In this case $\tau_L = 0$ and $\Gamma_L = -1$. Thus $V_2^- = -V_2^+$ and $I_{o(sc)}$ is given by

$$I_{o(sc)} = \frac{V_2^+}{Z_c} - \frac{V_2^-}{Z_c} = \frac{V_s}{Z_c}\frac{2\tau_s e^{-j\beta\ell}}{1+\Gamma_s e^{-j2\beta\ell}} \tag{5.58}$$

It follows that Z_{out} is given by

$$Z_{\text{out}} = \frac{V_{o(oc)}}{I_{o(sc)}} = Z_c \frac{1+\Gamma_s e^{-j2\beta\ell}}{1-\Gamma_s e^{-j2\beta\ell}} \tag{5.59}$$

With $z = 0$, this is one of the expressions in Eq. (5.24).

5.3.2 Series and Shunt Noise Sources

Figure 5.7(a) shows a section of line with a series noise voltage source and a shunt noise current source. Forward and reverse propagating waves are assumed at each end of the line. If the length of the line is considered to be negligible, we can write

$$V_1^+ + V_1^- + V_n = V_2^+ + V_2^- \tag{5.60}$$

$$\frac{V_1^+}{Z_c} - \frac{V_1^-}{Z_c} + I_n = \frac{V_2^+}{Z_c} - \frac{V_2^-}{Z_c} \tag{5.61}$$

These equations can be solved for V_2^+ and V_1^- to obtain

$$V_2^+ = V_1^+ + \frac{V_n + I_n Z_c}{2} \tag{5.62}$$

$$V_1^- = V_2^- + \frac{V_n - I_n Z_c}{2} \tag{5.63}$$

It follows from these equations that the voltage V_n appears as a common-mode component of the V_2^+ and V_1^- waves and $I_n Z_c/2$ appears as a differential component. Fig. 5.7(b) shows the flow graph for the circuit. The nodes labeled V_a and V_b are summing nodes which are not in the circuit of Fig. 5.7(a).

5.4 S-Parameter Representation

The classical circuit models for two-port amplifiers are the chain-matrix, the y-matrix, and the z-matrix models. These are based primarily on an

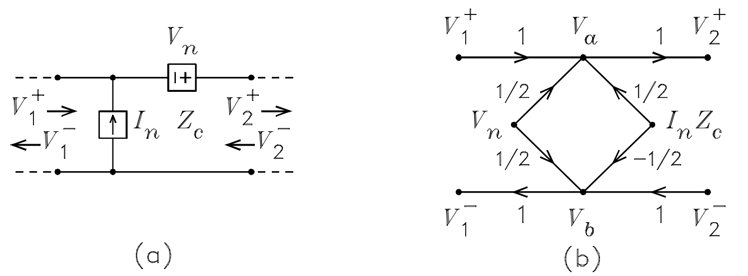

Figure 5.7: (a) Section of line with a series voltage source and a shunt current source. (b) Signal flow graph.

impedance description of the circuits. As frequency is increased, voltage and current become increasingly difficult to measure. The problems can be circumvented with the scattering matrix or s-parameter two-port model, which is based on transmission line concepts. The s parameters are measured at rf and microwave frequencies with a network analyzer.

Consider a device, e.g. an amplifier, with transmission lines having a characteristic impedance Z_c connected to its input and to its output as shown in Fig. 5.8. At the input, denote the phasor voltages propagating in the $+z$ and $-z$ directions, respectively, by V_1^+ and V_1^-. Similarly, at the output, denote the phasor voltages by V_2^+ and V_2^-. We can write

$$V_1^- = s_{11}V_1^+ + s_{12}V_2^- \tag{5.64}$$

$$V_2^+ = s_{21}V_1^+ + s_{22}V_2^- \tag{5.65}$$

The coefficients in these equations are called the s parameters. Note that s_{11} is simply the reflection coefficient at the input. Similarly s_{22} is the reflection coefficient at the output. The forward transmission coefficient is s_{21}. The reverse transmission coefficient is s_{12}. Note that the second subscript of each parameter refers to the input wave and the first refers to the output wave.

The convention used here is to label all waves traveling in the forward or $+z$ direction with a plus superscript and all waves traveling in the reverse or $-z$ direction with a minus superscript. A more commonly used convention is to label all waves traveling into a device with a plus superscript and all waves traveling out of the device with a minus superscript. When this convention is used in the analysis of multistage devices, the reverse wave at the output of one stage corresponds to the

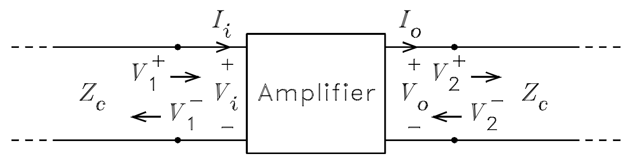

Figure 5.8: Amplifier with input and output transmission lines.

forward wave at the input to the following stage. The confusion of reversal of the directions of the forward and reverse directions between stages is eliminated with the notation used here.

5.4.1 Single-Stage Amplifier

Figure 5.9 shows an amplifier driven from a source having an output impedance Z_s. The amplifier load impedance is Z_L. Transmission lines of characteristic impedance Z_c connect the source and load to the amplifier. We assume that the s parameters of the amplifier are known. The amplitude of the $+z$ wave at the source is calculated as if the source is connected to an infinitely long transmission line having a characteristic impedance Z_c. It can be written $V_1^+ = \tau_s V_s$ where τ_s is the voltage division ratio given by

$$\tau_s = \frac{V_1^+}{V_s} = \frac{Z_c}{Z_s + Z_c} \tag{5.66}$$

The load transmission coefficient τ_L is calculated under the assumption that the load is driven from a transmission line having a characteristic impedance Z_c. It is given by Eq. (5.19).

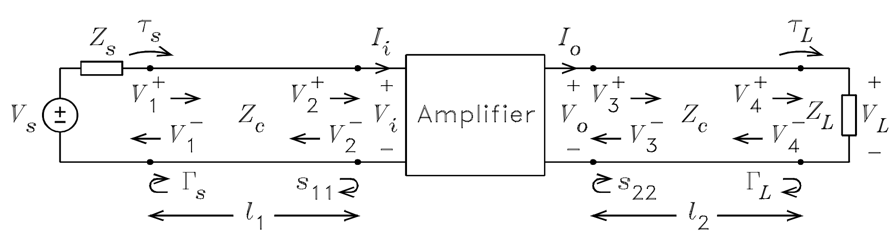

Figure 5.9: Single-stage amplifier.

The reflection coefficients Γ_s at the source and Γ_L at the load are calculated for incident waves on lines of characteristic impedance Z_c. Γ_L is given by Eq. (5.14) and Γ_s is given by

$$\Gamma_s = \frac{V_1^+}{V_1^-} = \frac{Z_s - Z_c}{Z_s + Z_c} \tag{5.67}$$

The flow graph for the system is shown in Fig. 5.10. There are three loops in the graph, two of which do not touch. By Eq. (5.43), the determinant is given by

$$\begin{aligned}\Delta \;=\; & 1 - \left[s_{11}\Gamma_s e^{-j2\beta\ell_1} + s_{22}\Gamma_L e^{-j2\beta\ell_2} + s_{21}\Gamma_L s_{12}\Gamma_s e^{-j2\beta(\ell_1+\ell_2)}\right] \\ & + s_{11}\Gamma_s s_{22}\Gamma_L e^{-j2\beta(\ell_1+\ell_2)} \end{aligned} \tag{5.68}$$

By Eq. (5.42), the voltage gain is given by

$$\frac{V_L}{V_s} = \frac{1}{\Delta}\tau_s s_{21}\tau_L e^{-j\beta(\ell_1+\ell_2)} \tag{5.69}$$

By the third relation in Eq. (5.21), the power delivered to the load can be written

$$P_L = \overline{v_L^2}\,\mathrm{Re}\,(Y_L^*) = \overline{v_s^2}\left|\frac{\tau_s s_{21}\tau_L}{\Delta}\right|^2 \mathrm{Re}\,(Y_L^*)$$

where $Y_L^* = 1/Z_L^*$.

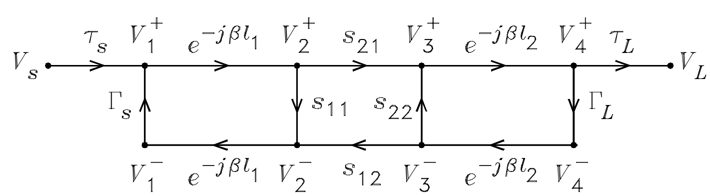

Figure 5.10: Amplifier signal-flow graph.

The input impedance looking into the amplifier input terminals is given by

$$Z_{\mathrm{in}} = \frac{V_i}{I_i} = Z_c\frac{V_2^+ + V_2^-}{V_2^+ - V_2^-} = Z_c\frac{V_2^+/V_s + V_2^-/V_s}{V_2^+/V_s - V_2^-/V_s} \tag{5.70}$$

From the flow graph, we have

$$\frac{V_2^+}{V_s} = \frac{1}{\Delta}\tau_s e^{-j\beta\ell_1}\left(1 - s_{22}\Gamma_L e^{-j2\beta\ell_2}\right) \tag{5.71}$$

$$\frac{V_2^-}{V_s} = \frac{1}{\Delta}\left[\tau_s s_{11} e^{-j\beta\ell_1}\left(1 - s_{22}\Gamma_L e^{-j2\beta\ell_2}\right) + \tau_s s_{21}\Gamma_L s_{12} e^{-j\beta(\ell_1+2\ell_2)}\right] \tag{5.72}$$

When these are substituted into Eq. (5.70), we obtain

$$Z_{\text{in}} = Z_c \frac{(1+s_{11})\left(1 - s_{22}\Gamma_L e^{-j2\beta\ell_2}\right) + s_{21}\Gamma_L s_{12} e^{-j2\beta\ell_2}}{(1-s_{11})\left(1 - s_{22}\Gamma_L e^{-j2\beta\ell_2}\right) - s_{21}\Gamma_L s_{12} e^{-j2\beta\ell_2}} \tag{5.73}$$

By symmetry, the output impedance seen looking into the amplifier output terminals can be obtained by a simple change of subscripts to obtain

$$Z_{\text{out}} = Z_c \frac{(1+s_{22})\left(1 - s_{11}\Gamma_s e^{-j2\beta\ell_1}\right) + s_{12}\Gamma_s s_{21} e^{-j2\beta\ell_1}}{(1-s_{22})\left(1 - s_{11}\Gamma_s e^{-j2\beta\ell_1}\right) - s_{12}\Gamma_s s_{21} e^{-j2\beta\ell_1}} \tag{5.74}$$

The Thévenin equivalent circuit seen looking into the amplifier output terminals consists of the impedance Z_{out} in series with a voltage source $V_{o(oc)}$. This voltage is simply the value of V_L with $R_L = \infty$ and $\ell_2 = 0$. For $R_L = \infty$, we have $\tau_L = 2$. In this case, $V_{o(oc)}$ and $\Delta_{(oc)}$ are given by

$$V_{o(oc)} = \frac{2\tau_s s_{21} e^{-j\beta\ell_1}}{\Delta_{(oc)}} V_s \tag{5.75}$$

$$\Delta_{(oc)} = 1 - \left[s_{11}\Gamma_s e^{-j2\beta\ell_1} + 2s_{22} + 2s_{21}s_{12}\Gamma_s e^{-j2\beta\ell_1}\right] + 2s_{11}\Gamma_s s_{22} e^{-j2\beta\ell_1} \tag{5.76}$$

From the flow graph and Eqs. (5.71) and (5.72), the amplifier input voltage is given by

$$\begin{aligned} V_i &= V_2^+ + V_2^- \\ &= V_s \frac{\tau_s e^{-j\beta\ell_1}}{\Delta}\left[(1+s_{11})\left(1 - s_{22}\Gamma_L e^{-j2\beta\ell_2}\right) + s_{21}\Gamma_L s_{12} e^{-j2\beta\ell_2}\right] \end{aligned} \tag{5.77}$$

The power delivered by the source to the amplifier input is

$$\begin{aligned} P_i &= \overline{v_i^2}\,\mathrm{Re}\left(Y_{\text{in}}^*\right) \\ &= \overline{v_s^2}\left|\frac{\tau_s}{\Delta}\right|^2 \left|(1+s_{11})\left(1 - s_{22}\Gamma_L e^{-j2\beta\ell_2}\right) + s_{21}\Gamma_L s_{12} e^{-j2\beta\ell_2}\right|^2 \mathrm{Re}\left(Y_{\text{in}}^*\right) \end{aligned} \tag{5.78}$$

where $Y_{\text{in}} = 1/Z_{\text{in}}$. It follows that the amplifier power gain is

$$\begin{aligned} G_p &= \frac{P_L}{P_i} \\ &= \frac{|s_{21}\tau_L|^2 \operatorname{Re}(Y_L^*) / \operatorname{Re}(Y_{\text{in}}^*)}{|(1+s_{11})(1-s_{22}\Gamma_L e^{-j2\beta\ell_2}) + s_{21}\Gamma_L s_{12} e^{-j2\beta\ell_2}|^2} \end{aligned} \tag{5.79}$$

The equivalent expressions for the case where the source is connected directly to the amplifier with no transmission line can be obtained from those above by setting $\ell_1 = 0$. Similarly, setting $\ell_2 = 0$ yields the expressions for the case where the load is connected directly to the amplifier. Setting $\ell_1 = \ell_2 = 0$ yields the expressions for the case with neither transmission line. The expressions for τ_s, Γ_s, τ_L, and Γ_L remain functions of Z_c, even if the transmission lines are not present. This is because the s parameters are measured or calculated with transmission lines at both the input and the output. The value of Z_c used in the expressions for τ_s, Γ_s, τ_L, and Γ_L must be the same as that used in the measurement or calculation of the s parameters. For example, if $\ell_1 = \ell_2 = 0$, the expressions for Z_{in}, Z_{out}, and G_p are

$$Z_{\text{in}} = Z_c \frac{(1+s_{11})(1-s_{22}\Gamma_L) + s_{21}\Gamma_L s_{12}}{(1-s_{11})(1-s_{22}\Gamma_L) - s_{21}\Gamma_L s_{12}} \tag{5.80}$$

$$Z_{\text{out}} = Z_c \frac{(1+s_{22})(1-s_{11}\Gamma_s) + s_{12}\Gamma_s s_{21}}{(1-s_{22})(1-s_{11}\Gamma_s) - s_{12}\Gamma_s s_{21}} \tag{5.81}$$

$$G_p = \frac{|s_{21}\tau_L|^2}{|(1+s_{11})(1-s_{22}\Gamma_L) + s_{21}\Gamma_L s_{12}|^2} \frac{\operatorname{Re}(Y_L^*)}{\operatorname{Re}(Y_{\text{in}}^*)} \tag{5.82}$$

Example 2 *At the frequency $f = 900$ MHz, the measured s-parameters of an amplifier are $s_{11} = 0.85\angle{-41^o}$, $s_{21} = 19.6\angle 72^o$, $s_{12} = 0.001\angle 107^o$, and $s_{22} = 0.10\angle - 11^o$. The test fixture transmission line used to measure the s-parameters has a characteristic impedance $Z_c = 50\ \Omega$. The amplifier is driven by a source with an output impedance $Z_s = 50\ \Omega$. The load impedance is $Z_L = 50\ \Omega$. The source and load are connected directly to the amplifier, i.e. with no transmission lines. Calculate Z_{in}, Z_{out}, and G_p.*

Solution. Γ_s, τ_L, and Γ_L are calculated assuming zero-length 50 Ω transmission lines connect the source and load to the amplifier. The values are $\Gamma_s = 0$, $\tau_L = 1$, and $\Gamma_L = 0$. Although τ_s is not required, its value is $\tau_s = 0.5$. Eqs. (5.80) – (5.82) give

$$Z_{\text{in}} = 50\frac{1 + 0.85\angle - 41^{\text{o}}}{1 - 0.85\angle - 41^{\text{o}}} = 31.6 - j127$$

$$Z_{\text{out}} = 50\frac{1 + 0.10\angle - 11^{\text{o}}}{1 - 0.10\angle - 11^{\text{o}}} = 60.8 - j2.35$$

$$G_p = \frac{|19.6\angle 72^{\text{o}}|^2}{|1 + 0.85\angle - 41^{\text{o}}|^2} \frac{0.02}{\text{Re}\left[(31.6 + j127)^{-1}\right]} = 9.47 \times 10^3$$

The decibel value of G_p is 39.8 dB. G_p can be increased by adding matching networks at the amplifier input and output. Example matching networks are covered in the following.

5.4.2 $v_n - i_n$ Noise Model

Figure 5.11 shows a source and a load connected to an amplifier with noise sources at its input. We desire the equivalent noise input voltage V_{ni} in series with the source when the amplifier is modeled by its s-parameter model. A transmission line is assumed between the source and the amplifier. For the case where the output is connected to the load by a transmission line, Z_L is the impedance seen looking into the transmission line.

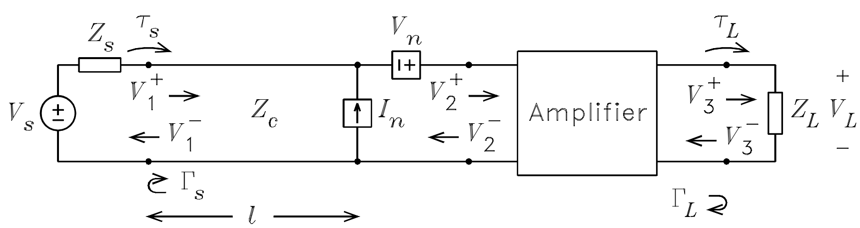

Figure 5.11: Amplifier with noise sources at its input.

The flow graph for the circuit is shown in Fig. 5.12, where the noise is modeled by the flow graph of Fig. 5.7(b). There are three loops in the graph, two which do not touch. The determinant is

$$\begin{aligned}\Delta \;=\; & 1 - \left(s_{11}\Gamma_s e^{-j2\beta\ell} + s_{22}\Gamma_L + s_{21}\Gamma_L s_{12}\Gamma_s e^{-j2\beta\ell}\right) \\ & + s_{11}\Gamma_s s_{22}\Gamma_L e^{-j2\beta\ell}\end{aligned} \tag{5.83}$$

There are three sources in the graph, all of which touch both loops. The total output voltage is given by

$$V_L = \frac{1}{\Delta}\left[(V_s + V_{ts})\,\tau_s s_{21}\tau_L e^{-j\beta\ell} + \frac{V_n + I_n Z_c}{2} s_{21}\Gamma_L + \frac{V_n - I_n Z_c}{2}\Gamma_s s_{21}\Gamma_L e^{-j2\beta\ell}\right] \quad (5.84)$$

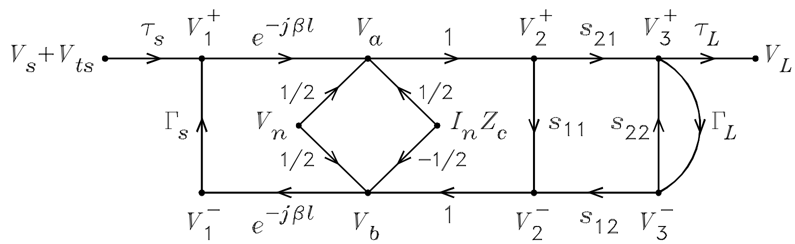

Figure 5.12: Flow graph for the amplifier.

To obtain V_{ni}, we factor the coefficient of V_s from the brackets. After doing this, V_{ni} consists of all terms in the brackets following V_s. It is given by

$$V_{ni} = V_{ts} + \frac{V_n e^{j\beta\ell}}{2\tau_s}\left(1 + \Gamma_s e^{-j2\beta\ell}\right) + \frac{I_n Z_c e^{j\beta\ell}}{2\tau_s}\left(1 - \Gamma_s e^{-j2\beta\ell}\right) \quad (5.85)$$

The mean-square value is

$$\overline{v_{ni}^2} = 4kT\,\mathrm{Re}\,(Z_s)\,\Delta f + \frac{\overline{v_n^2}}{4\left|\tau_s\right|^2}\left|1 + \Gamma_s e^{-j2\beta\ell}\right|^2 + \frac{\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}}{2\left|\tau_s\right|^2}\mathrm{Re}\left\{\gamma\left[1 + j2\,\mathrm{Im}\left(\Gamma_s e^{-j\beta\ell}\right) - \left|\Gamma_s\right|^2\right] Z_c\right\} + \frac{\overline{i_n^2} Z_c^2}{4\left|\tau_s\right|^2}\left|1 - \Gamma_s e^{-j2\beta\ell}\right|^2 \quad (5.86)$$

A case of special interest is for $Z_s = Z_c$. In this case, $\Gamma_s = 0$, $\tau_s = 0.5$, and $\overline{v_{ni}^2}$ reduces to

$$\overline{v_{ni}^2} = 4kT\,\mathrm{Re}\,(Z_s)\,\Delta f + \overline{v_n^2} + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\,(\gamma Z_s^*) + \overline{i_n^2}\left|Z_s\right|^2 \quad (5.87)$$

This is identical to Eq. (5.2) for the case with no input transmission line.

5.5 Maximum Power Theorems

Figure 5.13 shows a source with an open-circuit voltage V_s and an output impedance $Z_s = R_s + jX_s$ connected to the input of an amplifier. The power delivered to Z_i is given by

$$P_i = \left| \frac{V_s}{Z_s + Z_i} \right|^2 \mathrm{Re}\,(Z_i) = \frac{\overline{v_s^2} R_i}{(R_s + R_i)^2 + (X_s + X_i)^2} \tag{5.88}$$

We seek the conditions which maximize P_i. There are two ways. One is to vary Z_s. The other is to vary Z_i.

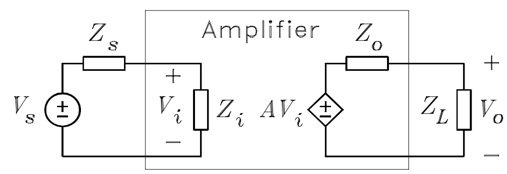

Figure 5.13: Amplifier with source connected to its input.

Suppose that Z_s is to be varied. Maximum power occurs when $R_s = 0$ and $X_s = -X_i$ and is given by

$$P_i = \frac{\overline{v_s^2}}{R_i} \tag{5.89}$$

If Z_i is varied, maximum power occurs when $R_i = R_s$ and $X_i = -X_s$. In this case, the power is called the available input power and is given by

$$P_{ai} = \frac{\overline{v_s^2}}{4R_s} \tag{5.90}$$

In this case, the input impedance is equal to the conjugate of the source impedance, i.e. $Z_i = Z_s^*$ or $Z_s = Z_i^*$. Such a condition is called a conjugate impedance match.

5.6 Conjugate Matching Networks

In rf design, the source impedance is usually fixed and a matching network between the source and the load is used to maximize the power absorbed by the load. For minimum power loss, the matching network must be lossless. In order for the power delivered by the source to be a maximum, the input impedance to the matching network must be equal to the conjugate of the source impedance. For a lossless matching network, the power absorbed by the load is equal to the power delivered by the source.

Because the matching network maximizes the power delivered by the source, it follows that it also maximizes the power absorbed by the load. This could not be true unless the source impedance seen by the load is equal to the conjugate of the load impedance. Otherwise, the load impedance could be changed to increase the power absorbed by it, which would cause the power delivered by the source to increase.

Based on the above argument for a single matching network, a general impedance-transforming theorem for a cascade connection of lossless networks can be described. It is as follows:

Impedance-Transforming Theorem Let a cascade of two-port networks containing only pure reactances connect between a signal source and a load. If there is a conjugate match of impedances at any junction between two of the networks, there will be a conjugate match of impedances at every other junction between the two networks.

This theorem is used in the following to simplify the design of lossless transforming networks.

5.6.1 Lumped-Parameter Matching Networks

The simplest matching network is a two element network consisting of a series element and a shunt element. We assume that both the source impedance and the load impedance are real, i.e. resistive. This is not a restrictive assumption because a series or parallel reactance can always be used to cancel the imaginary part of an impedance. Fig. 5.14 shows a source connected to a load through two different matching networks. The network of Fig. 5.14(a) has the shunt element on the source side of the circuit. This network can be used to obtain a conjugate match for $R_s > R_i$. The network of Fig. 5.14(b) has the shunt element on the load side of the network. It can be used to obtain a conjugate match for $R_s < R_i$.

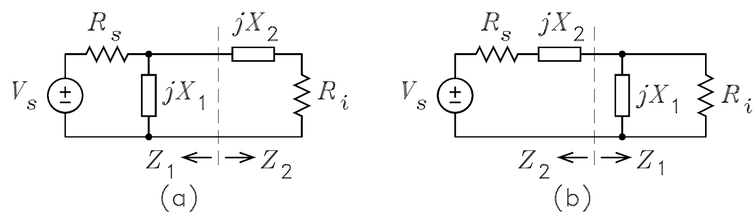

Figure 5.14: Lossless two-element matching networks.

Consider the network in Fig. 5.14(a). The impedances Z_1 and Z_2 are given by

$$Z_1 = R_s \| jX_1 = \frac{R_s \times jX_1}{R_s + jX_1} = \frac{R_s X_1^2}{R_s^2 + X_1^2} + j\frac{R_s^2 X_1}{R_s^2 + X_1^2} \tag{5.91}$$

$$Z_2 = R_i + jX_2 \tag{5.92}$$

Maximum power is delivered to R_i if the condition $Z_2 = Z_1^*$ holds. This leads to the equations

$$X_1 = \pm R_s \sqrt{\frac{R_i}{R_s - R_i}} \tag{5.93}$$

$$X_2 = \frac{-R_i R_s}{X_1} \tag{5.94}$$

Note that R_s must be greater than R_i for X_1 and X_2 to be real. There are two solutions. If the positive solution is taken for X_1, then X_2 is negative. The matching network has a high-pass response. Conversely, if the negative solution is taken for X_1, then X_2 is positive. The matching network has a low-pass response.

Consider the network in Fig. 5.14(b). The impedances Z_1 and Z_2 are given by

$$Z_1 = R_i \| jX_1 = \frac{R_i \times jX_1}{R_i + jX_1} = \frac{R_i X_1^2}{R_i^2 + X_1^2} + j\frac{R_i^2 X_1}{R_i^2 + X_1^2} \tag{5.95}$$

$$Z_2 = R_s + jX_2 \tag{5.96}$$

Maximum power is delivered to R_i if the condition $Z_1 = Z_2^*$ holds. This leads to the equations

$$X_1 = \pm R_i \sqrt{\frac{R_s}{R_i - R_s}} \tag{5.97}$$

$$X_2 = \frac{-R_i R_s}{X_1} \tag{5.98}$$

Note that R_i must be greater than R_s for X_1 and X_2 to be real. As for the matching network of Fig. 5.14(a), there are two solutions. If the positive solution is taken for X_1, then X_2 is negative. The matching network has a low-pass response. Conversely, if the negative solution is taken for X_1, then X_2 is positive. The matching network has a high-pass response.

Input-End Reflection Coefficient

Figure 5.15 shows a source connected to a matching network with an input impedance $Z_{im} = R_{im} + jX_{im}$. The maximum available power to drive the network is $P_{ai} = \overline{v_s^2}/4R_s$. This occurs when $Z_{im} = R_s$. Under this condition, the input voltage to the network is $V_1 = V_s/2$. The amount by which V_1 deviates from $V_s/2$ is an indication of the amount by which the power delivered to the network deviates from the maximum. We define the input-end reflection coefficient to the matching network by

$$\Gamma_{im} = \frac{V_1 - V_s/2}{V_s/2} = \frac{I_1 Z_{im} - I_1 (R_s + Z_{im})/2}{I_1 (R_s + Z_{im})/2} = \frac{Z_{im} - R_s}{Z_{im} + R_s} \tag{5.99}$$

The power delivered to the network is given by

$$P_{im} = \left| \frac{V_s}{R_s + Z_{im}} \right|^2 \mathrm{Re}(Z_{im}) = \frac{\overline{v_s^2}}{|R_s + Z_{im}|^2} R_{im} \tag{5.100}$$

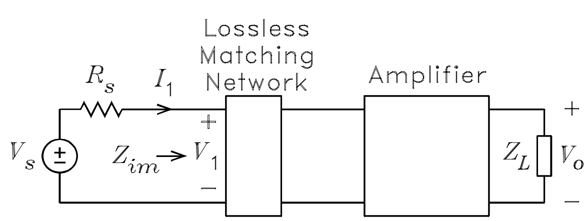

Figure 5.15: Matching network between a source and an amplifier.

The difference between the maximum power and the actual power into the network is

$$
\begin{aligned}
P_{ai} - P_{im} &= \frac{\overline{v_s^2}}{4R_s} - \frac{\overline{v_s^2}}{|R_s + Z_{im}|^2} R_{im} \\
&= \frac{(R_{im} - R_s)^2 + X_{im}^2}{(R_{im} + R_s)^2 + X_{im}^2} \frac{\overline{v_s^2}}{4R_s} \\
&= |\Gamma_{im}|^2 P_{ai} \qquad (5.101)
\end{aligned}
$$

This can be solved for P_{im} to obtain

$$P_{im} = \left(1 - |\Gamma_{im}|^2\right) P_{ai} \qquad (5.102)$$

For maximum power delivered to the network, it follows that $|\Gamma_{im}|^2 = 0$. This occurs when $R_{im} = R_s$ and $X_{im} = 0$. With single-stage matching networks, this condition can be achieved at a single frequency. With two or more matching networks in cascade, an impedance match can be obtained over a wider frequency band. This is illustrated in the next section.

If the source impedance is complex, maximum power is delivered to the network when $Z_{im} = Z_s^*$. In this case, $\Gamma_{im} \neq 0$. To use the concept of an input-end reflection coefficient with a complex source impedance, the reactive part of Z_s must be considered to be part of the matching network. Alternately, a reactance can be put in series with the source to cancel the reactive component of Z_s.

Matching Network Examples

Consider the case where a source having the output resistance $R_s = 50\ \Omega$ is to be matched to an amplifier having the input resistance $R_i = 25\ \Omega$ at the frequency $f = 10$ MHz. Without a matching network, the input-end reflection coefficient is $\Gamma_{im} = -1/3$. Thus, without a matching network, the ratio of the power delivered to the amplifier to the maximum available power is $1 - |\Gamma_{im}|^2 = 8/9$ or 88.9%.

The matching network of Fig. 5.14(a) with $X_1 > 0$ is to be used to match the source to the amplifier. The circuit is shown in Fig. 5.15(a). The element values are calculated from Eqs. (5.93) and (5.94) as follows:

$$X_1 = 50\sqrt{\frac{25}{50 - 25}} = 50\ \Omega$$

$$X_2 = \frac{-25 \times 50}{50} = -25\ \Omega$$

$$L_1 = \frac{X_1}{2\pi 10^7} = 0.796\ \mu\text{H}$$

$$C_2 = \frac{-1}{2\pi 10^7 X_2} = 0.637\ \text{nF}$$

Curve a in Fig. 5.17 shows the plot of the magnitude of the input-end reflection coefficient as a function of frequency for the single-section matching network.

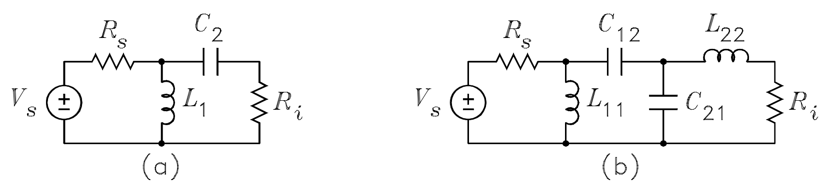

Figure 5.16: Circuits for example matching network calculations. (a) Single-section network. (b) Two-section network.

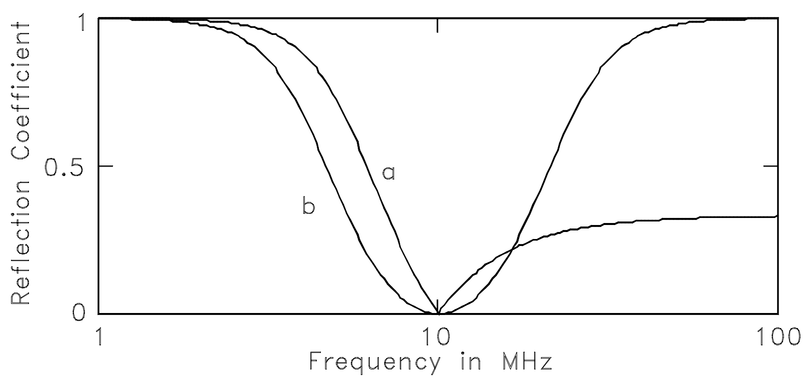

Figure 5.17: Reflection coefficient versus frequency for the circuits in Fig. 5.16.

The bandwidth of the matching network can be increased by cascading two or more networks. A two-section network is shown in Fig. 5.15(b). The network consists of a high-pass section in cascade with a low-pass section. For optimal bandwidth, the input resistance to the second network should be the geometrical mean of R_s and R_i, i.e. $\sqrt{R_s R_i}$.

For the present example, this is $\sqrt{50 \times 25} = 35.36$. The element values in the matching network are calculated as follows:

$$X_{21} = -35.4\sqrt{\frac{25}{35.4 - 25}} = -54.9\ \Omega$$
$$X_{22} = \frac{25 \times 35.36}{54.9} = 16.1\ \Omega$$
$$C_{21} = \frac{-1}{2\pi 10^7 X_{21}} = 2.90\ \text{nF}$$
$$L_{22} = \frac{X_{22}}{2\pi 10^7} = 2.56\ \mu\text{H}$$
$$X_{11} = 50\sqrt{\frac{35.4}{50 - 35.4}} = 77.7\ \Omega$$
$$X_{12} = \frac{-35.36 \times 50}{77.7} = -22.8\ \Omega$$
$$L_{11} = \frac{X_{11}}{2\pi 10^7} = 1.24\ \mu\text{H}$$
$$C_{12} = \frac{-1}{2\pi 10^7 X_{12}} = 0.699\ \text{nF}$$

Curve b in Fig. 5.17 shows the plot of $|\Gamma_{im}|^2$ as a function of frequency. Note the increased bandwidth compared to the single-section network. For $|\Gamma_{im}|^2 \leq 0.01$, the bandwidth of the two-stage network is approximately twice that of the single-stage network. The bandwidth can be increased by increasing the number of stages. For example, two of the matching networks shown in Fig. 5.15(b) can be cascaded to form a four-stage network. The input resistance of each stage should be the geometric mean of the input resistances of the stages at its input and at its output. For four stages, it follows that the input resistance of the third stage should be $R_{i3} = \sqrt{R_s R_i}$, the input resistance of the second stage should be $R_{i2} = \sqrt{R_s R_{i3}}$, and the input resistance of the fourth stage should be $R_{i4} = \sqrt{R_{i3} R_i}$.

Broadband matching networks can be designed so that $|\Gamma_{im}|^2$ is less than a specified value over a desired frequency band. The approximating functions are often taken to be those used in network synthesis. For example, a Butterworth approximation is one for which $\partial^m |\Gamma_{im}|^2 / \partial\omega^m = 0$ for $1 \leq m \leq 2n - 1$, where n is the order of the transfer function. Curve (b) in Fig. 5.17 is a Butterworth approximation. For a Chebyshev approximation, $|\Gamma_{im}|^2$ ripples between 0 and a specified maximum value over the band. The design of the Chebyshev networks can be tedious without computer design tools.

5.6.2 Transmission Line Matching Networks

At higher frequencies, inductors and capacitors become increasingly difficult to realize as lumped-parameter elements. These problems are often circumvented by the use of microstrip transmission line matching networks. Two such networks are described below.

The Quarter-Wave Transformer

The quarter-wave transformer consists of a section of line having a length equal to $\lambda/4$. Consider the line shown in Fig. 5.1(c). For $\ell = \lambda/4$, we have $\beta\ell = \pi/2$. It follows from Eq. (5.24) that the input impedance at $z = 0$ is given by

$$Z(0) = Z_c \frac{1}{\widehat{Z}_L} = \frac{Z_c^2}{Z_L} \tag{5.103}$$

The quarter-wave transformer is often used as a matching network. The characteristic impedance of the line must be equal to the geometric mean of the load impedance and the desired input impedance to the transformer. Broadband matching networks can be designed that consist of a cascade of quarter-wave transformers, each section having a different characteristic impedance.

Two-Line Matching Networks

A transmission line of length ℓ with a short circuit for its load exhibits an input impedance given by

$$Z = jZ_c \tan(\beta\ell) = jZ_c \tan\left(\frac{2\pi\ell}{\lambda}\right) \tag{5.104}$$

where Z_c is the characteristic impedance of the line. This can be made to be either inductive or capacitive by varying ℓ. The shorted line is used as the shunt element in a matching network with its load end is connected to signal ground. The series element is usually realized with a transmission line that is used not as an impedance element but as an impedance transforming element. Such a network is illustrated in Fig. 5.18. The design of this network is described below for a conjugate match at the amplifier input.

It is desired to calculate the lengths of the two transmission lines in Fig. 5.18 so that the amplifier sees a source impedance equal to Z_i^*. We assume that the source impedance is real and equal to R_s. Let the

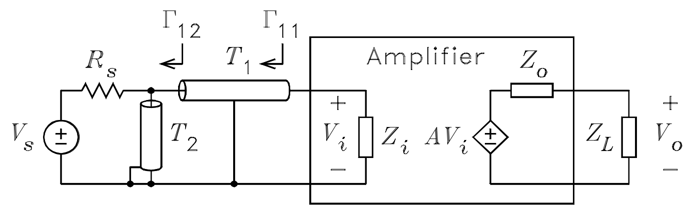

Figure 5.18: Amplifier with transmission line matching network.

characteristic impedance of T_1 be Z_{c1} and that of T_2 be Z_{c2}. Looking out of the amplifier input into T_1, the desired reflection coefficient is given by

$$\Gamma_{11} = \frac{Z_i^* - Z_{c1}}{Z_i^* + Z_{c1}} \tag{5.105}$$

Let the reflection coefficient at the source end of T_1 be Γ_{12}. This is given by

$$\Gamma_{12} = \frac{R_s \| jX_2 - Z_{c1}}{R_s \| jX_2 + Z_{c1}} \tag{5.106}$$

where jX_2 is the reactance seen looking into T_2.

Because the magnitude of the reflection coefficient is independent of position on the line, it follows that $|\Gamma_{12}| = |\Gamma_{11}|$. This leads to the equation

$$\left| \frac{R_s \| jX_2 - Z_{c1}}{R_s \| jX_2 + Z_{c1}} \right| = |\Gamma_{11}| \tag{5.107}$$

This can be solved for X_2 to obtain

$$X_2 = \pm Z_{c1} \left[\frac{1 - |\Gamma_{11}|^2}{|\Gamma_{11}|^2 (1 + Z_{c1}/R_s)^2 - (1 - Z_{c1}/R_s)^2} \right]^{1/2} \tag{5.108}$$

where the positive solution minimizes the length of T_2. It follows that the electrical length of T_2 is given by

$$\beta \ell_2 = \tan^{-1} \left(\frac{\pm X_2}{Z_{c2}} \right) \tag{5.109}$$

To obtain a realizable solution, the quantity in the brackets in Eq. (5.108) must be positive. This is true if

$$|\Gamma_{11}| > \left| \frac{R_s - Z_{c1}}{R_s + Z_{c1}} \right| \tag{5.110}$$

The reflection coefficient at the amplifier end of T_1 is equal to the reflection coefficient at the source end multiplied by $\exp(-j2\beta\ell_1)$. It follows that

$$\Gamma_{11} = \Gamma_{12} \exp(-j2\beta\ell_1) \tag{5.111}$$

Solution for the electrical length of T_1 yields

$$\beta\ell_1 = \frac{1}{2} \arg\left(\frac{\Gamma_{12}}{\Gamma_{11}}\right) \tag{5.112}$$

Example 3 *At the frequency $f = 900$ MHz, the input impedance of an amplifier is measured to be $Z_i = 31.6 - j127$ Ω. Determine the lengths ℓ_1 and ℓ_2 of the transmission lines in the network of Fig. 5.18 which cause the source impedance seen by the amplifier to be the conjugate of Z_i. The source impedance is $R_s = 50$ Ω. The characteristic impedance of the two transmission lines in the network is 75 Ω.*

Solution. The desired impedance seen looking out of the amplifier input is $Z_i^* = 31.6 + j127$. By Eq. (5.105), the reflection coefficient at the amplifier end of T_1 is

$$\Gamma_{11} = \frac{31.6 + j127 - 75}{31.6 + j127 + 75} = 0.418 + j0.693$$

This has the magnitude

$$|\Gamma_{11}| = 0.809$$

The reactance of T_2 is calculated from Eq. (5.108) to obtain

$$X_2 = 75\left[\frac{1 - 0.809^2}{0.809^2(1 + 75/50)^2 - (1 - 75/50)^2}\right]^{1/2} = 22.5$$

where the positive solution has been used. By Eq. (5.109), the electrical length of T_2 is

$$\beta\ell_2 = \tan^{-1}\left(\frac{22.5}{75}\right) = 16.7^\circ$$

The reflection coefficient at the source end of T_1 is calculated from Eq. (5.106) to obtain

$$\Gamma_{12} = \frac{50\|j22.5 - 75}{50\|j22.5 + 75} = -0.713 + j0.384$$

By Eq. (5.112), the electrical length of T_1 is

$$\beta\ell_1 = \frac{1}{2}\arg\left(\frac{\Gamma_{12}}{\Gamma_{11}}\right) = \frac{1}{2}\arg\left(\frac{-0.713 + j0.384}{0.418 + j0.693}\right) = 46.4^{\text{o}}$$

As a check of the impedance seen looking into T_1, we have

$$\begin{aligned} Z &= Z_{c1}\frac{R_s \| jX_2 + jZ_{c1}\tan(\beta\ell_2)}{Z_{c1} + j\left(R_s \| jX_2\right)\tan(\beta\ell_2)} \\ &= 75\frac{21.02 + j122.0}{42.96 + j27.28} = 31.6 + j127 \end{aligned}$$

This is the desired value.

Chapter 6

Noise Specifications

6.1 Signal-to-Noise Ratio

The signal-to-noise ratio is usually measured at the output of an amplifier where the signal and noise voltages are larger and easier to measure. It is given by $SNR = \overline{v_{so}^2}/\overline{v_{no}^2}$, where $\overline{v_{so}^2}$ is the mean-square signal output voltage and $\overline{v_{no}^2}$ is the mean-square noise output voltage. It is usually specified in dB by the relation $10\log\left(\overline{v_{so}^2}/\overline{v_{no}^2}\right)$. When calculating the SNR with the $v_n - i_n$ noise model of the amplifier, it is convenient to make the calculation at the amplifier input. When the source is modeled by a Thévenin source, the SNR is given by $SNR = \overline{v_s^2}/\overline{v_{ni}^2}$, where $\overline{v_s^2}$ is the mean-square source voltage and $\overline{v_{ni}^2}$ is the mean-square equivalent input noise voltage. When it is modeled by a Norton source, it is given by $SNR = \overline{i_s^2}/\overline{i_{ni}^2}$, where $\overline{i_s^2}$ is the mean-square source current and $\overline{i_{ni}^2}$ is the mean-square equivalent input noise current. Expressions are derived below for the SNR for both cases. The source impedance and admittance which maximizes the SNR are also derived.

6.1.1 Thévenin Source

When the source is modeled by a Thévenin equivalent circuit as in Fig. 6.1, the signal-to-noise ratio is given by $SNR = \overline{v_s^2}/\overline{v_{ni}^2}$. When Eq. (4.3) is used for $\overline{v_{ni}^2}$, it follows that the SNR is given by

$$SNR = \frac{\overline{v_s^2}}{\overline{v_{ni}^2}} = \frac{\overline{v_s^2}}{4kTR_s\Delta f + \overline{v_n^2} + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\left(\gamma Z_s^*\right) + \overline{i_n^2}\left|Z_s\right|^2} \tag{6.1}$$

where $Z_s = R_s + jX_s$ and $\gamma = \gamma_r + j\gamma_i$. It is expressed in dB by the relation $10\log\left(\overline{v_s^2}/\overline{v_{ni}^2}\right)$. It is maximized by minimizing $\overline{v_{ni}^2}$. The source impedance which minimizes $\overline{v_{ni}^2}$ can be obtained by setting $\partial\overline{v_{ni}^2}/\partial R_s = 0$ and $\partial\overline{v_{ni}^2}/\partial X_s = 0$ and solving for R_s and X_s. The solution for R_s is negative. Because this is not realizable, $R_s = 0$ is the realizable solution for the least noise. The source impedance which minimizes $\overline{v_{ni}^2}$ is given by

$$Z_s = R_s + jX_s = 0 - j\gamma_i\sqrt{\frac{\overline{v_n^2}}{\overline{i_n^2}}} \tag{6.2}$$

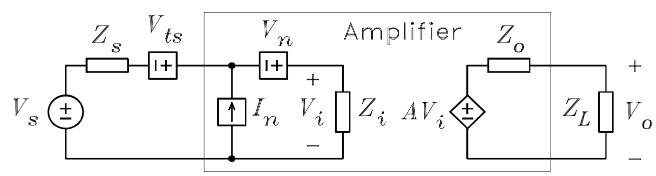

Figure 6.1: Amplifier with Thévenin source.

Because minimum noise occurs for $R_s = 0$, it can be concluded that a resistor should never be connected in series with a source at an amplifier input if noise performance is a design criterion. If a series resistor is required, e.g. for stability, it should be much smaller than R_s. Although the output impedance of a source is usually fixed, the SNR can be improved by adding a reactance in series with the source which makes the total series reactance equal to the imaginary part of Z_s in Eq. (6.2). When this is the case, $\overline{v_{ni}^2}$ is given by

$$\overline{v_{ni}^2} = 4kTR_s\Delta f + \overline{v_n^2}\left(1-\gamma_i^2\right) + 2\gamma_r R_s\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}} + \overline{i_n^2}R_s^2 \tag{6.3}$$

6.1.2 Norton Source

When the source is modeled by a Norton equivalent circuit as in Fig. 6.2, the signal-to-noise ratio is given by $SNR = \overline{i_s^2}/\overline{i_{ni}^2}$. When Eq. (4.22) is used for $\overline{i_{ni}^2}$, it follows that the SNR is given by

$$SNR = \frac{\overline{i_s^2}}{\overline{i_{ni}^2}} = \frac{\overline{i_s^2}}{4kTG_s\Delta f + \overline{v_n^2}\left|Y_s\right|^2 + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\left(\gamma Y_s\right) + \overline{i_n^2}} \tag{6.4}$$

where $Y_s = G_s + jB_s$ and $\gamma = \gamma_r + j\gamma_i$. It is expressed in dB by the relation $10\log\left(\overline{i_s^2}/\overline{i_{ni}^2}\right)$. It is maximized by minimizing $\overline{i_{ni}^2}$. The source admittance which minimizes $\overline{i_{ni}^2}$ can be obtained by setting $\partial\overline{i_{ni}^2}/\partial G_s = 0$ and $\partial\overline{i_{ni}^2}/\partial B_s = 0$ and solving for G_s and B_s. The solution for G_s is negative. Because this is not realizable, $G_s = 0$ is the realizable solution for the least noise. The source admittance which minimizes $\overline{i_{ni}^2}$ is given by

$$Y_s = G_s + jB_s = 0 + j\gamma_i\sqrt{\frac{\overline{i_n^2}}{\overline{v_n^2}}} \tag{6.5}$$

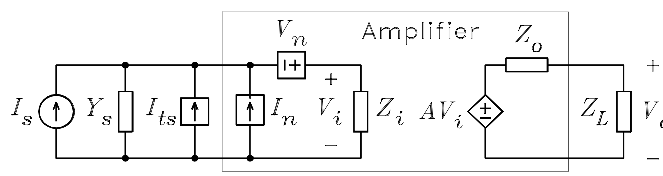

Figure 6.2: Amplifier with Norton source.

Because minimum noise occurs for $G_s = 0$, it can be concluded that a resistor should never be connected in parallel with a source at an amplifier input if noise performance is a design criterion. If a parallel resistor is required, e.g. as part of a bias network, it should be much larger than $1/G_s$. Although the output admittance of a source is usually fixed, the SNR can be improved by adding a susceptance in parallel with the source which makes the total parallel susceptance equal to the imaginary part of Y_s in Eq. (6.5). When this is the case, $\overline{i_{ni}^2}$ is given by

$$\overline{i_{ni}^2} = 4kTG_s\Delta f + \overline{v_n^2}G_s^2 + 2\gamma_r G_s\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}} + \overline{i_n^2}\left(1-\gamma_i^2\right) \tag{6.6}$$

6.2 Noise Factor and Noise Figure

The noise factor F of an amplifier is defined as the ratio of its actual SNR and the SNR if the amplifier is noiseless, where the temperature is taken to be the standard temperature T_0. When it is expressed in dB, it is called noise figure and is given by $NF = 10\log(F)$. In this section, the noise factor is derived for an amplifier driven by a Thévenin source and by a Norton source. Often, it is convenient to express F in terms of the amplifier noise resistance R_n and noise conductance G_n defined in Eqs. (2.18) and (2.19) and the correlation impedance Z_γ and correlation admittance Y_γ defined in Eqs. (3.12) and (3.13). For reference, the expressions for these are repeated below.

$$R_n = \frac{\overline{v_n^2}}{4kT_0\Delta f} \tag{6.7}$$

$$G_n = \frac{\overline{i_n^2}}{4kT_0\Delta f} \tag{6.8}$$

$$Z_\gamma = R_\gamma + jX_\gamma = \gamma\sqrt{\frac{R_n}{G_n}} = (\gamma_r + j\gamma_i)\sqrt{\frac{R_n}{G_n}} \tag{6.9}$$

$$Y_\gamma = G_\gamma + jB_\gamma = \gamma\sqrt{\frac{G_n}{R_n}} = (\gamma_r - j\gamma_i)\sqrt{\frac{G_n}{R_n}} \tag{6.10}$$

Note that R_n and G_n, respectively, represent normalized values of $\overline{v_n^2}$ and $\overline{i_n^2}$, where the normalization factor is $4kT_0\Delta f$.

6.2.1 Thévenin Source

Consider the amplifier model in Fig. 6.1. If the amplifier is noiseless, the signal-to-noise ratio given by $SNR = \overline{v_s^2}/\overline{v_{ts}^2}$, where $\overline{v_s^2}$ is the mean-square source voltage and $\overline{v_{ts}^2}$ is the mean-square thermal noise voltage generated by the source impedance. The noise factor F is obtained by dividing this equation by Eq. (6.1) to obtain

$$F = \frac{\left(\overline{v_s^2}/\overline{v_{ts}^2}\right)}{\left(\overline{v_s^2}/\overline{v_{ni}^2}\right)} = \frac{\overline{v_{ni}^2}}{\overline{v_{ts}^2}} = 1 + \frac{\overline{v_n^2} + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\left(\gamma Z_s^*\right) + \overline{i_n^2}\left|Z_s\right|^2}{4kT_0R_s\Delta f} \tag{6.11}$$

It follows from this expression that a noiseless amplifier has the noise factor $F = 1$. An alternate expression for F is obtained when the amplifier noise parameters are expressed in terms of R_n, G_n, and Z_γ. It is

$$F = 1 + \frac{R_n + 2G_n\,\mathrm{Re}\left(Z_\gamma Z_s^*\right) + G_n\left|Z_s\right|^2}{R_s} \tag{6.12}$$

The value of Z_s which minimizes F is called the optimum source impedance and is denoted by Z_{so}. It is obtained by setting $\partial F/\partial R_s = 0$ and $\partial F/\partial X_s = 0$ and solving for R_s and X_s. The impedance is given by

$$Z_{so} = R_{so} + jX_{so} = \left[\sqrt{1-\gamma_i^2} - j\gamma_i\right]\sqrt{\frac{\overline{v_n^2}}{\overline{i_n^2}}} = \sqrt{\frac{R_n}{G_n} - X_\gamma^2} - jX_\gamma \tag{6.13}$$

Note that the imaginary part of Z_{so} is equal to the imaginary part of Z_s in Eq. (6.2) which maximizes the signal-to-noise ratio. The corresponding minimum value of the noise factor is given by

$$F_0 = 1 + \frac{\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}}{2kT_0\Delta f}\left(\gamma_r + \sqrt{1-\gamma_i^2}\right) = 1 + 2G_n\left(R_\gamma + R_{so}\right) \tag{6.14}$$

We next wish to express F in terms of F_0 and Z_{so}. It follows from Eqs. (6.12) and (6.14) that the difference $F - F_0$ is given by

$$\begin{aligned} F - F_0 &= \frac{R_n + 2G_n\left(R_\gamma R_s + X_\gamma X_s\right) + G_n\left|Z_s\right|^2}{R_s} - 2G_n\left(R_\gamma + R_{so}\right) \\ &= \frac{R_n - 2G_n\left(R_{so}R_s - X_\gamma X_s\right) + G_n\left|Z_s\right|^2}{R_s} \\ &= \frac{R_n - 2G_n\left(R_{so}R_s + X_{so}X_s\right) + G_n\left|Z_s\right|^2}{R_s} \end{aligned} \tag{6.15}$$

The square in the numerator of this expression can be completed by adding and subtracting the term $G_n\left(R_{so}^2 + X_{so}^2\right) = G_n\left|Z_{so}\right|^2$. This leads to the equation

$$F - F_0 = \frac{R_n + G_n\left[\left(R_s - R_{so}\right)^2 + \left(X_s - X_{so}\right)^2 - \left|Z_{so}\right|^2\right]}{R_s} \tag{6.16}$$

From Eq. (6.13), we have $\left|Z_{so}\right|^2 = R_n/G_n$. It follows that F can be written

$$\begin{aligned} F &= F_0 + \frac{G_n}{R_s}\left[\left(R_s - R_{so}\right)^2 + \left(X_s - X_{so}\right)^2\right] \\ &= F_0 + \frac{G_n}{R_s}\left|Z_s - Z_{so}\right|^2 \end{aligned} \tag{6.17}$$

Example 1 *With a* 50 Ω *transmission line test fixture, the reflection coefficient seen looking out of the input terminals of an amplifier that minimizes its noise factor* F *at* 900 *MHz is determined to be* $\Gamma_{so} = 0.7\angle 23.7^o$. *Determine the lengths* ℓ_1 *and* ℓ_2 *of the transmission lines in the network of Fig. 6.3 which cause the source impedance seen by the amplifier to be the optimum source impedance* Z_{so}. *The source impedance is* $R_s = 50$ Ω. *The characteristic impedance of the two transmission lines in the network is* 75 Ω.

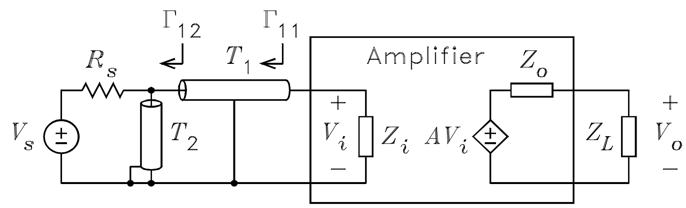

Figure 6.3: Transmission line noise matching network.

Solution. With $Z_c = 50\ \Omega$, i.e. the line impedance of the text fixture used to measure Γ_{so}, the optimum source impedance is given by

$$\begin{aligned} Z_{so} &= Z_c \frac{1+\Gamma_{so}}{1-\Gamma_{so}} = 50 \frac{1+0.7\cos 23.7^{\text{o}} + j0.7\sin 23.7^{\text{o}}}{1-0.7\cos 23.7^{\text{o}} - j0.7\sin 23.7^{\text{o}}} \\ &= 122.6 + j135.2 \end{aligned}$$

By Eq. (5.105), the reflection coefficient at the amplifier end of T_1 is

$$\Gamma_{11} = \frac{122.6 + j135.2 - 75}{122.6 + j135.2 + 75} = 0.4230 + j0.3539$$

This has the magnitude

$$|\Gamma_{11}| = 0.5988$$

The reactance of T_2 is calculated from Eq. (5.108) to obtain

$$X_2 = 75 \left[\frac{1 - 0.5988^2}{0.5988^2 \left(1 + 75/50\right)^2 - \left(1 - 75/50\right)^2} \right]^{1/2} = 42.58$$

where the positive solution has been used. By Eq. (5.109), the electrical length of T_2 is

$$\beta\ell_2 = \tan^{-1}\left(\frac{42.58}{75}\right) = 29.58^{\text{o}}$$

The reflection coefficient at the source end of T_1 is calculated from Eq. (5.106) to obtain

$$\Gamma_{12} = \frac{50\|j42.58 - 75}{50\|j42.58 + 75} = -0.4654 + j0.3767$$

By Eq. (5.112), the electrical length of T_1 is

$$\beta\ell_1 = \frac{1}{2}\arg\left(\frac{\Gamma_{12}}{\Gamma_{11}}\right) = \frac{1}{2}\arg\left(\frac{-0.4654 + j0.3767}{0.4830 + j0.3539}\right) = 52.39^{\text{o}}$$

As a check of the impedance seen looking into T_1, we have

$$\begin{aligned} Z &= Z_{c1} \frac{R_s \| jX_2 + jZ_{c1} \tan(\beta \ell_2)}{Z_{c1} + j(R_s \| jX_2) \tan(\beta \ell_2)} \\ &= 75 \frac{21.02 + j122.0}{42.96 + j27.28} = 122.6 + j135.2 \end{aligned}$$

This is the desired value.

6.2.2 Norton Source

Consider the amplifier model in Fig. 6.2. If the amplifier is noiseless, the signal-to-noise ratio given by $SNR = \overline{i_s^2}/\overline{i_{ts}^2}$, where $\overline{i_s^2}$ is the mean-square source current and $\overline{i_{ts}^2}$ is the mean-square thermal noise current generated by the source impedance. The noise factor F is obtained by dividing this equation by Eq. (6.4) to obtain

$$F = \frac{\left(\overline{i_s^2}/\overline{i_{ts}^2}\right)}{\left(\overline{i_s^2}/\overline{i_{ni}^2}\right)} = \frac{\overline{i_{ni}^2}}{\overline{i_{ts}^2}} = 1 + \frac{\overline{v_n^2}\,|Y_s|^2 + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}(\gamma Y_s) + \overline{i_n^2}}{4kT_0 G_s \Delta f} \tag{6.18}$$

An alternate expression for F is obtained when the amplifier noise parameters are expressed in terms of R_n, G_n, and Y_γ. It is

$$F = 1 + \frac{R_n |Y_s|^2 + 2R_n \,\mathrm{Re}(Y_\gamma Y_s) + G_n}{G_s} \tag{6.19}$$

The value of Y_s that minimizes F is called the optimum source admittance and is denoted by Y_{so}. It is obtained by setting $\partial F/\partial G_s = 0$ and $\partial F/\partial B_s = 0$ and solving for G_s and B_s. The admittance is given by

$$Y_{so} = G_{so} + jB_{so} = \left[\sqrt{1-\gamma_i^2} + j\gamma_i\right]\sqrt{\frac{\overline{i_n^2}}{\overline{v_n^2}}} = \sqrt{\frac{G_n}{R_n} - B_\gamma^2} + jB_\gamma \tag{6.20}$$

Note that this is the reciprocal of the optimum source impedance, i.e. $Y_0 = 1/Z_{so}$. Also, the imaginary part of Y_0 is equal to the imaginary part of Y_s in Eq. (6.5) which maximizes the signal-to-noise ratio. The corresponding minimum value of the noise factor is given by

$$F_0 = 1 + \frac{\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}}{2kT_0\Delta f}\left(\gamma_r + \sqrt{1-\gamma_i^2}\right) = 1 + 2R_n(G_\gamma + G_{so}) \tag{6.21}$$

We next wish to express F in terms of F_0 and Y_0. It follows from Eqs. (6.19) and (6.21) that the difference $F - F_0$ is given by

$$\begin{aligned} F - F_0 &= \frac{R_n \left|Y_s\right|^2 + 2R_n \left(G_\gamma G_s - B_\gamma B_s\right) + G_n}{G_s} - 2R_n \left(G_\gamma + G_{so}\right) \\ &= \frac{R_n \left|Y_s\right|^2 - 2R_n \left(G_{so} G_s + B_\gamma B_s\right) + G_n}{G_s} \\ &= \frac{R_n \left|Y_s\right|^2 - 2R_n \left(G_{so} G_s + B_0 B_s\right) + G_n}{G_s} \end{aligned} \tag{6.22}$$

The square in the numerator of this expression can be completed by adding and subtracting the term $R_n \left(G_{so}^2 + B_{so}^2\right) = R_n \left|Y_{so}\right|^2$. This leads to the equation

$$F - F_0 = \frac{R_n \left[\left(G_s - G_{so}\right)^2 + \left(B_s - B_{so}\right)^2 - \left|Y_0\right|^2\right] + G_n}{G_s} \tag{6.23}$$

From Eq. (6.20), we have $\left|Y_{so}\right|^2 = G_n/R_n$. It follows that F can be written

$$\begin{aligned} F &= F_0 + \frac{R_n}{G_s} \left[\left(G_s - G_{so}\right)^2 + \left(B_s - B_{so}\right)^2\right] \\ &= F_0 + \frac{R_n}{G_s} \left|Y_s - Y_{so}\right|^2 \end{aligned} \tag{6.24}$$

6.2.3 The Noise Factor Fallacy

The noise factor can be a misleading specification. If an attempt is made to minimize F by adding resistors either in series or in parallel with the source at the input of an amplifier, the SNR is always decreased. This is referred to as the noise factor fallacy or the noise figure fallacy. Potential confusion can be avoided if low-noise amplifiers are designed to maximize the SNR. This is accomplished by minimizing the equivalent noise input voltage. If a series resistor must be included at the amplifier input, its value should be much smaller than the source impedance. If a parallel resistor must be included at the amplifier input, its value should be much larger than the source impedance.

Example 2 *An amplifier has an input resistance* $R_i = 150\ \Omega$. *For* $\Delta f = 1$ *Hz, its noise parameters are* $\sqrt{\overline{v_n^2}} = 2\ nV$, $\sqrt{\overline{i_n^2}} = 10\ pA$, *and* $\rho = 0.1$. *It is driven from a source having an output resistance* $R_s = 50$

Ω. *(a) Calculate* $\overline{v_{ni}^2}$*,* F*, and* NF*. (c) A resistance is added in series with the source impedance to minimize* F*. Calculate the new* $\overline{v_{ni}^2}$*,* F*, and* NF*. (e) Calculate the change in the signal-to-noise ratio.*

Solution. (a) $\overline{v_{ni}^2} = 4kTR_s\Delta f + \overline{v_n^2} + 2\rho\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}R_s + \overline{i_n^2}R_s^2 = 5.25 \times 10^{-18}\ \mathrm{V}^2$. (b) $F = \overline{v_{ni}^2}/4kTR_s\Delta f = 6.56$, $NF = 10\log F = 8.17$ dB. (c) $R_{so} = \sqrt{\overline{v_n^2}/\overline{i_n^2}} = 200\ \Omega$, $\overline{v_{ni}^2} = 4kTR_{so}\Delta f + \overline{v_n^2} + 2\rho\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}R_{so} + \overline{i_n^2}R_{so}^2 = 1.2 \times 10^{-17}\ \mathrm{V}^2$, $F = \overline{v_{ni}^2}/4kTR_{so}\Delta f = 3.75$, $NF = 10\log F = 5.74$ dB. (e) The dB change in the SNR is $10\log\left(5.25 \times 10^{-18}/1.2 \times 10^{-17}\right) = -3.59$ dB.

The above example illustrates how the noise figure appears to be increased but the signal-to-noise ratio is decreased by adding resistance in series with an amplifier input. The fallacy comes from treating the added resistance as part of the source rather than part of the amplifier. In reality, the amplifier noise is increased by the added resistor, but the source noise remains constant. The correct way to calculate the noise factor with the added resistor is $F = \overline{v_{ni}^2}/4kTR_s\Delta f$ which gives $F = 15$ and $NF = 11.8$ dB. Thus the noise figure is degraded by the same amount as the signal-to-noise ratio.

6.3 Noise Temperature

The internal noise generated by an amplifier can be expressed as an equivalent input-termination noise temperature. When the source is represented by a Thévenin equivalent circuit, the noise temperature T_n is the temperature of the source resistance that generates a thermal noise voltage equal to the internal noise generated in the amplifier when referred to its input. For the Thévenin source, the noise temperature is defined by

$$4kT_nR_s\Delta f = \overline{v_n^2} + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\left(\gamma Z_s^*\right) + \overline{i_n^2}\left|Z_s\right|^2 \tag{6.25}$$

where $R_s = \mathrm{Re}\left(Z_s\right)$. It follows that the noise temperature is given by

$$T_n = \frac{\overline{v_n^2} + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\left(\gamma Z_s^*\right) + \overline{i_n^2}\left|Z_s\right|^2}{4kR_s\Delta f} = \frac{\overline{v_{ni}^2}}{4kR_s\Delta f} - T \tag{6.26}$$

where $\overline{v_{ni}^2}$ is the mean-square equivalent input noise voltage in the band Δf and T is the temperature of R_s.

When the source is represented by a Norton equivalent circuit, the noise temperature is the temperature of the source conductance that generates a thermal noise current equal to the internal noise generated in the amplifier when referred to its input. It is given by

$$T_n = \frac{\overline{v_n^2}\,|Y_s|^2 + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\,(\gamma Y_s) + \overline{i_n^2}}{4kG_s\Delta f} = \frac{\overline{i_{ni}^2}}{4kG_s\Delta f} - T \tag{6.27}$$

where $G_s = \mathrm{Re}\,(Y_s)$, $\overline{i_{ni}^2}$ is the mean-square short-circuit noise current generated by the source in the band Δf, and T is the temperature of G_s.

If the temperature of the source is T_0, the noise temperature is related to the noise factor by

$$T_n = T_0\,(F - 1) \tag{6.28}$$

This holds for either the Thévenin or the Norton source.

6.4 Noise Factor of a Multistage Amplifier

The circuit model for a multi-stage amplifier is shown in Fig. 6.4. It is shown in Section 4.2 that the equivalent noise input voltage is given by

$$\begin{aligned} V_{ni} \;=\; & V_{ts} + V_{n1} + \frac{V_{n2}}{G_{m1}Z_{o1}} + \frac{V_{n3}}{G_{m1}Z_{o1}G_{m2}Z_{o2}} \\ & + \cdots + \frac{V_{nN}}{G_{m1}Z_{o1}G_{m2}Z_{o2}\cdots G_{m(N-1)}Z_{o(N-1)}} \\ & + I_{n1}Z_s + \frac{I_{n2}}{G_{m1}} + \frac{I_{n3}}{G_{m1}Z_{o1}G_{m2}} \\ & + \cdots + \frac{I_{nN}}{G_{m1}Z_{o1}G_{m2}Z_{o2}\cdots G_{m(N-1)}} \end{aligned} \tag{6.29}$$

The equivalent transconductance G_{mj} is the ratio of the short-circuit output current from the jth stage to its open-circuit input current. It is given by

$$G_{mj} = \frac{I_{oj}}{V_{ij(oc)}} = \frac{g_{mj}Z_{ij}}{Z_{o(j-1)} + Z_{ij}} \tag{6.30}$$

where $g_{mj} = I_{oj}/V_{ij}$ andV_{ij} is the input voltage across Z_{ij}. The noise factor of the amplifier is given by

$$\begin{aligned} F &= \frac{\overline{v_{ni}^2}}{4kTR_s\Delta f} \\ &= \frac{1}{4kTR_s\Delta f}\left[\overline{v_{ni1}^2} + \frac{\overline{v_{ni2}^2}}{\left|G_{m1}Z_{o1}\right|^2} + \cdots \right. \\ &\left. + \frac{\overline{v_{niN}^2}}{\left|G_{m1}Z_{o1}G_{m2}Z_{o2}\cdots G_{m(N-1)}Z_{o(N-1)}\right|^2}\right] \end{aligned} \tag{6.31}$$

where $R_s = \mathrm{Re}\,(Z_s)$.

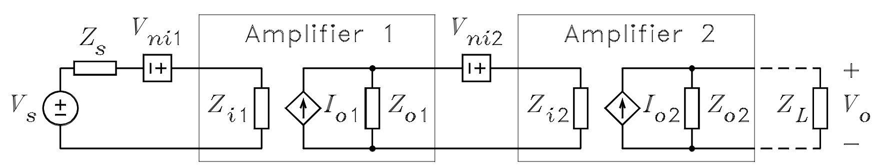

Figure 6.4: Multi-stage amplifier.

We wish to express F as a function of the noise factor of each stage. The noise factors are given by

$$\begin{aligned} F_1 &= \frac{\overline{v_{ni1}^2}}{4kTR_s\Delta f} \\ F_2 &= \frac{\overline{v_{ni2}^2} + 4kTR_{o1}\Delta f}{4kTR_{o1}\Delta f} \\ &\vdots \\ F_N &= \frac{\overline{v_{niN}^2} + 4kTR_{o(N-1)}\Delta f}{4kTR_{o(N-1)}\Delta f} \end{aligned} \tag{6.32}$$

where $R_{oj} = \mathrm{Re}\,(Z_{oj})$. It follows from Eq. (6.31) and (6.32) that F can be written

$$\begin{aligned} F &= F_1 + \frac{(F_2 - 1)\,R_{o1}}{\left|G_{m1}Z_{o1}\right|^2 R_s} + \cdots \\ &+ \frac{(F_N - 1)\,R_{o(N-1)}}{\left|G_{m1}Z_{o1}G_{m2}Z_{o2}\cdots G_{m(N-1)}Z_{o(N-1)}\right|^2 R_s} \end{aligned} \tag{6.33}$$

We next express F as a function of the available power gains of the stages. For the first stage, the current through Z_{i1} is $I_{i1} = V_s/(Z_s + Z_{i1})$. The power delivered by the source to Z_{i1} is given by

$$P_{i1} = \overline{i_{i1}^2} R_{i1} = \frac{\overline{v_s^2} R_{i1}}{|Z_s + Z_{i1}|^2} = \frac{\overline{v_s^2} R_{i1}}{(R_s + R_{i1})^2 + (X_s + X_{i1})^2} \tag{6.34}$$

The maximum value of P_{i1} is called the available input power P_{ai1}. It is solved for by setting $\partial P_{i1}/\partial R_{i1} = 0$ and $\partial P_{i1}/\partial X_{i1} = 0$ and solving for R_{i1} and X_{i1}. The solutions are $R_{i1} = R_s$ and $X_{i1} = -X_s$, i.e. $Z_{i1} = Z_s^*$. It follows that P_{ai1} is given by

$$P_{ai1} = \frac{\overline{v_s^2}}{4R_s} \tag{6.35}$$

For the second stage, the current through Z_{i2} is $I_{i2} = I_{o1}Z_{o1}/(Z_{o1} + Z_{i2})$, where $I_{o1} = G_{m1}V_s$. The power delivered to Z_{i2} is the power output from the first stage. It is given by

$$\begin{aligned} P_{o1} &= \overline{i_{i2}^2} R_{i2} = \frac{\overline{i_{o1}^2}\,|Z_{o1}|^2}{|Z_{o1} + Z_{i2}|^2} R_{i2} \\ &= \frac{|G_{m1}Z_{o1}|^2\,\overline{v_s^2}}{(R_{o1} + R_{i2})^2 + (X_{o1} + X_{i2})^2} R_{i2} \end{aligned} \tag{6.36}$$

The maximum value of P_{o1} is called the available output power P_{ao1} from the first stage. It is solved for by setting $\partial P_{o1}/\partial R_{i2} = 0$ and $\partial P_{o1}/\partial X_{i2} = 0$ and solving for R_{i2} and X_{i2}. The solutions are $R_{i2} = R_{o1}$ and $X_{i2} = -X_{o1}$, i.e. $Z_{i2} = Z_{o1}^*$. It follows that P_{ao1} is given by

$$P_{ao1} = \frac{|G_{m1}Z_{o1}|^2\,\overline{v_s^2}}{4R_{o1}} \tag{6.37}$$

The available power gain G_{a1} of the first stage is given by

$$G_{a1} = \frac{P_{ao1}}{P_{ai1}} = \frac{|G_{m1}Z_{o1}|^2\,R_s}{R_{o1}} \tag{6.38}$$

Similarly, it can be shown that the available power gain G_{aj} of the jth stage is given by

$$G_{aj} = \frac{P_{aoj}}{P_{aij}} = \frac{|G_{mj}Z_{oj}|^2\,R_{o(j-1)}}{R_{oj}} \tag{6.39}$$

With these definitions, it follows that Eq. (6.33) for F can be written

$$F = F_1 + \frac{F_2 - 1}{G_{a1}} + \cdots + \frac{F_N - 1}{G_{a1} G_{a2} \cdots G_{a(N-1)}} \tag{6.40}$$

This is the desired result.

If G_{a1} can be made large enough, the above equation implies that $F \simeq F_1$. However, increasing G_{a1} may not make $(F_2 - 1)/G_{a1}$ approach zero. For example, consider the case where $Z_{o1} = R_{o1} + j0$. In this case, G_{a1} is given by

$$G_{a1} = |G_{m1}|^2 R_s R_{o1} \tag{6.41}$$

If R_{o1} is increased, G_{a1} can be made arbitrarily large. The contribution to F by the second-stage noise is given by

$$\frac{F_2 - 1}{G_{a1}} = \frac{1}{4kTR_s \Delta f} \left[\frac{\overline{v_{n2}^2}}{G_{m1}^2 R_{o1}^2} + \frac{\overline{i_{n2}^2}}{G_{m1}^2} \right] \tag{6.42}$$

This cannot be made arbitrarily small by making R_{o1} arbitrarily large unless $\overline{i_{n2}^2}$ is negligible.

Let T_n be the noise temperature of the overall amplifier and T_{nj} the noise temperature of the jth stage. Eq. (6.28) can be used to express the noise factors in Eq. (6.40) in terms of the noise temperatures. It follows that the noise temperature of the multistage amplifier is given by

$$T_n = T_{n1} + \frac{T_{n2}}{G_{a1}} + \cdots + \frac{T_{nN}}{G_{a1} G_{a2} \cdots G_{a(N-1)}} \tag{6.43}$$

6.5 Effect of a Matching Network on Noise

6.5.1 Thévenin Source

Figure 6.5 shows a matching network between a signal source and the input to an amplifier. We assume that the matching network is lossless. Denote its input and output impedances, respectively, by $Z_{im} = R_{im} + jX_{im}$ and $Z_{om} = R_{om} + jX_{om}$. The power delivered to the matching network by the source is

$$P_{im} = \left| \frac{V_s}{Z_s + Z_{im}} \right|^2 \operatorname{Re}(Z_{im}) = \frac{\overline{v_s^2}}{|Z_s + Z_{im}|^2} R_{im} \tag{6.44}$$

Let V_{is} be the voltage across the amplifier input due to V_s and let $Z_i = R_i + jX_i$. The power delivered to the amplifier by the matching network is

$$P_{om} = \left|\frac{V_{is}}{Z_i}\right|^2 \mathrm{Re}\left(Z_i\right) = \frac{\overline{v_{is}^2}}{|Z_i|^2} R_i \tag{6.45}$$

Because the matching network is lossless, we have $P_{im} = P_{om}$. This leads to the relation

$$\overline{v_{is}^2} = \frac{R_{im}}{R_i}\left|\frac{Z_i}{Z_s + Z_{im}}\right|^2 \overline{v_s^2} \tag{6.46}$$

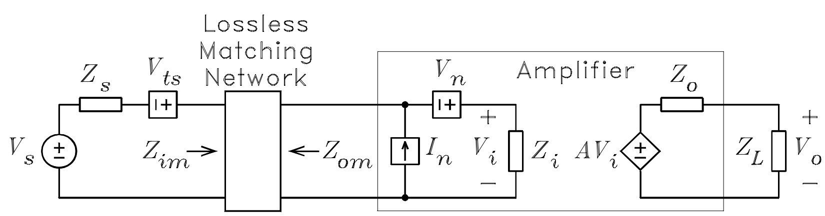

Figure 6.5: Amplifier with a Thévenin source and an input matching network.

Let V_{i1} be the voltage across the amplifier input due to V_s and V_{ts}. Its mean-square value is obtained by adding the mean square value of V_{ts} to that of V_s in Eq. (6.46) to obtain

$$\overline{v_{i1}^2} = \frac{R_{im}}{R_i}\left|\frac{Z_i}{Z_s + Z_{im}}\right|^2 \left(\overline{v_s^2} + 4kTR_s\Delta f\right) \tag{6.47}$$

This equation can be rewritten

$$\begin{aligned}\overline{v_{i1}^2} &= \left|\frac{Z_i}{Z_i + Z_{om}}\right|^2 \left[\frac{R_{im}}{R_i}\left|\frac{Z_i + Z_{om}}{Z_s + Z_{im}}\right|^2 \left(\overline{v_s^2} + 4kTR_s\Delta f\right)\right] \\ &= \left|\frac{Z_i}{Z_i + Z_{om}}\right|^2 \left(\overline{v_{is(oc)}^2} + \overline{v_{its(oc)}^2}\right) \end{aligned} \tag{6.48}$$

where $\overline{v_{is(oc)}^2}$ is the mean-square open-circuit signal input voltage due to V_s and $\overline{v_{its(oc)}^2}$ is the mean-square open-circuit thermal noise input voltage due to R_s. The latter is given by

$$\overline{v_{its(oc)}^2} = \frac{R_{im}}{R_i}\left|\frac{Z_i + Z_{om}}{Z_s + Z_{im}}\right|^2 4kTR_s\Delta f \tag{6.49}$$

This equation must be of the form $\overline{v_{its(oc)}^2} = 4kTR_{om}\Delta f$, where $R_{om} = \mathrm{Re}\,(Z_{om})$. It follows that R_{om} is given by

$$R_{om} = \frac{\overline{v_{its(oc)}^2}}{4kT\Delta f} = \frac{R_{im}}{R_i}\left|\frac{Z_i + Z_{om}}{Z_s + Z_{im}}\right|^2 R_s \tag{6.50}$$

When this equation is solved for R_{im}/R_i and the result used in Eq. (6.47), it follows that

$$\overline{v_{i1}^2} = \left|\frac{Z_i}{Z_i + Z_{om}}\right|^2 \frac{R_{om}}{R_s}\left(\overline{v_s^2} + 4kTR_s\Delta f\right) \tag{6.51}$$

It might seem a contradiction that R_{om} in Eq. (6.50) is a function of Z_i. However, the dependence cancels because Z_{im} is also a function of Z_i.

Let V_{i2} be the voltage across the amplifier input due to the amplifier noise sources V_n and I_n. It is given by

$$V_{i2} = V_n\frac{Z_i}{Z_i + Z_{om}} + I_n Z_i \| Z_{om} \tag{6.52}$$

The mean-square value is

$$\overline{v_{i2}^2} = \left|\frac{Z_i}{Z_i + Z_{om}}\right|^2\left[\overline{v_n^2} + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\,(\gamma Z_{om}^*) + \overline{i_n^2}\,|Z_{om}|^2\right] \tag{6.53}$$

The total mean-square input voltage to the amplifier is given by

$$\begin{aligned}\overline{v_i^2} &= \overline{v_{i1}^2} + \overline{v_{i2}^2} \\ &= \left|\frac{Z_i}{Z_i + Z_{om}}\right|^2 \frac{R_{om}}{R_s}\Bigg[\overline{v_s^2} + 4kTR_s\Delta f \\ &\quad + \frac{R_s}{R_{om}}\left(\overline{v_n^2} + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\,(\gamma Z_{om}^*) + \overline{i_n^2}\,|Z_{om}|^2\right)\Bigg]\end{aligned} \tag{6.54}$$

It follows from this expression that the mean-square equivalent noise input voltage in series with v_s is

$$\overline{v_{ni}^2} = 4kTR_s\Delta f + \frac{R_s}{R_{om}}\left[\overline{v_n^2} + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\,(\gamma Z_{om}^*) + \overline{i_n^2}\,|Z_{om}|^2\right] \tag{6.55}$$

The noise factor is given by

$$F = \frac{\overline{v_{ni}^2}}{4kTR_s\Delta f} = 1 + \frac{\overline{v_n^2} + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\,(\gamma Z_{om}^*) + \overline{i_n^2}\,|Z_{om}|^2}{4kTR_{om}\Delta f} \tag{6.56}$$

This is the same as the noise factor calculated at the output of the matching network. The basic reason that the noise factors at the source and at the output of the matching network are equal is because a lossless matching network cannot add noise. Thus it follows that the signal-to-noise ratio is also the same at the input to the matching network as it is at the input to the amplifier. These conclusions do not hold for a lossy matching network. Eq. (6.56) can be used to predict the noise factor for any arbitrary matching network. For example, Z_{om} might be chosen to be the optimum source impedance to minimize F. Alternately, it could be chosen for a conjugate impedance match to maximize the power gain. Such calculations are illustrated in Example 3.

For a conjugate match, the condition $Z_{om} = Z_i^*$ must hold. In this case, $R_{om} = R_i$ and the expression for $\overline{v_{ni}^2}$ in Eq. (6.55) reduces to

$$\overline{v_{ni}^2} = 4kTR_s\Delta f + \frac{R_s}{R_i}\left[\overline{v_n^2} + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\,(\gamma Z_i) + \overline{i_n^2}\,|Z_i^*|^2\right] \tag{6.57}$$

The noise factor is

$$F = \frac{\overline{v_{ni}^2}}{4kTR_s\Delta f} = 1 + \frac{\overline{v_n^2} + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\,(\gamma Z_i) + \overline{i_n^2}\,|Z_i^*|^2}{4kTR_i\Delta f} \tag{6.58}$$

Because of the dependence of $\overline{v_{ni}^2}$ and F on Z_i, it is difficult to predict from these equations how changes in Z_i affect the noise. This is because V_n, I_n, and γ in the noise model are, in general, related to Z_i. For example, V_n, I_n, γ, and Z_i may all be functions of the bias current in the amplifier input stage. A change in the bias current to vary Z_i can cause a change in V_n, I_n, and γ. Thus the effects cannot be examined in detail unless the relations between the variables are known.

Example 3 *An amplifier is driven from a source with a resistive output impedance $R_s = 50\ \Omega$. At the operating frequency of $f = 10$ MHz, the amplifier has a resistive input impedance $R_i = 25\ \Omega$ and the noise parameters $\sqrt{\overline{v_n^2}/\Delta f} = 0.447\ nV/\sqrt{Hz}$, $\sqrt{\overline{i_n^2}} = 31\ pA/\sqrt{Hz}$, and $\gamma = 0.12 - j0.44$. (a) Calculate the noise figure with a conjugate impedance matching network between the source and the amplifier. (b) Calculate the noise figure if the matching network is designed so that the amplifier sees its optimum source impedance. (c) Calculate the decrease in power gain with the second matching network.*

Solution. (a)

$$\begin{aligned} NF &= 10\log\left[1+\frac{\overline{v_n^2}+2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\,(\gamma Z_i)+\overline{i_n^2}\,|Z_i^*|^2}{4kTR_i\Delta f}\right] \\ &= 5.062\ \mathrm{dB} \end{aligned}$$

(b) The optimum source impedance is given by Eq. (6.13). It is

$$\begin{aligned} Z_{so} &= \left[\sqrt{1-\gamma_i^2}-j\gamma_i\right]\sqrt{\frac{\overline{v_n^2}}{\overline{i_n^2}}} \\ &= 13+j6.35 \end{aligned}$$

Thus the noise figure is

$$\begin{aligned} NF &= 10\log\left[1+\frac{\overline{v_n^2}+2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\,(\gamma Z_{so}^*)+\overline{i_n^2}\,|Z_{so}|^2}{4kT\,\mathrm{Re}\,(Z_{so})\,\Delta f}\right] \\ &= 4.414\ \mathrm{dB} \end{aligned}$$

This is 0.648 dB lower than for part (a).
(c) By Eq. (6.48), $\overline{v_{is(oc)}^2}/\overline{v_s^2} = 25/50 = 0.5$ for part (a) and $\overline{v_{is(oc)}^2}/\overline{v_s^2} = 13/50 = 0.26$ for part (b). In general, the power delivered to the amplifier input is

$$P_i = \left|\frac{Z_i}{Z_{om}+Z_i}\right|^2 \frac{\overline{v_{is(oc)}^2}}{R_i}$$

For part (a), we have

$$P_{i(a)} = \left|\frac{25}{25+25}\right|^2 \frac{0.5\overline{v_s^2}}{25} = 0.005\overline{v_s^2}$$

For part (b)

$$P_{i(b)} = \left|\frac{25}{13-j6.35+25}\right|^2 \frac{0.26\overline{v_s^2}}{25} = 0.00439\overline{v_s^2}$$

It follows that the power gain of the amplifier drops by 12.2%, or by 0.567 dB, with the optimum source impedance.

6.5.2 Norton Source

Figure 6.6 shows an amplifier with a Norton source at its input. The solutions for the mean-squared noise current in parallel with I_s and the noise factor follow the derivations of Eqs. (6.55) and (6.56) for the amplifier with a Thévenin source. The mean-square input noise current is given by

$$\overline{i_{ni}^2} = 4kTG_s\Delta f + \frac{G_s}{G_{om}}\left[\overline{v_n^2}\,|Y_{om}|^2 + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\,(\gamma Y_{om}) + \overline{i_n^2}\right] \quad (6.59)$$

where $G_s = \mathrm{Re}\,(Y_s)$ and $Y_{om} = G_{om} + jB_{om}$. The noise factor is given by

$$F = \frac{\overline{i_{ni}^2}}{4kTG_s\Delta f} = 1 + \frac{\overline{v_n^2}\,|Y_{om}|^2 + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\,(\gamma Y_{om}) + \overline{i_n^2}}{4kTG_{om}\Delta f} \quad (6.60)$$

For a conjugate match, $Y_{om} = Y_i^*$. In this case $\overline{i_{ni}^2}$ and F are given by

$$\overline{i_{ni}^2} = 4kTG_s\Delta f + \frac{G_s}{G_i}\left[\overline{v_n^2}\,|Y_i^*|^2 + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\,(\gamma Y_i^*) + \overline{i_n^2}\right] \quad (6.61)$$

$$F = \frac{\overline{i_{ni}^2}}{4kTG_s\Delta f} = 1 + \frac{\overline{v_n^2}\,|Y_i^*|^2 + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\,(\gamma Y_i^*) + \overline{i_n^2}}{4kTG_i\Delta f} \quad (6.62)$$

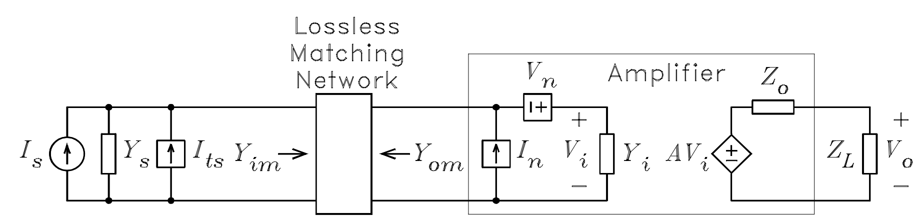

Figure 6.6: Amplifier with a Norton source and an input matching network.

6.6 Noise Circles

In rf design, the contours of constant F on the Smith chart for the reflection coefficient seen looking out of an amplifier input are important.

These contours are circles. Thus they are called noise circles. These circles are developed below for both the impedance and admittance Smith charts.

6.6.1 Thévenin Source

Fig. 6.7 shows an amplifier with a Thévenin source and a lossless input matching network. Let the impedance seen looking into the output of the matching network be Z_{om}. Imagine a zero length transmission line of characteristic impedance Z_c connected between the matching network and the amplifier input. Let Γ_{om} be the reflection coefficient seen looking out of the amplifier into this line. It is given by

$$\Gamma_{om} = \frac{Z_{om} - Z_c}{Z_{om} + Z_c} = \frac{\widehat{Z}_{om} - 1}{\widehat{Z}_{om} + 1} \tag{6.63}$$

where $\widehat{Z}_{om} = Z_{om}/Z_c$.

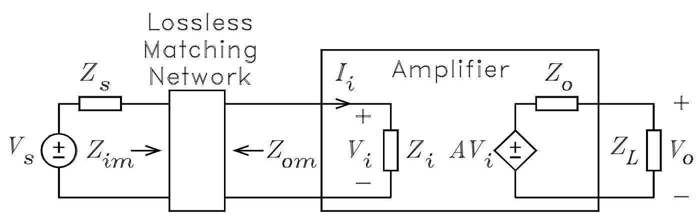

Figure 6.7: Amplifier with a Thévenin source and a lossless input matching network.

Equation (6.63) can be solved for $\widehat{Z}_{om}$ to obtain

$$\widehat{Z}_{om} = \frac{1 + \Gamma_{om}}{1 - \Gamma_{om}} \tag{6.64}$$

Let Γ_0 be the reflection coefficient corresponding to $Z_{om} = Z_{so}$, where Z_{so} is the optimum source impedance which minimizes F. We can write

$$\widehat{Z}_{om} - \widehat{Z}_{so} = \frac{1 + \Gamma_{om}}{1 - \Gamma_{om}} - \frac{1 + \Gamma_0}{1 - \Gamma_0} = 2\frac{\Gamma_{om} - \Gamma_0}{(1 - \Gamma_{om})(1 - \Gamma_0)} \tag{6.65}$$

Let $\Delta\widehat{Z} = \widehat{Z}_{om} - \widehat{Z}_{so}$, $\Gamma_{om} = p + jq$, and $\Gamma_0 = p_0 + jq_0$. It follows that

$$\left|\Delta\widehat{Z}\right|^2 = 4\frac{(p - p_0)^2 + (q - q_0)^2}{\left[(1 - p)^2 + q^2\right] |1 - \Gamma_0|^2} \tag{6.66}$$

The above equation can be put into the form

$$(p-a)^2 + (q-b)^2 = c^2 \tag{6.67}$$

where

$$a = \frac{4p_0 - |1-\Gamma_0|^2 \left|\Delta\widehat{Z}\right|^2}{4 - |1-\Gamma_0|^2 \left|\Delta\widehat{Z}\right|^2} \tag{6.68}$$

$$b = \frac{4q_0}{4 - |1-\Gamma_0|^2 \left|\Delta\widehat{Z}\right|^2} \tag{6.69}$$

$$c^2 = a^2 + b^2 - \frac{4|\Gamma_0|^2 - |1-\Gamma_0|^2 \left|\Delta\widehat{Z}\right|^2}{4 - |1-\Gamma_0|^2 \left|\Delta\widehat{Z}\right|^2} \tag{6.70}$$

For $\left|\Delta\widehat{Z}\right|$ a constant, Eq. (6.67) represents a circle of radius c that is centered at the point $\Gamma = a + jb$ on the Smith chart.

By Eq. (6.17), the noise factor can be written

$$F = F_0 + \frac{G_n}{R_s}|\Delta Z|^2 = F_0 + \frac{G_n Z_c^2}{R_s}\left|\Delta\widehat{Z}\right|^2 \tag{6.71}$$

Because F is constant for $\left|\Delta\widehat{Z}\right|$ a constant, it follows that the contours of constant F on the Smith chart are the circles defined by Eq. (6.67). For $\Delta\widehat{Z} = 0$, the circles degenerate into a point located at the point $\Gamma = \Gamma_0$.

Figure 6.8(a) shows an example impedance Smith chart with the point $\Gamma_0 = 0.5\angle - 135^{\text{o}}$ and three surrounding noise circles labeled Γ_1, Γ_2, and Γ_3, corresponding to $\left|\Delta\widehat{Z}\right| = 0.15$, $\left|\Delta\widehat{Z}\right| = 0.3$, and $\left|\Delta\widehat{Z}\right| = 0.45$, respectively.

Example 4 *An amplifier is driven from a source with an output impedance* $Z_s = 60\ \Omega$. *The amplifier has an input spot noise current* $\sqrt{i_n^2/\Delta f} = 20\ pA/\sqrt{Hz}$. *The optimum noise figure is* $NF_0 = 1.5\ dB$ *when the source impedance is* $30 - j20\ \Omega$. *(a) If a zero-length transmission line having a characteristic impedance* $Z_c = 50\ \Omega$ *is connected between the source and the amplifier, calculate the reflection coefficient* Γ *seen looking out of the amplifier input. (b) Calculate the optimum reflection coefficient* Γ_0. *(c) Use Eq. (6.17) to calculate the noise figure when the amplifier is driven from the source with an impedance* $Z_s = 60$

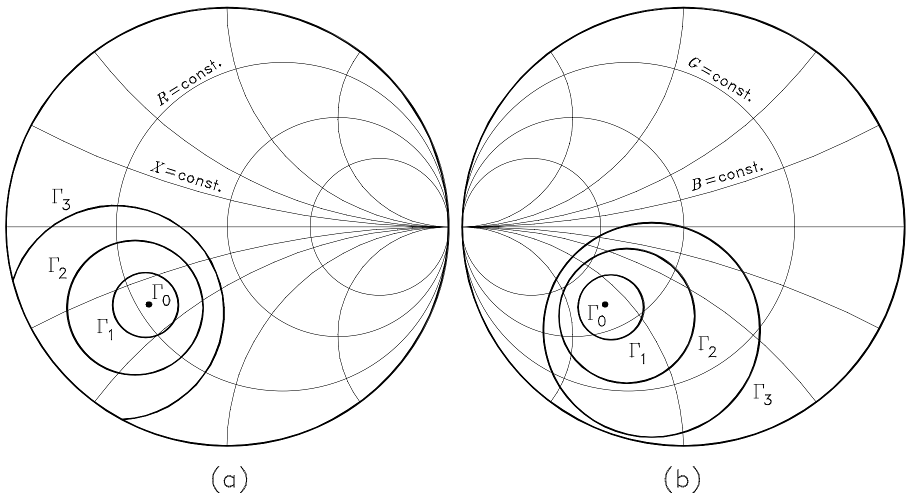

Figure 6.8: Smith charts showing the optimum reflection coefficient Γ_0 and three noise circles. (a) Impedance chart. (b) Admittance chart.

Ω. *(d) Calculate the center coordinates and the radius of the noise circle that* Γ_{om} *lies on. (e) Calculate the equivalent spot noise input voltage* $\sqrt{\overline{v_{ni}^2}/\Delta f}$.

Solution. (a) Looking into the zero-length transmission line, the reflection coefficient is

$$\Gamma = \frac{60 - 50}{60 + 50} = 0.091$$

(b) For $Z_s = Z_{so}$, the optimum reflection coefficient is

$$\Gamma_0 = \frac{30 - j20 - 50}{30 - j20 + 50} = -0.176 - j0.294$$

(c) $G_n = \left(20 \times 10^{-9}\right)^2 / 1.6 \times 10^{-20} = 0.025$, $F_0 = 10^{1.5/10} = 1.413$, $|Z_s - Z_0|^2 = 1300$

$$\begin{aligned} F &= 1.413 + \frac{0.025}{60} 1300 = 1.954 \\ NF &= 10 \log 1.954 = 2.91 \text{ dB} \end{aligned}$$

(d) $x_0 = -0.176$, $y_0 = -0.294$, $|\Gamma_0|^2 = 0.118$, $|1-\Gamma_0|^2 = 1.471$

$$\begin{aligned}
\left|\frac{\Delta Z}{Z_c}\right|^2 &= \left|\frac{60-(30-j20)}{50}\right|^2 = 0.52 \\
a &= \frac{4\times(-0.176) - 1.471\times 0.52}{4-1.471\times 0.52} = -0.455 \\
b &= \frac{4\times(-0.294)}{4-1.471\times 0.52} = -0.364 \\
c &= \left[0.455^2 + 0.364^2 - \frac{4\times 0.118 - 1.471\times 0.52}{4-1.471\times 0.52}\right]^{1/2} = 0.656
\end{aligned}$$

(e) $\sqrt{\overline{v_{ni}^2}/\Delta f} = \sqrt{F4kT_0R_s} = 1.37\ \mathrm{nV}/\sqrt{\mathrm{Hz}}$.

6.6.2 Norton Source

Figure 6.9 shows an amplifier and input matching network with the source modeled by a Norton equivalent circuit. Let a zero length transmission line with characteristic admittance Y_c be connected between the network and the amplifier input. The reflection coefficient Γ_{om} seen looking out of the amplifier input can be expressed in terms of the admittance Y_{mo} seen looking into the output of the matching network. It is given by

$$\Gamma_{om} = \frac{Y_c - Y_{om}}{Y_c + Y_{om}} = \frac{1-\widehat{Y}_{om}}{1+\widehat{Y}_{om}} \tag{6.72}$$

where $\widehat{Y}_{om} = Y_{om}/Y_c$. This equation can be solved for $\widehat{Y}_{om}$ to obtain

$$\widehat{Y}_{om} = \frac{1-\Gamma_{om}}{1+\Gamma_{om}} \tag{6.73}$$

Let Γ_0 be the reflection coefficient corresponding to $Y_{om} = Y_{so}$, where Y_{so} is the optimum source admittance which minimizes F. We can write

$$\widehat{Y}_{om} - \widehat{Y}_{so} = \frac{1-\Gamma_{om}}{1+\Gamma_{om}} - \frac{1-\Gamma_0}{1+\Gamma_0} = 2\frac{\Gamma_0 - \Gamma_{om}}{(1+\Gamma_{om})(1+\Gamma_0)} \tag{6.74}$$

Let $\Delta\widehat{Y} = \widehat{Y}_{om} - \widehat{Y}_{so}$, $\Gamma_{om} = p + jq$, and $\Gamma_0 = p_0 + jq_0$. It follows that

$$\left|\Delta\widehat{Y}\right|^2 = 4\frac{(p_0-p)^2 + (q_0-q)^2}{\left[(1+p)^2 + q^2\right]|1+\Gamma_0|^2} \tag{6.75}$$

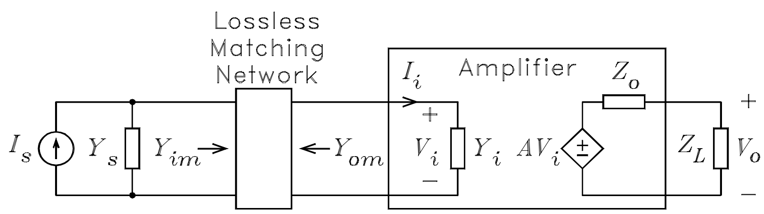

Figure 6.9: Amplifier with a Norton source and a lossless input matching network.

This equation can be put into the form of Eq. (6.67) where the constants a, b, and c are given by

$$a = \frac{4p_0 + |1+\Gamma_0|^2 \left|\Delta\widehat{Y}\right|^2}{4 - |1+\Gamma_0|^2 \left|\Delta\widehat{Y}\right|^2} \tag{6.76}$$

$$b = \frac{4q_0}{4 - |1+\Gamma_0|^2 \left|\Delta\widehat{Y}\right|^2} \tag{6.77}$$

$$c^2 = a^2 + b^2 - \frac{4\,|\Gamma_0|^2 - |1+\Gamma_0|^2 \left|\Delta\widehat{Y}\right|^2}{4 - |1+\Gamma_0|^2 \left|\Delta\widehat{Y}\right|^2} \tag{6.78}$$

For $\left|\Delta\widehat{Y}\right|$ a constant, Eq. (6.67) represents a circle of radius c that is centered at the point $\Gamma = a + jb$ on the Smith chart.

By Eq. (6.24), the noise factor can be written

$$F = F_0 + \frac{R_n}{G_s}\,|\Delta Y|^2 = F_0 + \frac{R_n Y_c^2}{G_s}\left|\Delta\widehat{Y}\right|^2 \tag{6.79}$$

Because F is constant for $\left|\Delta\widehat{Y}\right|$ a constant, it follows that the contours of constant F on the admittance Smith chart are the circles defined by Eq. (6.67). For $\Delta\widehat{Y} = 0$, the circles degenerate into a point located at the point $\Gamma = \Gamma_0$.

Figure 6.8(b) shows an example admittance Smith chart with the point $\Gamma_0 = 0.5\angle - 135^{\mathrm{o}}$ and three surrounding noise circles labeled Γ_1, Γ_2, and Γ_3, corresponding to $\left|\Delta\widehat{Y}\right| = 0.525$, $\left|\Delta\widehat{Y}\right| = 0.983$, and $\left|\Delta\widehat{Y}\right| =$

1.354, respectively. These values are chosen to give the same radii of the corresponding circles on the two charts. It follows from Eqs. (6.71) and (6.79), however, that the value of F on corresponding circles is not the same unless $\left|\Delta\widehat{Z}\right|$ and $\left|\Delta\widehat{Y}\right|$ satisfy the relation

$$\frac{G_n Z_c^2}{R_s}\left|\Delta\widehat{Z}\right|^2 = \frac{R_n Y_c^2}{G_s}\left|\Delta\widehat{Y}\right|^2 \tag{6.80}$$

A common use of the Smith chart is to rotate 180$^{\text{o}}$ around the center to convert a normalized impedance into a normalized admittance and vice versa. The above results show that rotating the noise circles by 180$^{\text{o}}$ does not necessarily convert the circles on an impedance chart into the circles on an admittance chart and vice versa.

6.7 Gain Circles

Compromises are often made in rf amplifier design between lowest noise and highest gain. In Sec. 6.6, the contours on the Smith chart of constant noise factor are derived. In this section, the contours of constant gain are derived. Like the noise factor contours, the constant gain contours are circles. Thus they are called gain circles. Given the Smith chart with both the noise circles and the gain circles plotted for a particular amplifier, the effect of the input matching network on both noise and gain can be easily visualized.

6.7.1 Thévenin Source

Consider the amplifier model of Fig. 6.7. By Eq. (6.51), the mean-square signal voltage across Z_i is given by

$$\overline{v_i^2} = \left|\frac{Z_i}{Z_i + Z_{om}}\right|^2 \frac{R_{om}}{R_s}\overline{v_s^2} \tag{6.81}$$

where $R_{om} = \text{Re}\left(Z_{om}\right)$ and $R_s = \text{Re}\left(Z_s\right)$. The power delivered to Z_i is given by

$$P_i = \overline{i_i^2}\,\text{Re}\left(Z_i\right) = \frac{\overline{v_i^2}}{\left|Z_i\right|^2}R_i = \frac{\overline{v_s^2}}{\left|Z_i + Z_{om}\right|^2}\frac{R_i R_{om}}{R_s} \tag{6.82}$$

The maximum value of P_i occurs when $Z_{om} = Z_i^*$ and is given by

$$P_{i(\max)} = \frac{\overline{v_s^2}}{4R_s} \tag{6.83}$$

The power gain G_p of the input matching network is given by

$$G_p = \frac{P_i}{P_{i(\max)}} = \frac{4R_i R_{om}}{|Z_i + Z_{om}|^2} \tag{6.84}$$

Let us write $Z_{om} = Z_i^* + \Delta Z$, where $\Delta Z = \Delta R + j\Delta X$ represents the amount by which Z_{om} deviates from Z_i^*. Thus we can write $R_{om} = R_i + \Delta R$ and $Z_i + Z_{om} = 2R_i + \Delta Z$. It follows that G_p can be written

$$G_p = \frac{4R_i (R_i + \Delta R)}{|2R_i + \Delta Z|^2} = \frac{4R_i (R_i + \Delta R)}{4R_i (R_i + \Delta R) + |\Delta Z|^2} \tag{6.85}$$

This expression is of the form

$$G_p = \frac{1}{1+h} \tag{6.86}$$

where h is given by

$$h = \frac{|\Delta Z|^2}{4R_i (R_i + \Delta R)} = \frac{\Delta R^2 + \Delta X^2}{4R_i (R_i + \Delta R)} \tag{6.87}$$

On a contour for which G_p is constant, h must be independent of ΔZ. Eq. (6.87) can be rewritten $\Delta R^2 - 4hR_i\Delta R + \Delta X^2 = 4R_i^2 h$. After completing the square, this equation can be put into the form

$$(\Delta R - 2hR_i)^2 + \Delta X^2 = |\Delta Z - 2hR_i|^2 = 4R_i^2 h (1+h) \tag{6.88}$$

For h a constant, it follows from this equation that G_p is constant on the contours for which $|\Delta Z - 2hR_i|^2$ is constant.

To solve for the gain contours, we can write

$$\Delta Z - 2hR_i = (Z_{om} - Z_i^*) - 2hR_i = Z_{om} - Z_h \tag{6.89}$$

where Z_h is the impedance given by

$$Z_h = (1+2h) R_i - jX_i \tag{6.90}$$

Note that $Z_h = Z_i^*$ and G_p is a maximum for the case $h = 0$. Define the normalized impedances $\widehat{Z}_{om} = Z_{om}/Z_c$, $\widehat{Z}_h = Z_h/Z_c$, $\Delta\widehat{Z} = \Delta Z/Z_c$, and $\widehat{R}_i = R_i/Z_c$. It follows from Eqs. (6.88) and (6.89) that

$$\left|\widehat{Z}_{om} - \widehat{Z}_h\right|^2 = \left|\Delta\widehat{Z} - 2h\widehat{R}_i\right|^2 = 4\widehat{R}_i^2 h (1+h) \tag{6.91}$$

Let a zero length transmission line having a characteristic impedance Z_c be connected between the matching network and the amplifier input. The reflection coefficient $\Gamma_{om} = p + jq$ seen looking out of the amplifier input is given by Eq. (6.63). Let $\Gamma_h = p_h + jq_h$ be the value of Γ_{om} if the matching network is removed and replaced with the impedance Z_h. Following Eq. (6.65) for the derivation of the noise circles, we can relate $\widehat{Z}_{om} - \widehat{Z}_h$ to Γ_{om} and Γ_h as follows:

$$\widehat{Z}_{om} - \widehat{Z}_h = \frac{1+\Gamma_{om}}{1-\Gamma_{om}} - \frac{1+\Gamma_h}{1-\Gamma_h} = 2\frac{\Gamma_{om} - \Gamma_h}{(1-\Gamma_{om})(1-\Gamma_h)} \tag{6.92}$$

The squared magnitude is given by

$$\left|\widehat{Z}_{om} - \widehat{Z}_h\right|^2 = 4\frac{(p-p_h)^2 + (q-q_h)^2}{\left[(1-p)^2 + q^2\right]|1-\Gamma_h|^2} \tag{6.93}$$

When Eq. (6.88) is used for $\left|\widehat{Z}_{om} - \widehat{Z}_h\right|^2$, we can write

$$\widehat{R}_i^2 h(1+h) = \frac{(p-p_h)^2 + (q-q_h)^2}{\left[(1-p)^2 + q^2\right]|1-\Gamma_h|^2} \tag{6.94}$$

By analogy to Eqs. (6.66) – (6.70) for the noise circles, it follows from Eq. (6.94) that the contours of constant G_p on the Smith chart are circles of radius c centered at the point $\Gamma = a + jb$, where a, b, and c are given by

$$a = \frac{p_h - |1-\Gamma_h|^2 \widehat{R}_i^2 h(1+h)}{1 - |1-\Gamma_h|^2 \widehat{R}_i^2 h(1+h)} \tag{6.95}$$

$$b = \frac{q_h}{1 - |1-\Gamma_h|^2 \widehat{R}_i^2 h(1+h)} \tag{6.96}$$

$$c^2 = a^2 + b^2 - \frac{|\Gamma_h|^2 - |1-\Gamma_h|^2 \widehat{R}_i^2 h(1+h)}{1 - |1-\Gamma_h|^2 \widehat{R}_i^2 h(1+h)} \tag{6.97}$$

Figure 6.10(a) shows an example impedance Smith chart with the point $\Gamma_i^* = 0.3\angle 45^{\mathrm{o}}$ labeled G_0 and three surrounding gain circles labeled G_1, G_2, and G_3, corresponding to gains lower than G_0 by 1 dB ($h = 0.26$), 2 dB ($h = 0.58$), and 3 dB ($h = 1$), respectively.

Example 5 *The amplifier of Example 2 in Chapter 5 is driven from a source with the output impedance $Z_s = 50\ \Omega$. The amplifier has the input*

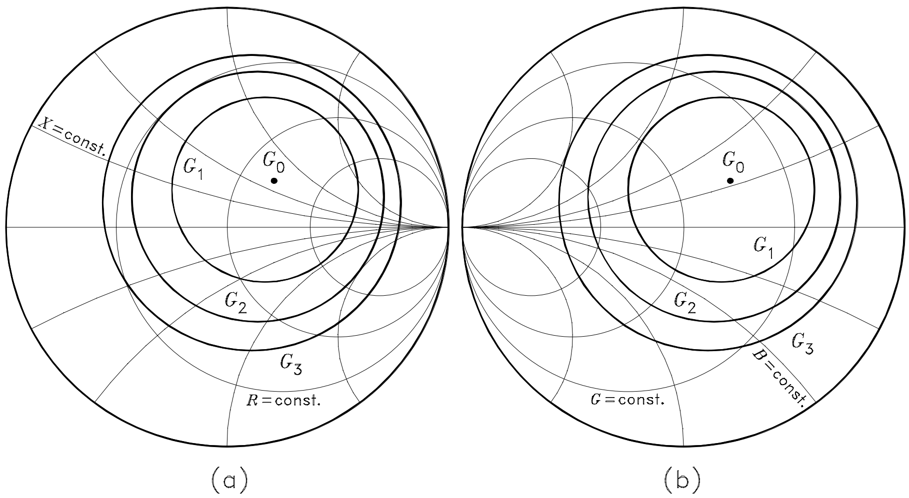

Figure 6.10: (a) Impedance Smith chart showing example gain circles corresponding to 0, −1, −2, and −3 dB. (b) Corresponding admittance Smith chart.

impedance $Z_{in} = 31.6 - j127$ *and a gain of* 39.8 *dB. (a) Calculate* G_p *for the input network. (b) Calculate a, b, and c for the gain circle on which* G_p *lies. (c) Calculate the new overall gain of the amplifier if a conjugate matching network is added between the source and the amplifier. Assume zero-length transmission lines with the characteristic impedance* $Z_c = 50$ Ω *for the Smith chart calculations.*

Solution. (a) Because there is no input matching network, $Z_{om} = Z_s = 50\ \Omega$. Thus $\Delta Z = Z_{om} - Z_{\text{in}}^* = 18.4 - j127$ and

$$h = \frac{18.4^2 + 127^2}{4 \times 31.6 \times (31.6 + 18.4)} = 2.61$$

$$G_p = \frac{1}{1 + 2.61} = 0.277 \text{ or } -5.57 \text{ dB}$$

(b) $\widehat{Z}_h = \left[(1 + 2 \times 2.61) \times 31.6 + j127\right]/50 = 3.93 + j2.54$,
$\Gamma_h = \left(\widehat{Z}_h - 1\right) / \left(\widehat{Z}_h + 1\right) = 0.679 + j0.165$, $|\Gamma_h| = 0.699$,

$|1-\Gamma_h| = 0.361$, $\widehat{R}_i = 31.6/50 = 0.632$

$$a = \frac{0.679 - 0.361^2 \times 0.632^2 \times 2.61 \times 3.61}{1 - 0.361^2 \times 0.632^2 \times 2.61 \times 3.61} = 0.373$$

$$b = \frac{0.165}{1 - 0.361^2 \times 0.632^2 \times 2.61 \times 3.61} = 0.324$$

$$\begin{aligned} c &= \left[a^2 + b^2 - \frac{0.699^2 - 0.361^2 \times 0.632^2 \times 2.61 \times 3.61}{1 - 0.361^2 \times 0.632^2 \times 2.61 \times 3.61}\right]^{1/2} \\ &= 0.493 \end{aligned}$$

(c) The addition of a conjugate matching network at the input would increase the gain to $39.8 + 5.6 = 45.4$ dB.

6.7.2 Norton Source

Consider the amplifier model of Fig. 6.9, where the source is represented by a Norton equivalent. With $\overline{v_s^2} = \overline{i_s^2}\,|Z_s|^2$, $Y_s = 1/Z_s$, $\overline{v_i^2} = \overline{i_i^2}\,|Z_i|^2$, $Y_i = 1/Z_i$, and $Y_{om} = 1/Z_{om}$, it follows from Eq. (6.51) that the mean-square signal current through Y_i is given by

$$\overline{i_i^2} = \left|\frac{Y_i}{Y_i + Y_{om}}\right|^2 \frac{G_{om}}{G_s}\overline{i_s^2} \tag{6.98}$$

where $G_{om} = \mathrm{Re}\,(Y_{om})$ and $G_s = \mathrm{Re}\,(Y_s)$. The power delivered to Y_i is given by

$$P_i = \overline{i_i^2}\,\mathrm{Re}\left(\frac{1}{Y_i}\right) = \frac{\overline{i_i^2}}{|Y_i|^2}G_i = \frac{\overline{i_s^2}}{|Y_i + Y_{om}|^2}\frac{G_i G_{om}}{G_s} \tag{6.99}$$

The maximum value of P_i occurs when $Y_{om} = Y_i^*$ and is given by

$$P_{i(\max)} = \frac{\overline{i_s^2}}{4G_s} \tag{6.100}$$

The power gain G_p of the input matching network is given by

$$G_p = \frac{P_i}{P_{i(\max)}} = \frac{4G_i G_{om}}{|Y_i + Y_{om}|^2} \tag{6.101}$$

Let us write $Y_{om} = Y_i^* + \Delta Y$, where $\Delta Y = \Delta G + j\Delta B$ represents the amount by which Y_{om} deviates from Y_i^*. Thus we can write $G_{om} = G_i + \Delta G$ and $Y_i + Y_{om} = 2G_i + \Delta Y$. It follows that G_p can be written

$$G_p = \frac{4G_i\,(G_i + \Delta G)}{|2G_i + \Delta Y|^2} = \frac{4G_i\,(G_i + \Delta G)}{4G_i\,(G_i + \Delta G) + |\Delta Y|^2} \tag{6.102}$$

This expression is of the form

$$G_p = \frac{1}{1+h} \tag{6.103}$$

where h is given by

$$h = \frac{|\Delta Y|^2}{4G_i\,(G_i + \Delta G)} = \frac{\Delta G^2 + \Delta B^2}{4G_i\,(G_i + \Delta G)} \tag{6.104}$$

On a contour for which G_p is constant, h must be independent of ΔY. Eq. (6.104) can be rewritten $\Delta G^2 - 4hG_i\Delta G + \Delta B^2 = 4G_i^2 h$. After completing the square, this equation can be put into the form

$$(\Delta G - 2hG_i)^2 + \Delta B^2 = |\Delta Y - 2hG_i|^2 = 4G_i^2 h\,(1+h) \tag{6.105}$$

For h a constant, it follows from this equation that G_p is constant on the contours for which $|\Delta Y - 2hG_i|^2$ is constant.

To solve for the gain contours, we can write

$$\Delta Y - 2hG_i = (Y_{om} - Y_i^*) - 2hG_i = Y_{om} - Y_h \tag{6.106}$$

where Y_h is the admittance given by

$$Y_h = (1 + 2h)\,G_i - jB_i \tag{6.107}$$

Note that $Y_h = Y_i^*$ and G_p is a maximum for the case $h = 0$. Define the normalized admittances $\widehat{Y}_{om} = Y_{om}/Y_c$, $\widehat{Y}_h = Y_h/Y_c$, $\Delta\widehat{Y} = \Delta Y/Y_c$, and $\widehat{G}_i = G_i/Y_c$. It follows from Eqs. (6.105) and (6.106) that

$$\left|\widehat{Y}_{om} - \widehat{Y}_h\right|^2 = \left|\Delta\widehat{Y} - 2h\widehat{G}_i\right|^2 = 4\widehat{G}_i^2 h\,(1+h) \tag{6.108}$$

Let a zero length transmission line having a characteristic admittance Y_c be connected between the matching network and the amplifier input. The reflection coefficient $\Gamma_{om} = p + jq$ seen looking out of the amplifier input is given by Eq. (6.72). Let $\Gamma_h = p_h + jq_h$ be the value of Γ_{om} if the matching network is removed and replaced with the admittance Y_h. Following Eq. (6.74) for the derivation of the noise circles, we can relate $\widehat{Y}_{om} - \widehat{Y}_h$ to Γ_{om} and Γ_h as follows:

$$\widehat{Y}_{om} - \widehat{Y}_h = \frac{1 - \Gamma_{om}}{1 + \Gamma_{om}} - \frac{1 - \Gamma_h}{1 + \Gamma_h} = 2\frac{\Gamma_h - \Gamma_{om}}{(1 - \Gamma_{om})(1 - \Gamma_h)} \tag{6.109}$$

The squared magnitude is given by

$$\left|\widehat{Y}_{om} - \widehat{Y}_h\right|^2 = 4\frac{(p_h - p)^2 + (q_h - q)^2}{\left[(1 + p)^2 + q^2\right]|1 + \Gamma_h|^2} \tag{6.110}$$

When Eq. (6.108) is used for $\left|\widehat{Y}_{om} - \widehat{Y}_h\right|^2$, we can write

$$\widehat{G}_i^2 h(1 + h) = \frac{(p_h - p)^2 + (q_h - q)^2}{\left[(1 + p)^2 + q^2\right]|1 + \Gamma_h|^2} \tag{6.111}$$

By analogy to Eqs. (6.75) – (6.78) for the noise circles, it follows from Eq. (6.111) that the contours of constant G_p on the Smith chart are circles of radius c centered at the point $\Gamma = a + jb$, where a, b, and c are given by

$$a = \frac{p_h + |1 + \Gamma_h|^2 \widehat{G}_i^2 h(1 + h)}{1 - |1 + \Gamma_h|^2 \widehat{G}_i^2 h(1 + h)} \tag{6.112}$$

$$b = \frac{q_h}{1 - |1 + \Gamma_h|^2 \widehat{G}_i^2 h(1 + h)} \tag{6.113}$$

$$c^2 = a^2 + b^2 - \frac{|\Gamma_h|^2 - |1 + \Gamma_h|^2 \widehat{G}_i^2 h(1 + h)}{1 - |1 + \Gamma_h|^2 \widehat{G}_i^2 h(1 + h)} \tag{6.114}$$

For the same values of h, the gain circles for an admittance chart are the same as for an impedance chart. Fig. 6.10(b) shows the admittance chart circles corresponding to the impedance chart circles in Fig. 6.10(a). On corresponding circles having the same value of h on the two charts, $|\Delta Z|$ and $|\Delta Y|$ satisfy

$$\frac{|\Delta Z|^2}{R_i(R_i + \Delta R)} = \frac{|\Delta Y|^2}{G_i(G_i + \Delta G)} \tag{6.115}$$

6.8 Measuring the Noise Factor

6.8.1 Method 1

This method is the most general one because it does not require knowledge of either the amplifier gain or its noise bandwidth. Consider the noise model of an amplifier given in Fig. 6.11. Consider the source to be a white noise source having the spectral density $S_v(f) = \overline{V_s V_s^*}/\Delta f = \overline{v_s^2}/\Delta f$. The total noise voltage at the output can be written

$$\begin{aligned} V_o &= A\left[(V_s + V_{ts} + V_n)\frac{Z_i}{Z_s + Z_i} + I_n(Z_s \| Z_i)\right] \\ &= \frac{AZ_i}{Z_s + Z_i}(V_s + V_{ts} + V_n + I_n Z_s) \end{aligned} \tag{6.116}$$

The mean-square value is given by

$$\begin{aligned} \overline{v_o^2} &= \left|\frac{AZ_i}{Z_s + Z_i}\right|^2 \Bigg[S_v(f) B_n + 4kTR_s B_n + \overline{v_n^2} \\ &\quad + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}(\gamma Z_s^*) + \overline{i_n^2}|Z_s|^2\Bigg] \end{aligned} \tag{6.117}$$

where B_n is the amplifier noise bandwidth.

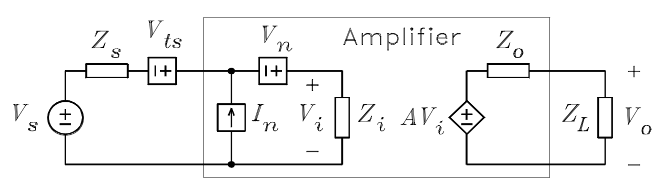

Figure 6.11: Amplifier driven by a white noise source.

Let $\overline{v_{o1}^2}$ be the value of $\overline{v_o^2}$ with the noise source at the input set to zero, i.e. $S_v(f) = 0$. Now, let $S_v(f)$ be increased until the rms output voltage increases by a factor r, i.e. $\sqrt{\overline{v_o^2}} = r\sqrt{\overline{v_{o1}^2}}$. It follows by taking the ratio of the two mean-square voltages that

$$\begin{aligned} r^2 &= 1 + \frac{S_v(f) B_n}{4kTR_s B_n + \overline{v_n^2} + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}(\gamma Z_s^*) + \overline{i_n^2}|Z_s|^2} \\ &= 1 + \frac{S_v(f)}{F \times 4kT_0 R_s} \end{aligned} \tag{6.118}$$

where F is the noise factor given by Eq. (6.11).

The above equation can be solved for F to obtain

$$F = \frac{S_v(f)}{(r^2 - 1) \times 4kT_0R_s} \tag{6.119}$$

In making measurements, a commonly used value for r is $r = \sqrt{2}$. In this case, the output noise voltage increases by 3 dB when the source is activated. Note that the expression for F is independent of B_n, A, and Z_i.

6.8.2 Method 2

This method replaces the white noise source in Fig. 6.11 with a sinusoidal source. The frequency should be chosen for maximum gain. The noise bandwidth B_n of the amplifier must be known. An often used alternative is to estimate B_n with the equation

$$B_n = \frac{\pi}{2} B_3 \tag{6.120}$$

where B_3 is the -3 dB bandwidth. This expression is exact if the amplifier has a first-order low-pass response or a second-order band-pass response.

The mean-square output voltage is given by

$$\begin{aligned} \overline{v_o^2} &= \left| \frac{AZ_i}{Z_s + Z_i} \right|^2 \Bigg[\overline{v_s^2} + 4kTR_sB_n + \overline{v_n^2} \\ &\quad + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\left(\gamma Z_s^*\right) + \overline{i_n^2}\left|Z_s\right|^2 \Bigg] \end{aligned} \tag{6.121}$$

where $\overline{v_s^2}$ is the mean-square open-circuit source voltage. With $V_s = 0$, the mean-square noise output voltage is measured. Denote this by $\overline{v_{o1}^2}$. Increase V_s until the rms output voltage increases by the factor r. It follows by taking the ratio of the two mean-square voltages that

$$\begin{aligned} r^2 &= 1 + \frac{\overline{v_s^2}}{4kTR_sB_n + \overline{v_n^2} + 2\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\left(\gamma Z_s^*\right) + \overline{i_n^2}\left|Z_s\right|^2} \\ &= 1 + \frac{\overline{v_s^2}}{F \times 4kT_0B_nR_s} \end{aligned} \tag{6.122}$$

Solution for F yields

$$F = \frac{\overline{v_s^2}}{(r^2 - 1) \times 4kT_0 B_n R_s} \tag{6.123}$$

6.8.3 Method 3

Unlike the above methods, this method requires knowledge of both B_n and the amplifier gain. The gain is measured with a sine-wave source having an open-circuit output voltage V_s and an output resistance R_s, where R_s is the value of the source resistance for which the noise factor is to be measured. With the source connected to the amplifier input, adjust the voltage to obtain a convenient voltage at the amplifier output. Denote this by V_{o1}. The frequency should be chosen for maximum gain. Next, disconnect the source from the amplifier input and measure its open-circuit output voltage. Denote this by V_{s1}. Let A_0 be the magnitude of the gain at the test frequency. With reference to the model in Fig. 6.11, it is given by

$$A_0 = \left| \frac{V_{o1}}{V_{s1}} \right| = \left| \frac{AZ_i}{R_s + Z_i} \right| \tag{6.124}$$

This is the gain including the loading effects at the input. The next step is to measure the noise bandwidth B_n. An often used alternative is to estimate B_n with Eq. (6.120).

The source is then replaced with a resistor of value R_s and the amplifier noise output voltage is measured. The mean-square value is given by

$$\overline{v_{no}^2} = \left| \frac{AZ_i}{Z_s + Z_i} \right|^2 \overline{v_{ni}^2} = 4kT_0 R_s B_n F A_0^2 \tag{6.125}$$

This equation can be solved for F to obtain

$$F = \frac{\overline{v_{no}^2}}{4kT_0 R_s B_n A_0^2} \tag{6.126}$$

6.9 Determination of Noise Parameters

Let the noise factor F be measured for N values of source admittance, where $N \geq 4$. Denote the noise factor values by F_i and the source admittance values by $Y_i = G_i + jB_i$. By Eq. (6.19), we can write

$$F_i = 1 + R_n \left(G_i + \frac{B_i^2}{G_i} \right) + 2R_n G_\gamma - 2R_n B_\gamma \frac{B_i}{G_i} + \frac{G_n}{G_i} \tag{6.127}$$

where $Y_\gamma = G_\gamma + jB_\gamma$ is the correlation admittance given by Eq. (6.10). The object is to use the measured values of F to determine the noise resistance R_n, the noise conductance G_n, and the correlation admittance Y_γ.

Define the mean-square error function

$$\epsilon^2 = \sum \left[(F_i - 1) - R_n \left(G_i + \frac{B_i^2}{G_i} \right) - 2R_n G_\gamma + 2R_n B_\gamma \frac{B_i}{G_i} - \frac{G_n}{G_i} \right]^2 \tag{6.128}$$

where the summation extends over the range $1 \le i \le N$. The values of R_n, G_n, G_γ, and B_γ which minimize ϵ^2 represent a best estimate of the noise parameters. These values can be obtained by simultaneous solution of the set of equations $\partial\epsilon^2/\partial R_n = 0$, $\partial\epsilon^2/\partial G_n = 0$, $\partial\epsilon^2/\partial (R_n G_\gamma) = 0$, and $\partial\epsilon^2/\partial (R_n B_\gamma) = 0$, where $R_n G_\gamma$ and $R_n B_\gamma$ are considered independent variables. This procedure leads to the following solution:

$$\begin{bmatrix} R_n \\ 2R_n G_\gamma \\ -2R_n B_\gamma \\ G_n \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} \sum \left(G_i + \frac{B_i^2}{G_i} \right) (F_i - 1) \\ \sum (F_i - 1) \\ \sum \frac{B_i}{G_i} (F_i - 1) \\ \sum \frac{1}{G_i} (F_i - 1) \end{bmatrix} \tag{6.129}$$

where the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} \sum \left(G_i + \frac{B_i^2}{G_i} \right)^2 & \sum G_i + \frac{B_i^2}{G_i} & \sum B_i + \frac{B_i^3}{G_i^2} & \sum 1 + \frac{B_i^2}{G_i^2} \\ \sum G_i + \frac{B_i^2}{G_i} & N & \sum \frac{B_i}{G_i} & \sum \frac{1}{G_i} \\ \sum B_i + \frac{B_i^3}{G_i^2} & \sum \frac{B_i}{G_i} & \sum \frac{B_i^2}{G_i^2} & \sum \frac{B_i}{G_i^2} \\ \sum 1 + \frac{B_i^2}{G_i^2} & \sum \frac{1}{G_i} & \sum \frac{B_i}{G_i^2} & \sum \frac{1}{G_i^2} \end{bmatrix} \tag{6.130}$$

The matrix $\mathbf{A}$ is singular if the ratios $a_{m_1,n}/a_{m_2,n}$ are equal for all elements in any two rows. With 4 rows, there are 6 combinations of two rows. It follows that the matrix is singular if the values of G_i and B_i lie on one of the contours defined by

$$G^2 + B^2 = k_1^2 \tag{6.131}$$

$$(G - k_2)^2 + B^2 = k_2^2 \tag{6.132}$$

$$G + (B - k_3)^2 = k_3^2 \tag{6.133}$$

$$B = k_4 G \tag{6.134}$$

$$G = k_5 \tag{6.135}$$

$$B = k_6 \tag{6.136}$$

where k_1 through k_6 are constants. Fig. 6.12 shows example plots of these equations on the (G, B) plane. The ks are chosen so that the curves intersect at two common points. The curves are labeled a through f, corresponding in order to Eqs. (6.131) through (6.136). Two curves are labeled c, d, and f, corresponding to positive and negative values of k_3, k_4, and k_6.

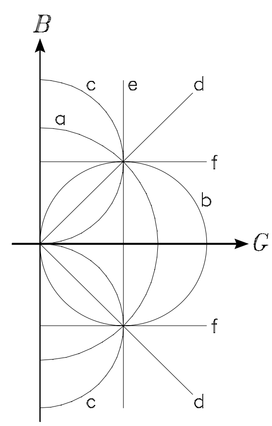

Figure 6.12: Example contours on which the matrix **A** is singular.

Given the solution for R_n, G_n, G_γ, and B_γ, the solutions for $\overline{v_n^2}$, $\overline{i_n^2}$, and γ are

$$\overline{v_n^2} = 4kT_0 R_n \Delta f \tag{6.137}$$

$$\overline{i_n^2} = 4kT_0 G_n \Delta f \tag{6.138}$$

$$\gamma = \sqrt{\frac{\overline{v_n^2}}{\overline{i_n^2}}} \left(G_\gamma - jB_\gamma\right) \tag{6.139}$$

Because the quantity $(F - 1)$ is involved in the calculations, large percentage errors in $(F - 1)$ can be caused by small percentage errors in

F when F has a value close to 1. Thus the solutions can be sensitive to experimental errors. Another problem lies in the choice of the values of G_i and B_i for which F is measured. If the values lie on or near one of the curves which makes the $\mathbf{A}$ matrix singular, the solutions can be unstable. To minimize this problem, the values of G_i and B_i should be chosen randomly.

Chapter 7

Noise in Diodes and BJTs

7.1 Junction Diode Noise Model

The current in a pn junction diode consists of two components – the forward diffusion current I_F and the reverse saturation current I_S. The total current is given by $I = I_F - I_S$. The forward diffusion current is a function of the diode voltage V and is given by $I_F = I_S \exp(V/nV_T)$, where n is the emission coefficient and V_T is the thermal voltage. For discrete silicon diodes $n \simeq 2$ whereas for integrated circuit silicon diodes and discrete germanium diodes $n \simeq 1$. Both I_F and I_S generate uncorrelated shot noise. The total mean-square noise current is given by

$$\overline{i_n^2} = 2q\left(I_F + I_S\right)\Delta f = 2q\left(I + 2I_S\right)\Delta f \simeq 2qI\Delta f \tag{7.1}$$

where the approximation holds for a forward biased diode for which $I >> I_S$. Fig. 7.1(a) shows the diode noise model. In Fig. 7.1(b), the diode is replaced by its small-signal resistance given by

$$r_d = \frac{nV_T}{I + I_S} \simeq \frac{nV_T}{I} \tag{7.2}$$

The mean-square open-circuit noise voltage across the diode is

$$\overline{v_n^2} = \overline{i_n^2} r_d^2 = 2qI\Delta f \left(\frac{nV_T}{I + I_S}\right)^2 \simeq \frac{2qn^2V_T^2\Delta f}{I} \tag{7.3}$$

At low-frequencies, the diode exhibits flicker noise. When this is included, the total mean-square noise current is given by

$$\overline{i_n^2} = 2q\left(I + 2I_S\right)\Delta f + \frac{K_f I \Delta f}{f} \simeq 2qI\Delta f + \frac{K_f I \Delta f}{f} \tag{7.4}$$

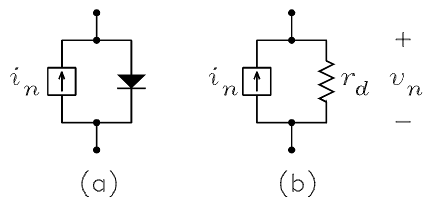

Figure 7.1: (a) Diode noise model. (b) Small-signal model.

A plot of $\overline{i_n^2}$ versus f for a constant Δf exhibits a slope of -10 dB/decade for very low frequencies and a slope of zero for higher frequencies. The two terms in Eq. (7.4) are equal at the frequency where the noise current is up 3 dB compared to its high-frequency limit. This frequency is called the flicker noise corner frequency.

Diodes are often used as noise sources in circuits. Specially processed zener diodes are fabricated as solid-state noise diodes. The noise mechanism in these is called avalanche noise and it is associated with the diode reverse breakdown current. For a given breakdown current, avalanche noise is much greater than the shot noise in the same current.

7.2 BJT Device Equations

Figure 7.2 shows the circuit symbols for the npn and pnp BJTs. In the active mode, the collector-base junction is reverse biased and the base-emitter junction is forward biased. Because of recombinations of the minority and majority carriers, the equations for the currents can be divided into three regions: low, mid, and high. For the npn device, the currents are given by

Low Level:

$$i_B = I_{SE}\left[\exp\left(\frac{v_{BE}}{nV_T}\right) - 1\right] \tag{7.5}$$

$$i_C = I_S\left[\exp\left(\frac{v_{BE}}{V_T}\right) - 1\right] \tag{7.6}$$

Mid Level:

$$i_B = \frac{I_S}{\beta_F}\left[\exp\left(\frac{v_{BE}}{V_T}\right) - 1\right] \tag{7.7}$$

$$i_C = I_S\left[\exp\left(\frac{v_{BE}}{V_T}\right) - 1\right] \tag{7.8}$$

High Level:

$$i_B = \frac{I_S}{\beta_F}\left[\exp\left(\frac{v_{BE}}{V_T}\right) - 1\right] \tag{7.9}$$

$$i_C = I_S\sqrt{\frac{I_K}{I_{SO}}}\left[\exp\left(\frac{v_{BE}}{2V_T}\right) - 1\right] \tag{7.10}$$

where all leakage currents that are a function of v_{CB} have been neglected. In the current equations, I_S is the saturation current and β_F is the mid-level base-to-collector current gain. These are functions of the collector-base voltage and are given by

$$I_S = I_{SO}\left(1 + \frac{v_{CB}}{V_A}\right) = I_{SO}\left(1 + \frac{v_{CE} - v_{BE}}{V_A}\right) \tag{7.11}$$

$$\beta_F = \beta_{FO}\left(1 + \frac{v_{CB}}{V_A}\right) = \beta_{FO}\left(1 + \frac{v_{CE} - v_{BE}}{V_A}\right) \tag{7.12}$$

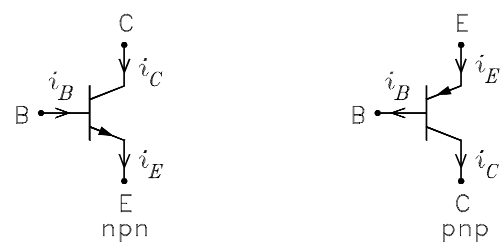

Figure 7.2: BJT circuit symbols.

In the equations for i_B and i_C, V_A is the Early voltage and I_{SO} and β_{FO}, respectively, are the zero bias values of I_S and β_F. The constant n is the emission coefficient or ideality factor of the base-emitter junction. It accounts for recombinations of holes and electrons in the base-emitter junction at low levels. Its value, typically in the range $1 \leq n \leq 4$, is determined by the slope of the plot of $\ln(i_C)$ versus v_{BE} at low levels. The default value in SPICE is $n = 1.5$. The constant I_{SE} is determined by the value of i_B where transition from the low-level to the mid-level region occurs. The constant I_K is determined by the value of i_C where transition from the mid-level to the high-level region occurs. Note that $I_S/\beta_F = I_{S0}/\beta_{F0}$ so that i_B is not a function of v_{CB} in the mid-level region. The equations apply to the pnp device if the subscripts BE and CB are reversed.

Figure 7.3 shows a typical plot of i_C versus v_{BE} for v_{CE} constant. The plot is called the transfer characteristics. There is a threshold voltage above which the current appears to increase rapidly. This voltage is typically 0.5 to 0.6 V. In the forward active region, the base-to-emitter voltage is typically 0.6 to 0.7 V. Figure 7.4 shows typical plots of i_C versus v_{CE} for i_B constant. The plots are called the output characteristics. Note that the slope approaches a constant as v_{CE} is increased. If the straight line portions of the curves are extended back so that they intersect the v_{CE} axis, they would intersect at the voltage $v_{CE} = -V_A + v_{BE} \simeq -V_A$. For v_{CE} small, $v_{BE} > v_{CE}$ and the BJT is in the saturation region.

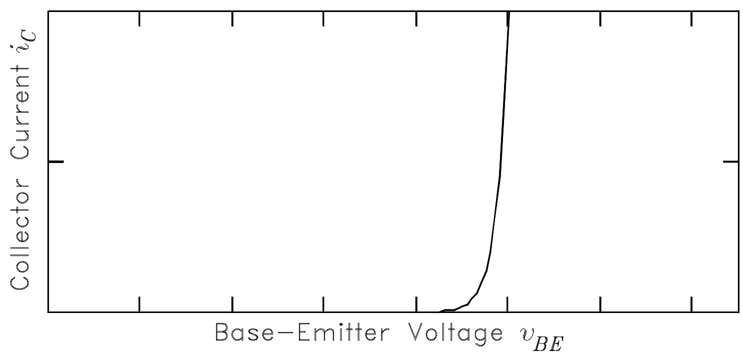

Figure 7.3: Typical plot of i_C versus v_{BE} for v_{CE} constant.

In the Gummel-Poon model of the BJT, the current equations are combined to write the general equations for i_B and i_C as follows:

$$i_B = I_{SE}\left[\exp\left(\frac{v_{BE}}{nV_T}\right) - 1\right] + \frac{I_{SO}}{\beta_{FO}}\left[\exp\left(\frac{v_{BE}}{V_T}\right) - 1\right] \tag{7.13}$$

$$i_C = \frac{I_S}{K_q}\left[\exp\left(\frac{v_{BE}}{V_T}\right) - 1\right] \tag{7.14}$$

where K_q is given by

$$K_q = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{I_S}{I_K}\left[\exp\left(\frac{v_{BE}}{V_T}\right) - 1\right]} = 1 + \frac{i_C}{I_K} \tag{7.15}$$

Fig. 7.5 illustrates typical plots of $\ln(i_C)$ and $\ln(i_B)$ versus v_{BE}, where it is assumed that v_{CB} is held constant. At low levels, the i_C curve exhibits a slope $m = 1$ while the i_B curve exhibits a slope $m = 1/n$,

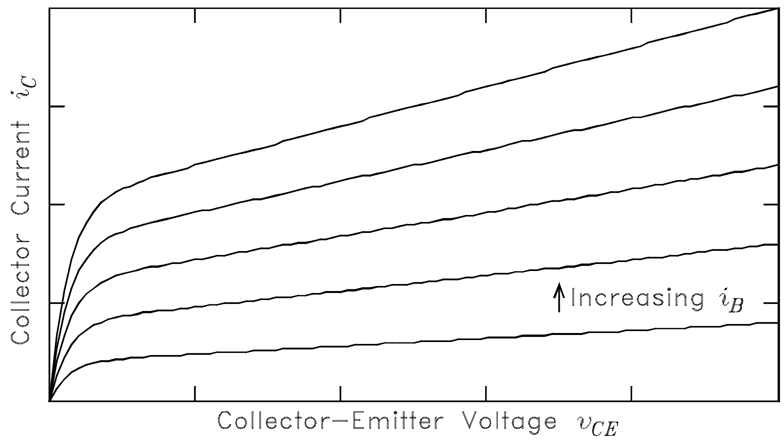

Figure 7.4: Plots of i_C versus v_{CE} for i_B constant.

where the value $n = 1.5$ has been used. At mid levels, both curves exhibit a slope $m = 1$. At high levels, the i_C curve exhibits a slope $m = 1/2$ while the i_B curve exhibits a slope $m = 1$. It follows that the ratio of i_C to i_B is approximately constant at mid levels and decreases at low and high levels.

7.2.1 Current Gains

Let the collector and base currents be written as the sum of a dc component and a small-signal ac component as follows:

$$i_C = I_C + i_c \tag{7.16}$$

$$i_B = I_B + i_b \tag{7.17}$$

The dc current gain $\beta_{F\text{dc}}$ is defined as the ratio of I_C to I_B. It is straightforward to show that it is given by

$$\beta_{F\text{dc}} = \frac{\beta_F}{1 + \dfrac{I_C}{I_K} + \dfrac{\beta_F I_{SE}}{I_C}\left[\left(1 + \dfrac{I_C}{I_S} + \dfrac{I_C^2}{I_S I_K}\right)^{1/n} - 1\right]} \tag{7.18}$$

Because β_F and I_S are functions of the collector-base voltage V_{CB}, it follows that $\beta_{F\text{dc}}$ is a function of both I_C and V_{CB}. If v_{CB} is held

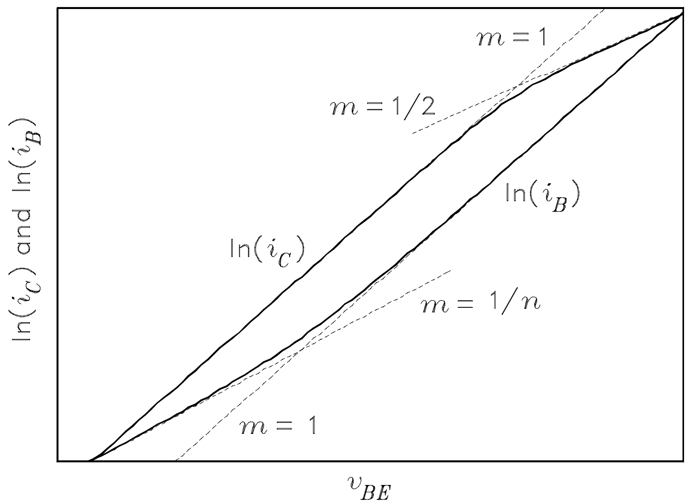

Figure 7.5: Example plots of $\ln(i_C)$ and $\ln(i_B)$ versus v_{BE}.

constant so that the change in i_C is due to a change in v_{BE}, the small-signal change in base current can be written

$$i_b = \frac{\partial I_B}{\partial I_C} i_c = \left[\frac{\partial}{\partial I_C} \left(\frac{I_C}{\beta_{F\mathrm{dc}}} \right) \right] i_c = \frac{i_c}{\beta_{F\mathrm{ac}}} \tag{7.19}$$

where $\beta_{F\mathrm{ac}}$ is the small-signal ac current gain given by

$$\begin{aligned} \beta_{F\mathrm{ac}} &= \left[\frac{\partial}{\partial I_C} \left(\frac{I_C}{\beta_{F\mathrm{dc}}} \right) \right]^{-1} \\ &= \frac{\beta_F}{\left(1 + \frac{2I_C}{I_K}\right) \left[1 + \frac{\beta_F I_{SE}}{n I_S} \left(1 + \frac{I_C}{I_S} + \frac{I_C^2}{I_S I_K} \right)^{\frac{1}{n}-1} \right]} \end{aligned} \tag{7.20}$$

Note that $\beta_{F\mathrm{ac}}$ is defined for a constant v_{CB}. In the small-signal models, it is common to define the small-signal ac current gain with a constant v_{CE}, i.e. $v_{ce} = 0$. This is defined in the next section, where the symbol β is used.

Typical plots of the two current gains as a function of I_C are shown in Fig. 7.6 where log scales are used. At low levels, the gains decrease with decreasing I_C because the base current decreases at a slower rate than the collector current. At high levels, the gains decrease with increasing

I_C because the collector current increases at a slower rate than the base current. At mid levels, both gains are approximately constant and have the same value. In the figure, the mid-level range is approximately two decades wide.

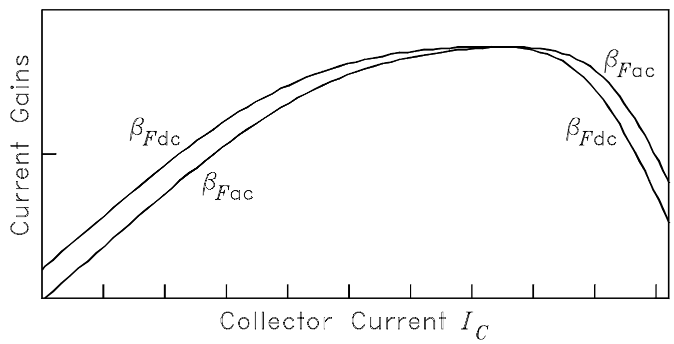

Figure 7.6: Log-log plots of $\beta_{F\mathrm{dc}}$ and $\beta_{F\mathrm{ac}}$ as functions of I_C.

The emitter-collector dc current gain $\alpha_{F\mathrm{dc}}$ is defined as the ratio of the dc collector current I_C to the dc emitter current I_E. To solve for this, we can write

$$I_E = I_B + I_C = \left(\frac{1}{\beta_{F\mathrm{dc}}} + 1\right) I_C = \frac{1 + \beta_{F\mathrm{dc}}}{\beta_{F\mathrm{dc}}} I_C \qquad (7.21)$$

It follows that

$$\alpha_{F\mathrm{dc}} = \frac{I_C}{I_E} = \frac{\beta_{F\mathrm{dc}}}{1 + \beta_{F\mathrm{dc}}} \qquad (7.22)$$

Thus the dc currents are related by the equations

$$I_C = \beta_{F\mathrm{dc}} I_B = \alpha_{F\mathrm{dc}} I_E \qquad (7.23)$$

7.3 Bias Equation

Figure 7.7(a) shows the BJT with the external circuits represented by Thévenin dc circuits. If the BJT is biased in the active region, we can write

$$\begin{aligned} V_{BB} - V_{EE} &= I_B R_{BB} + V_{BE} + I_E R_{EE} \\ &= \frac{I_C}{\beta_{F\mathrm{dc}}} R_{BB} + V_{BE} + \frac{I_C}{\alpha_{F\mathrm{dc}}} R_{EE} \end{aligned} \qquad (7.24)$$

This equation can be solved for I_C to obtain

$$I_C = \frac{V_{BB} - V_{EE} - V_{BE}}{R_{BB}/\beta_{F\mathrm{dc}} + R_{EE}/\alpha_{F\mathrm{dc}}} \tag{7.25}$$

It can be seen from Fig. 7.3 that large changes in I_C are associated with small changes in V_{BE}. This makes it possible to calculate I_C by assuming typical values of V_{BE}. Values in the range from 0.6 to 0.7 V are commonly used. In addition, $\beta_{F\mathrm{dc}}$ and $\alpha_{F\mathrm{dc}}$ are functions of I_C and V_{CB}. Mid-level values are commonly assumed for the current gains. Typical values are $\beta_{F\mathrm{dc}} = 100$ and $\alpha_{F\mathrm{dc}} = 1/1.01$.

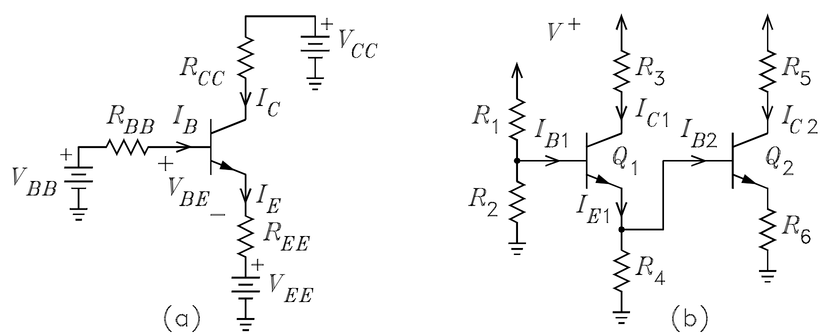

Figure 7.7: (a) BJT dc bias circuit. (b) Circuit for Example 1.

Example 1 *Figure 7.7(b) shows a BJT dc bias circuit. It is given that $V^+ = 15$ V, $R_1 = 20$ kΩ, $R_2 = 10$ kΩ, $R_3 = R_4 = 3$ kΩ, $R_5 = R_6 = 2$ kΩ. Solve for I_{C1} and I_{C2}. Assume $V_{BE} = 0.7$ V and $\beta_{Fdc} = 100$ for each transistor.*

Solution. For Q_1, we have $V_{BB1} = V^+ R_2/(R_1 + R_2)$, $R_{BB1} = R_1 \| R_2$, $V_{EE1} = -I_{B2}R_4 = -I_{C2}R_4/\beta$, $V_{EE1} = 0$, and $R_{EE1} = R_4$. For Q_2, we have $V_{BB2} = I_{E1}R_4 = I_{C1}R_4/\alpha_{F\mathrm{dc}}$, $R_{BB2} = R_4$, $V_{EE2} = 0$, $R_{EE2} = R_6$. Thus the bias equations are

$$V^+ \frac{R_2}{R_1 + R_2} + \frac{I_{C2}}{\beta_{F\mathrm{dc}}} R_4 = V_{BE} + \frac{I_{C1}}{\beta_{F\mathrm{dc}}} R_1 \| R_2 + \frac{I_{C1}}{\alpha_{F\mathrm{dc}}} R_4$$

$$\frac{I_{C1}}{\alpha_{F\mathrm{dc}}} R_4 = V_{BE} + \frac{I_{C2}}{\beta_{F\mathrm{dc}}} R_4 + \frac{I_{C2}}{\alpha_{F\mathrm{dc}}} R_6$$

These equations can be solved simultaneously to obtain $I_{C1} = 1.41$ mA and $I_{C2} = 1.74$ mA.

7.4 Small-Signal Models

There are two small-signal circuit models which are commonly used to analyze BJT circuits. These are the hybrid-π model and the T model. The two models are equivalent and give identical results. They are described below.

7.4.1 Hybrid-π Model

Let each current and voltage be written as the sum of a dc component and a small-signal ac component. The currents are given by Eqs. (7.16) and (7.17). The voltages can be written

$$v_{BE} = V_{BE} + v_{be} \tag{7.26}$$

$$v_{CB} = V_{CB} + v_{cb} \tag{7.27}$$

If the ac components are sufficiently small, i_c can be written

$$\begin{aligned} i_c &= \frac{\partial I_C}{\partial V_{BE}} v_{be} + \frac{\partial I_C}{\partial V_{CB}} v_{cb} = \frac{\partial I_C}{\partial V_{BE}} v_{be} + \frac{\partial I_C}{\partial V_{CB}} (v_{ce} - v_{be}) \\ &= \left(\frac{\partial I_C}{\partial V_{BE}} - \frac{\partial I_C}{\partial V_{CB}} \right) v_{be} + \frac{\partial I_C}{\partial V_{CB}} v_{ce} = g_m v_{be} + \frac{v_{ce}}{r_0} \end{aligned} \tag{7.28}$$

This equation defines the transconductance $g_m = \partial I_C/\partial V_{BE} - \partial I_C/\partial V_{CB}$ and the collector-emitter resistance $r_0 = (\partial I_C/\partial V_{CB})^{-1}$. With the aid of Eqs. (7.11) and (7.14), it follows that r_0 is given by

$$\begin{aligned} r_0 &= \left(\frac{\partial I_C}{\partial V_{CB}} \right)^{-1} = \left\{ \frac{K_q I_{SO}}{V_A} \left[\exp\left(\frac{v_{BE}}{V_T} \right) - 1 \right] \right\}^{-1} \\ &= \frac{V_A + V_{CB}}{I_C} \end{aligned} \tag{7.29}$$

To solve for g_m, we first solve for $\partial I_C/\partial V_{BE}$. Eqs. (7.14) and (7.15) can be combined to write

$$I_C + \frac{I_C^2}{I_K} = I_S \left[\exp\left(\frac{V_{BE}}{V_T} \right) - 1 \right] \tag{7.30}$$

It follows from this equation that

$$\left(1 + \frac{2I_C}{I_K} \right) \frac{\partial I_C}{\partial V_{BE}} = \frac{I_S}{V_T} \exp\left(V_{BE}/V_T \right) = \frac{I_C \left(1 + I_C/I_K \right) + I_S}{V_T} \tag{7.31}$$

which can be solved for $\partial I_C/\partial V_{BE}$. It follows that the transconductance is given by

$$g_m = \frac{\partial I_C}{\partial V_{BE}} - \frac{\partial I_C}{\partial V_{CB}} = \frac{I_C\left(1 + I_C/I_K\right) + I_S}{V_T\left(1 + 2I_C/I_K\right)} - \frac{1}{r_0} \tag{7.32}$$

It is clear from Eq. (7.13) that i_B is a function of v_{BE} only. We wish to solve for the small-signal ac base current given by $i_b = \left(\partial I_B/\partial V_{BE}\right) v_{be}$. This equation defines the small-signal ac base-emitter resistance $r_\pi = v_b/i_b = \left(\partial I_B/\partial V_{BE}\right)^{-1}$. Although Eq. (7.13) can be used to solve for this, we use a different approach. The small-signal ac collector current can be written

$$i_c = g_m v_{be} + \frac{v_{ce}}{r_0} = \left(g_m + \frac{1}{r_0}\right) v_{be} + \frac{v_{cb}}{r_0} = \beta_{F\mathrm{ac}} i_b + \frac{v_{cb}}{r_0} \tag{7.33}$$

It follows from this equation that

$$\left(g_m + \frac{1}{r_0}\right) v_{be} = \beta_{F\mathrm{ac}} i_b \tag{7.34}$$

Thus r_π is given by

$$r_\pi = \frac{v_{be}}{i_b} = \frac{\beta_{F\mathrm{ac}}}{g_m + 1/r_0} \tag{7.35}$$

The small-signal ac current gain β is defined as the ratio of i_c to i_b with v_{CE} constant, i.e. $v_{ce} = 0$. To solve for this, we can write for i_c

$$i_c = g_m v_{be} + \frac{v_{ce}}{r_0} = g_m i_b r_\pi + \frac{v_{ce}}{r_0} = \beta i_b + \frac{v_{ce}}{r_0} \tag{7.36}$$

It follows from this that β is given by

$$\beta = g_m r_\pi = \frac{g_m \beta_{F\mathrm{ac}}}{g_m + 1/r_0} \tag{7.37}$$

Thus far, we have neglected the base spreading resistance r_x. This is the ohmic resistance of the base contact in the BJT. When it is included in the model, it appears in series with the base lead. Because the base region is very narrow, the connection exhibits a resistance which often cannot be neglected. Fig. 7.8(a) shows the hybrid-π small-signal model with r_x included. The currents are given by

$$i_c = i_c' + \frac{v_{ce}}{r_0} \tag{7.38}$$

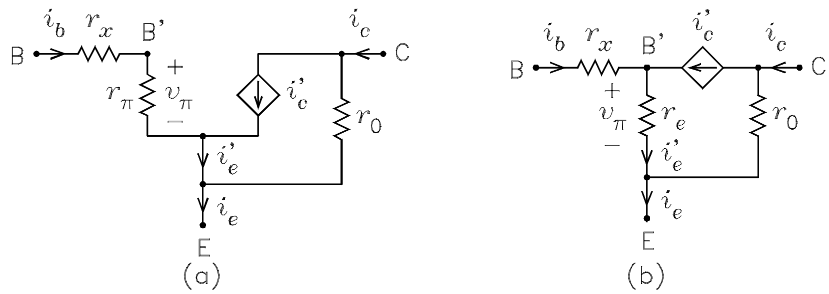

Figure 7.8: (a) Hybrid-π model. (b) T model.

$$i_c' = g_m v_\pi = \beta i_b \tag{7.39}$$

$$i_b = \frac{v_{be}}{r_\pi} \tag{7.40}$$

where r_0, g_m, β, and r_π are given above.

The equations derived above are based on the Gummel-Poon model of the BJT in the forward active region. The equations are often approximated by assuming that the mid-level current equations hold. In this case, $\beta_{F\text{dc}}$, $\beta_{F\text{ac}}$, r_0, g_m, r_π, and β are given by

$$\beta_{F\text{dc}} = \beta_{F\text{ac}} = \beta_F \tag{7.41}$$

$$r_0 = \frac{V_{CB} + V_A}{I_C} \tag{7.42}$$

$$g_m = \frac{I_C + I_S}{V_T} - \frac{1}{r_0} \simeq \frac{I_C}{V_T} \tag{7.43}$$

$$r_\pi = \frac{\beta_{F\text{ac}}}{g_m + 1/r_0} \simeq \frac{V_T}{I_B} \tag{7.44}$$

$$\beta = g_m r_\pi \simeq \frac{I_C}{I_B} = \beta_F \tag{7.45}$$

The approximations in these equations are commonly used for hand calculations.

7.4.2 T Model

The T model replaces the resistor r_π in series with the base with a resistor r_e in series with the emitter. This resistor is called the emitter intrinsic resistance. To solve for r_e, we first solve for the small-signal ac emitter-to-collector current gain α. In Fig. 7.8, the current i'_e can be written

$$i'_e = i_b + i'_c = \left(\frac{1}{\beta} + 1\right) i'_c = \frac{1+\beta}{\beta} i'_c = \frac{i'_c}{\alpha} \tag{7.46}$$

where α is given by

$$\alpha = \frac{i'_c}{i'_e} = \frac{\beta}{1+\beta} \tag{7.47}$$

Thus the current i'_c can be written

$$i'_c = \alpha i'_e \tag{7.48}$$

The voltage v_π can be related to i'_e as follows:

$$v_\pi = i_b r_\pi = \frac{i'_c}{\beta} r_\pi = \frac{\alpha i'_e}{\beta} r_\pi = i'_e \frac{r_\pi}{1+\beta} \tag{7.49}$$

It follows that the intrinsic emitter resistance is given by

$$r_e = \frac{v_\pi}{i'_e} = \frac{r_\pi}{1+\beta} \simeq \frac{V_T}{(1+\beta_F)\, I_B} = \frac{V_T}{I_E} \tag{7.50}$$

where the approximation is based on Eqs. (7.44) and (7.45). This approximation is often used for hand calculations. The T model of the BJT is shown in Fig. 7.8(b). The currents in both the π and T models are related by the equations

$$i'_c = g_m v_\pi = \beta i_b = \alpha i'_e \tag{7.51}$$

7.5 Small-Signal Equivalent Circuits

Several equivalent circuits are derived below which facilitate writing small-signal low-frequency equations for the BJT. We assume that the circuits external to the device can be represented by Thévenin equivalent circuits. The Norton equivalent circuit seen looking into the collector and the Thévenin equivalent circuits seen looking into the base and the emitter are derived. Several examples are given which illustrate use of the equivalent circuits.

7.5.1 Emitter Equivalent Circuit

Figure 7.9 shows the T model with a Thévenin source in series with the base. We wish to solve for an equivalent circuit in which the source $\alpha i'_e$ connects from the collector node to ground rather than from the collector node to the B' node. The first step is to replace the source with two identical series sources with the common node grounded. The circuit is shown in Fig. 7.10(a). The object is to absorb the left $\alpha i'_e$ source into the base-emitter circuit.

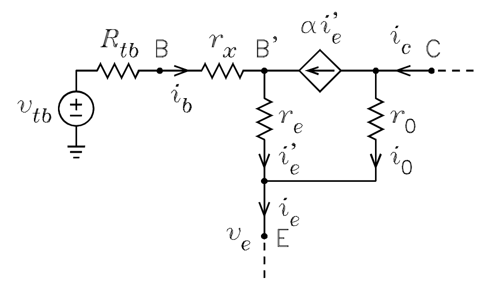

Figure 7.9: T model with Thévenin source connected to the base.

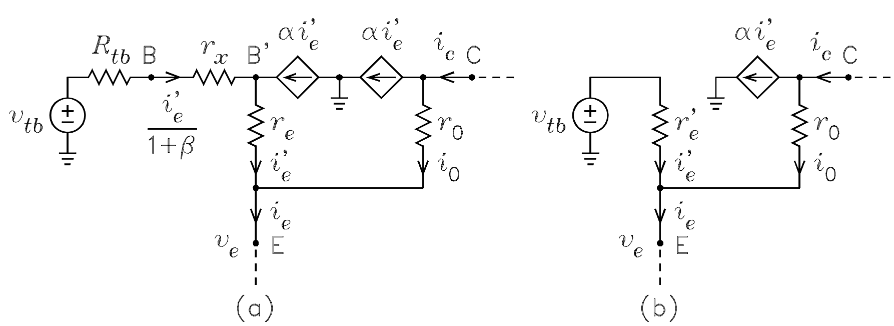

Figure 7.10: (a) Circuit with the i'_c source replaced by identical series sources. (b) Emitter equivalent circuit.

For the circuit in Fig. 7.10(a), we can write

$$v_e = v_{tb} - \frac{i'_e}{1+\beta}(R_{tb} + r_x) - i'_e r_e = v_{tb} - i'_e\left(\frac{R_{tb} + r_x}{1+\beta} + r_e\right) \quad (7.52)$$

Let us define the resistance r'_e by

$$r'_e = \frac{R_{tb} + r_x}{1+\beta} + r_e = \frac{R_{tb} + r_x + r_\pi}{1+\beta} \tag{7.53}$$

With this definition, v_e is given by

$$v_e = v_{tb} - i'_e r'_e \tag{7.54}$$

The circuit which models this equation is shown in Fig. 7.10(b). This will be called the emitter equivalent circuit. It predicts the same emitter and collector currents as the circuit in Fig. 7.9. Note that the resistors R_{tb} and r_x do not appear in this circuit. They are part of the resistor r'_e.

7.5.2 Norton Collector Circuit

The Norton equivalent circuit seen looking into the collector can be used to solve for the response of the common-emitter and common-base stages. It consists of a parallel current source $i_{c(sc)}$ and resistor r_{ic} from the collector to signal ground. Fig. 7.11(a) shows the BJT with Thévenin sources connected to its base and emitter. With the collector grounded, the collector current is the short-circuit or Norton collector current. To solve for this, we use the emitter equivalent circuit in Fig. 7.11(b). We use superposition of v_{tb} and v_{te} to solve for $i_{c(sc)}$.

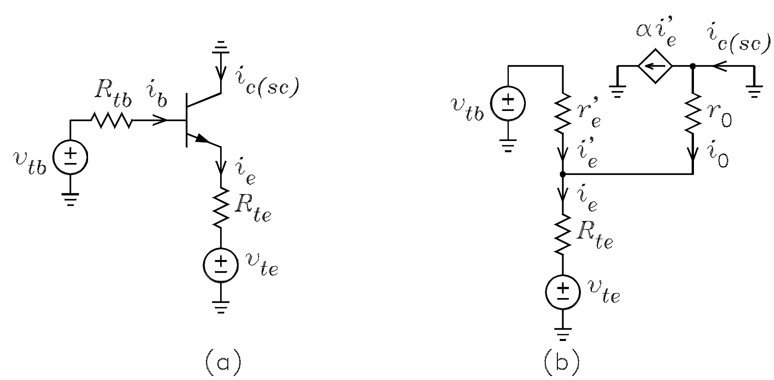

Figure 7.11: (a) BJT with Thevenin sources connected to the base and the emitter. (b) Emitter equivalent circuit.

With $v_{te} = 0$, it follows from Fig. 7.11(b) that

$$\begin{aligned} i_{c(sc)} &= \alpha i_e' + i_0 = \alpha i_e' - i_e' \frac{R_{te}}{r_0 + R_{te}} \\ &= \frac{v_{tb}}{r_e' + R_{te} \| r_0} \left(\alpha - \frac{R_{te}}{r_0 + R_{te}} \right) \end{aligned} \tag{7.55}$$

With $v_{tb} = 0$, we have

$$\begin{aligned} i_{c(sc)} &= \alpha i_e' + i_0 = \alpha i_e \frac{r_0}{r_0 + r_e'} + i_e \frac{r_e'}{r_0 + r_e'} \\ &= -\frac{v_{te}}{R_{te} + r_e' \| r_0} \frac{\alpha r_0 + r_e'}{r_0 + r_e'} \end{aligned} \tag{7.56}$$

These equations can be combined to obtain

$$i_{c(sc)} = \frac{v_{tb}}{r_e' + R_{te} \| r_0} \left(\alpha - \frac{R_{te}}{r_0 + R_{te}} \right) - \frac{v_{te}}{R_{te} + r_e' \| r_0} \frac{\alpha r_0 + r_e'}{r_0 + r_e'} \tag{7.57}$$

This equation is of the form

$$i_{c(sc)} = G_{mb} v_{tb} - G_{me} v_{te} \tag{7.58}$$

where

$$G_{mb} = \frac{1}{r_e' + R_{te} \| r_0} \left(\alpha - \frac{R_{te}}{r_0 + R_{te}} \right) = \frac{\alpha}{r_e' + R_{te} \| r_0} \frac{r_0 - R_{te}/\beta}{r_0 + R_{te}} \tag{7.59}$$

$$G_{me} = \frac{1}{R_{te} + r_e' \| r_0} \frac{\alpha r_0 + r_e'}{r_0 + r_e'} = \frac{\alpha}{r_e' + R_{te} \| r_0} \frac{r_0 + r_e'/\alpha}{r_0 + R_{te}} \tag{7.60}$$

The next step is to solve for the resistance seen looking into the collector with $v_{tb} = v_{te} = 0$. Figure 7.12(a) shows the emitter equivalent circuit with a test source connected to the collector. The resistance seen looking into the collector is given by $r_{ic} = v_t / i_c$. To solve for r_{ic}, we can write

$$\begin{aligned} i_c &= \alpha i_e' + i_0 = -\alpha i_0 \frac{R_{te}}{r_e' + R_{te}} + i_0 \\ &= \frac{v_t}{r_0 + r_e' \| R_{te}} \left(1 - \frac{\alpha R_{te}}{r_e' + R_{te}} \right) \end{aligned} \tag{7.61}$$

It follows that r_{ic} is given by

$$r_{ic} = \frac{v_t}{i_c} = \frac{r_0 + r_e' \| R_{te}}{1 - \alpha R_{te} / (r_e' + R_{te})} \tag{7.62}$$

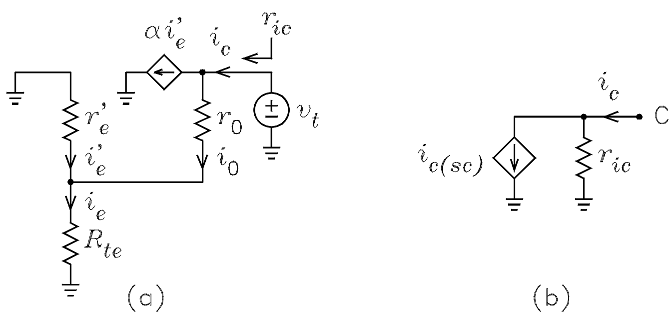

Figure 7.12: (a) Circuit for calculating r_{ic}. (b) Norton collector circuit.

The Norton equivalent circuit seen looking into the collector is shown in Fig. 7.12(b).

For the case $r_0 \gg R_{te}$ and $r_0 \gg r_e'$, we can write

$$i_{c(sc)} = G_m (v_{tb} - v_{te}) \tag{7.63}$$

where

$$G_m = \frac{\alpha}{r_e' + R_{te}} \tag{7.64}$$

The value of $i_{c(sc)}$ calculated with this approximation is simply the value of $\alpha i_e'$, where i_e' is calculated with r_0 considered to be an open circuit. The term "r_0 approximations" is used in the following when r_0 is neglected in calculating $i_{c(sc)}$ but not neglected in calculating r_{ic}.

7.5.3 Thévenin Emitter Circuit

The Thévenin equivalent circuit seen looking into the emitter is useful in calculating the response of common-collector stages. It consists of a voltage source $v_{e(oc)}$ in series with a resistor r_{ie} from the emitter node to signal ground. Fig. 7.13(a) shows the BJT symbol with a Thévenin source connected to the base. The resistor R_{tc} represents the external load resistance in series with the collector. With the emitter open circuited, we denote the emitter voltage by $v_{e(oc)}$. The voltage source in the Thévenin emitter circuit has this value. To solve for it, we use the emitter equivalent circuit in Fig. 7.13(b).

The current i_e' can be solved for by superposition of the sources v_{tb}

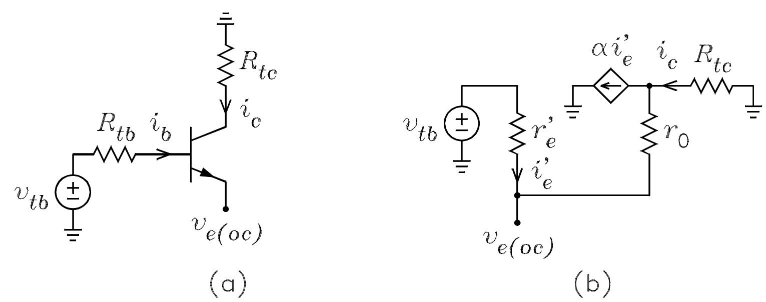

Figure 7.13: (a) BJT with Thévenin source connected to the base. (b) T model circuit for calculating $v_{e(oc)}$.

and $\alpha i'_e$. It is given by

$$i'_e = \frac{v_{tb}}{r'_e + r_0 + R_{tc}} + \alpha i'_e \frac{R_{tc}}{r'_e + r_0 + R_{tc}} \tag{7.65}$$

This can be solved for i'_e to obtain

$$i'_e = \frac{v_{tb}}{r'_e + r_0 + (1-\alpha) R_{tc}} = \frac{v_{tb}}{r'_e + r_0 + R_{tc}/(1+\beta)} \tag{7.66}$$

The open-circuit emitter voltage is given by

$$v_{e(oc)} = v_{tb} - i'_e r'_e = v_{tb} \frac{r_0 + R_{tc}/(1+\beta)}{r'_e + r_0 + R_{tc}/(1+\beta)} \tag{7.67}$$

We next solve for the resistance seen looking into the emitter node. It can be solved for as the ratio of the open-circuit emitter voltage $v_{e(oc)}$ to the short-circuit emitter current. The circuit for calculating the short-circuit current is shown in Fig. 7.14(a). By superposition of i'_e and $\alpha i'_e$, we can write

$$\begin{aligned} i_{e(sc)} &= i'_e - \alpha i'_e \frac{R_{tc}}{r_0 + R_{tc}} = i'_e \frac{r_0 + (1-\alpha) R_{tc}}{r_0 + R_{tc}} \\ &= \frac{v_{tb}}{r'_e} \frac{r_0 + R_{tc}/(1+\beta)}{r_0 + R_{tc}} \end{aligned} \tag{7.68}$$

The resistance seen looking into the emitter is given by

$$r_{ie} = \frac{v_{e(oc)}}{i_{e(sc)}} = r'_e \frac{r_0 + R_{tc}}{r'_e + r_0 + R_{tc}/(1+\beta)} \tag{7.69}$$

The Thévenin equivalent circuit seen looking into the emitter is shown in Fig. 7.14(b).

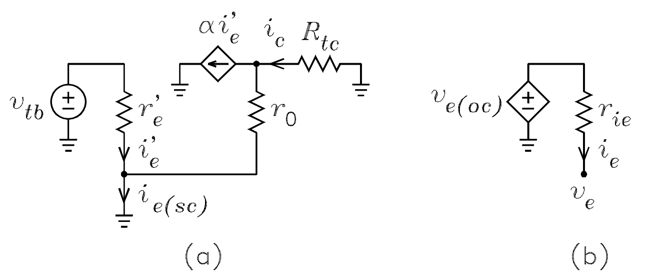

Figure 7.14: (a) Circuit for calculating $i_{e(sc)}$. (b) Thévenin emitter circuit.

7.5.4 Thévenin Base Circuit

Although the base is not an output terminal, the Thévenin equivalent circuit seen looking into the base is useful in calculating the base current. It consists of a voltage source $v_{b(oc)}$ in series with a resistor r_{ib} from the base node to signal ground. Fig. 7.15(a) shows the BJT symbol with a Thévenin source connected to its emitter. Fig. 7.15(b) shows the T model for calculating the open-circuit base voltage. Because $i_b = 0$, it follows that $i'_e = 0$. Thus there is no drop across r_x and r_e so that $v_{b(oc)}$ is given by

$$v_{b(oc)} = v_e = v_{te}\frac{r_0 + R_{tc}}{R_{te} + r_0 + R_{tc}} \tag{7.70}$$

The next step is to solve for the resistance seen looking into the base. It can be calculated by setting $v_{te} = 0$ and connecting a test current source i_t to the base. It is given by $r_{ib} = v_b/i_t$. Fig. 7.16(a) shows the T circuit for calculating v_b, where the current source βi_t has been divided into identical series sources with their common node grounded to simplify use of superposition. By superposition of i_t and the two βi_t sources, we can write

$$v_b = i_t r_x + (i_t + \beta i_t)\left[r_x + r_e + R_{te}\| \left(r_0 + R_{tc}\right)\right] - \beta i_t \frac{R_{tc}R_{te}}{R_{tc} + r_0} \tag{7.71}$$

This can be solved for r_{ib} to obtain

$$r_{ib} = \frac{v_b}{i_t} = r_x + (1 + \beta)\left[r_e + R_{te}\| \left(r_0 + R_{tc}\right)\right] - \frac{\beta R_{tc}R_{te}}{R_{tc} + r_0 + R_{te}} \tag{7.72}$$

The Thévenin base circuit is shown in Fig. 7.16(b).

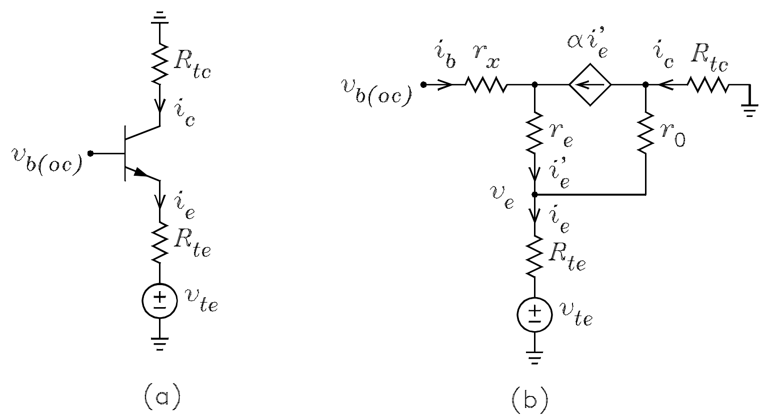

Figure 7.15: (a) BJT with Thevenin source connected to the emitter. (b) T model for calculating $v_{b(oc)}$.

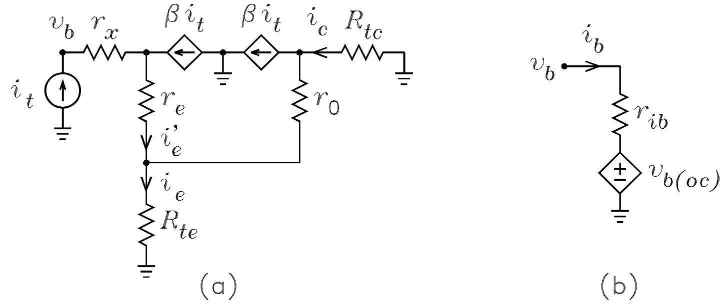

Figure 7.16: (a) Circuit for calculating v_b. (b) Thévenin base circuit.

7.5.5 Summary of Models

Figure 7.17 summarizes the four equivalent circuits derived above.

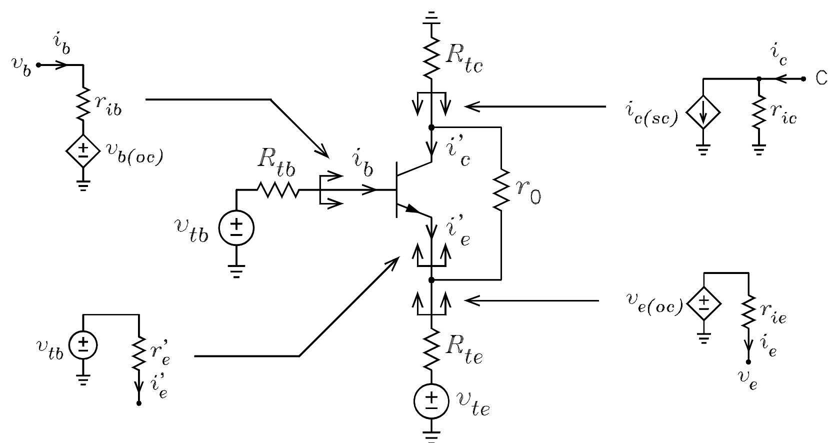

Figure 7.17: Summary of the small-signal equivalent circuits.

7.6 Example Amplifier Circuits

This section describes several examples which illustrate the use of the small-signal equivalent circuits derived above to write by inspection the voltage gain, the input resistance, and the output resistance of both single-stage and two-stage amplifiers.

7.6.1 The Common-Emitter Amplifier

Figure 7.18(a) shows the ac signal circuit of a common-emitter amplifier. We assume that the bias solution and the small-signal resistances r'_e and r_0 are known. The output voltage and output resistance can be calculated by replacing the circuit seen looking into the collector by the Norton equivalent circuit of Fig. 7.12(b). With the aid of this circuit, we can write

$$v_o = -i_{c(sc)} \left(r_{ic} \| R_{tc} \right) = -G_{mb} \left(r_{ic} \| R_{tc} \right) v_{tb} \tag{7.73}$$

$$r_{\text{out}} = r_{ic} \| R_{tc} \tag{7.74}$$

where G_{mb} and r_{ic}, respectively, are given by Eqs. (7.59) and (7.62). The input resistance is given by

$$r_{\text{in}} = R_{tb} + r_{ib} \tag{7.75}$$

where r_{ib} is given by Eq. (7.72).

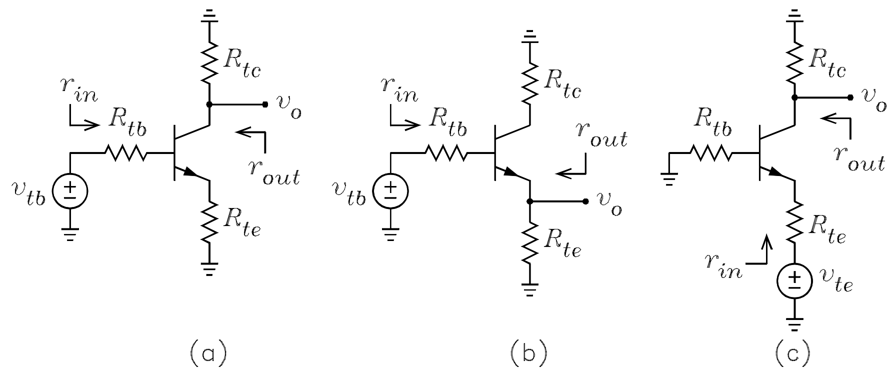

Figure 7.18: (a) Common-emitter amplifier. (b) Common-collector amplifier. (c) Common-base amplifier.

7.6.2 The Common-Collector Amplifier

Figure 7.18(b) shows the ac signal circuit of a common-collector amplifier. We assume that the bias solution and the small-signal resistances r'_e and r_0 are known. The output voltage and output resistance can be calculated by replacing the circuit seen looking into the emitter by the Thévenin equivalent circuit of Fig. 7.14(b). With the aid of this circuit, we can write

$$v_o = v_{e(oc)} \frac{R_{te}}{r_{ie} + R_{te}} = \frac{r_0 + R_{tc}/(1+\beta)}{r'_e + r_0 + R_{tc}/(1+\beta)} \frac{R_{te}}{r_{ie} + R_{te}} v_{tb} \tag{7.76}$$

$$r_{\text{out}} = r_{ie} \| R_{te} \tag{7.77}$$

where r_{ie} is given by Eq. (7.69). The input resistance is given by

$$r_{\text{in}} = R_{tb} + r_{ib} \tag{7.78}$$

where r_{ib} is given by Eq. (7.72).

7.6.3 The Common-Base Amplifier

Figure 7.18(c) shows the ac signal circuit of a common-base amplifier. We assume that the bias solution and the small-signal parameters r'_s and r_0 are known. The output voltage and output resistance can be calculated by replacing the circuit seen looking into the collector by the Norton equivalent circuit of Fig. 7.12(b). The input resistance can be calculated by replacing the circuit seen looking into the emitter by the Thévenin equivalent circuit of Fig. 7.14 with $v_{e(oc)} = 0$. With the aid of this circuit, we can write

$$v_o = -i_{c(sc)} \left(r_{ic} \| R_{tc} \right) = G_{me} \left(r_{ic} \| R_{tc} \right) v_{te} \tag{7.79}$$

$$r_{\text{out}} = r_{ic} \| R_{tc} \tag{7.80}$$

$$r_{\text{in}} = R_{te} + r_{ie} \tag{7.81}$$

where G_{me}, r_{ic}, and r_{ie}, respectively, are given by Eqs. (7.60), (7.62), and (7.69).

7.6.4 The CE/CC Amplifier

Figure 7.19(a) shows the ac signal circuit of a two-stage amplifier consisting of a CE stage followed by a CC stage. Such a circuit is used to obtain a high voltage gain and a low output resistance. The voltage gain can be written

$$\begin{aligned} \frac{v_o}{v_{tb1}} &= \frac{i_{c1(sc)}}{v_{tb1}} \frac{v_{tb2}}{i_{c1(sc)}} \frac{v_{e2(oc)}}{v_{tb2}} \frac{v_o}{v_{e2(oc)}} \\ &= G_{mb1} \left[- \left(r_{ic1} \| R_{C1} \right) \right] \frac{r_0}{r'_{e2} + r_0} \frac{R_{te2}}{r_{ie2} + R_{te2}} \end{aligned} \tag{7.82}$$

where r'_{e2} is calculated with $R_{tb2} = r_{ic1} \| R_{C1}$. The input and output resistances are given by

$$r_{\text{in}} = R_{tb1} + r_{ib1} \tag{7.83}$$

$$r_{\text{out}} = r_{ie2} \| R_{te2} \tag{7.84}$$

Although not a part of the solution, the resistance seen looking out of the collector of Q_1 is $R_{tc1} = R_{C1} \| r_{ib2}$.

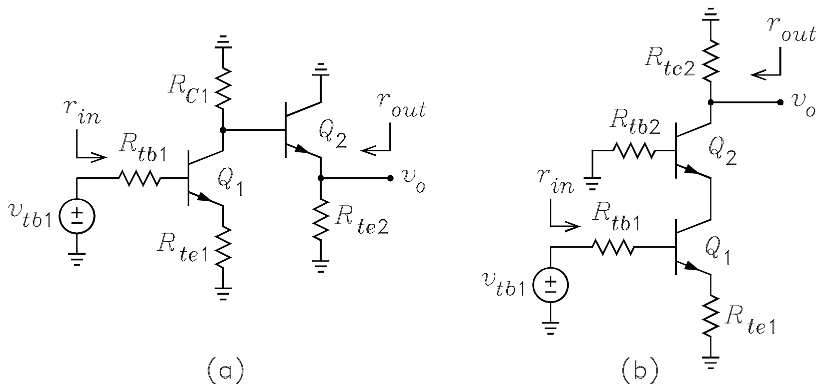

Figure 7.19: (a) CE-CC amplifier. (b) Cascode amplifier.

7.6.5 The Cascode Amplifier

Figure 7.19(b) shows the ac signal circuit of a cascode amplifier. The voltage gain can be written

$$\begin{aligned}\frac{v_o}{v_{tb1}} &= \frac{i_{c1(sc)}}{v_{tb1}} \frac{v_{te2}}{i_{c1(sc)}} \frac{i_{c2(sc)}}{v_{te2}} \frac{v_o}{i_{c2(sc)}} \\ &= G_{m1} \left(-r_{ic1}\right) \left(-G_{me2}\right) \left(-r_{ic2} \| R_{tc2}\right)\end{aligned}$$

where G_{me2} and r_{ic2} are calculated with $R_{te2} = r_{ic1}$. The input and output resistances are given by

$$r_{\text{in}} = R_{tb1} + r_{ib1}$$

$$r_{\text{out}} = R_{tc2} \| r_{ic2}$$

The resistance seen looking out of the collector of Q_1 is $R_{tc1} = r_{ie2}$.

A second cascode amplifier is shown in Fig. 7.20(a) where a pnp transistor is used for the second stage. The voltage gain is given by

$$\begin{aligned}\frac{v_o}{v_{tb1}} &= \frac{i_{c1(sc)}}{v_{tb1}} \frac{v_{te2}}{i_{c1(sc)}} \frac{i_{c2(sc)}}{v_{te2}} \frac{v_o}{i_{c2(sc)}} \\ &= G_{m1} \left(-r_{ic1} \| R_{C1}\right) \left(-G_{me2}\right) \left(-r_{ic2} \| R_{tc2}\right)\end{aligned}$$

where $R_{te2} = R_{C1} \| r_{ic1}$ and $R_{tc1} = R_{C1} \| r_{ie2}$. The expressions for r_{in} and r_{out} are the same as for the cascode amplifier in Fig. 7.19(b).

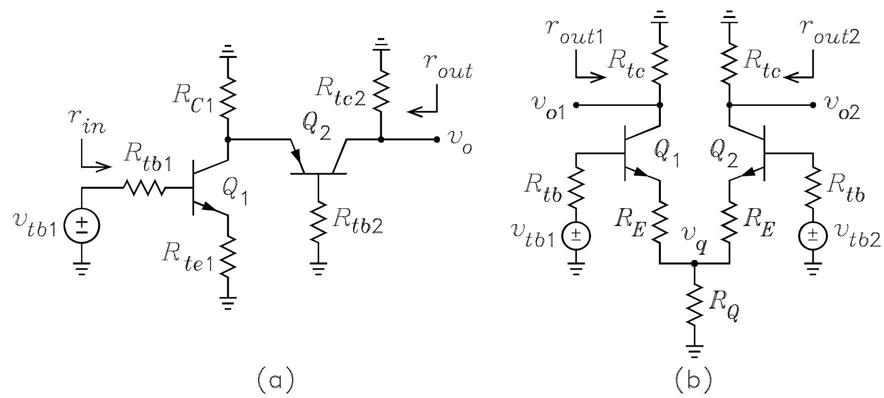

Figure 7.20: (a) Second cascode amplifier. (b) Differential amplifier.

7.6.6 The Differential Amplifier

Figure 7.20(b) shows the ac signal circuit of a differential amplifier. For the case of an active tail bias supply, the resistor R_Q represents its small-signal ac resistance. We assume that the transistors are identical, biased at the same currents and voltages, and have identical small-signal parameters. Looking out of the emitter of Q_1, the Thévenin voltage and resistance are given by

$$\begin{aligned} v_{te1} &= v_{e2(oc)} \frac{R_Q}{R_Q + R_E + r_{ie}} \\ &= v_{tb2} \frac{r_0 + R_{tc}/\left(1+\beta\right)}{r'_e + r_0 + R_{tc}/\left(1+\beta\right)} \frac{R_Q}{R_Q + R_E + r_{ie}} \end{aligned} \tag{7.85}$$

$$R_{te1} = R_E + R_Q \| \left(R_E + r_{ie}\right) \tag{7.86}$$

The small-signal collector voltage of Q_1 is given by

$$\begin{aligned} v_{o1} &= -i_{c1(sc)} \left(r_{ic} \| R_{tc}\right) = -\left(G_{mb} v_{tb1} - G_{me} v_{te1}\right) \left(r_{ic} \| R_{tc}\right) \\ &= -G_{mb} \left(r_{ic} \| R_{tc}\right) v_{tb1} \\ &\quad + G_{me} \frac{r_0 + R_{tc}/\left(1+\beta\right)}{r'_e + r_0 + R_{tc}/\left(1+\beta\right)} \frac{R_Q}{r_{ie} + R_E + R_Q} v_{tb2} \end{aligned} \tag{7.87}$$

By symmetry, v_{o2} is obtained by interchanging the subscripts 1 and 2 in this equation. The small-signal resistance seen looking into either output is

$$r_{\text{out}} = R_{tc} \| r_{ic} \tag{7.88}$$

where r_{ic} calculated from Eq. (7.62) with $R_{te} = R_E + R_Q \| (R_E + r_{ie})$. Although not labeled on the circuit, the input resistance seen by both v_{tb1} and v_{tb2} is $r_{\text{in}} = r_{ib}$.

A second solution of the diff amp can be obtained by replacing v_{tb1} and v_{tb2} with differential and common-mode components as follows:

$$v_{tb1} = v_{i(cm)} + \frac{v_{i(d)}}{2} \tag{7.89}$$

$$v_{tb2} = v_{i(cm)} - \frac{v_{i(d)}}{2} \tag{7.90}$$

where $v_{i(d)} = v_{tb1} - v_{tb2}$ and $v_{i(cm)} = (v_{tb1} + v_{tb2})/2$. Superposition of $v_{i(d)}$ and $v_{i(cm)}$ can be used to solve for v_{o1} and v_{o2}. With $v_{i(cm)} = 0$, the effects of $v_{tb1} = v_{i(d)}/2$ and $v_{tb2} = -v_{i(d)}/2$ are to cause $v_q = 0$. Thus the v_q node can be grounded and the circuit can be divided into two common-emitter stages in which $R_{te(d)} = R_E$ for each transistor. In this case, $v_{o1(d)}$ can be written

$$\begin{aligned} v_{o1(d)} &= \frac{i_{c1(sc)}}{v_{tb1(d)}} \frac{v_{o1(d)}}{i_{c1(sc)}} v_{tb1(d)} = G_{m(d)} \left(-r_{ic(d)} \| R_{tc}\right) \frac{v_{i(d)}}{2} \\ &= G_{m(d)} \left(-r_{ic(d)} \| R_{tc}\right) \frac{v_{tb1} - v_{tb2}}{2} \end{aligned} \tag{7.91}$$

By symmetry $v_{o2(d)} = -v_{o1(d)}$.

With $v_{i(d)} = 0$, the effects of $v_{tb1} = v_{tb2} = v_{i(cm)}$ are to cause the emitter currents in Q_1 and Q_2 to change by the same amounts. If R_Q is replaced by two parallel resistors of value $2R_Q$, it follows by symmetry that the circuit can be separated into two common-emitter stages each with $R_{te(cm)} = R_E + 2R_Q$. In this case, $v_{o1(cm)}$ can be written

$$\begin{aligned} v_{o1(cm)} &= \frac{i_{c1(sc)}}{v_{tb1(cm)}} \frac{v_{o1(cm)}}{i_{c1(sc)}} v_{tb1(cm)} = G_{m(cm)} \left(-r_{ic(cm)} \| R_{tc}\right) v_{i(cm)} \\ &= G_{m(cm)} \left(-r_{ic(cm)} \| R_{tc}\right) \frac{v_{tb1} + v_{tb2}}{2} \end{aligned} \tag{7.92}$$

By symmetry $v_{o2(cm)} = v_{o1(cm)}$.

Because R_{te} is different for the differential and common-mode circuits, G_m, r_{ic}, and r_{ib} are different. However, the total solution $v_{o1} = v_{o1(d)} + v_{o1(cm)}$ is the same as that given by Eq. (7.87), and similarly for v_{o2}. The small-signal base currents can be written $i_{b1} = v_{i(cm)}/r_{ib(cm)} + v_{i(d)}/r_{ib(d)}$ and $i_{b2} = v_{i(cm)}/r_{ib(cm)} - v_{i(d)}/r_{ib(d)}$. If $R_Q \to \infty$, the common-mode solutions are zero. In this case, the differential solutions can be used for the total solutions. If $R_Q \gg R_E + r_{ie}$, the common-mode solutions are often approximated by zero.

7.7 Small-Signal High-Frequency Models

Figure 7.21 shows the hybrid-π and T models for the BJT with the base-emitter capacitance c_π and the base-collector capacitance c_μ added. The capacitor c_{cs} is the collector-substrate capacitance which in present in monolithic integrated-circuit devices but is omitted in discrete devices. These capacitors model charge storage in the device which affects its high-frequency performance. They are given by

$$c_\pi = c_{je} + \frac{\tau_F I_C}{V_T} \tag{7.93}$$

$$c_\mu = \frac{c_{jc}}{[1 + V_{CB}/\phi_C]^{m_c}} \tag{7.94}$$

$$c_{cs} = \frac{c_{jcs}}{[1 + V_{CS}/\phi_C]^{m_c}} \tag{7.95}$$

where I_C is the dc collector current, V_{CB} is the dc collector to base voltage, V_{CS} is the dc collector to substrate voltage, c_{je} is the zero-bias junction capacitance of the base-emitter junction, τ_F is the forward transit time of the base-emitter junction, c_{jc} is the zero-bias junction capacitance of the base-collector junction, c_{jcs} is the zero-bias collector-substrate capacitance, ϕ_C is the built-in potential, and m_c is the junction exponential factor. For integrated circuit lateral pnp transistors, c_{cs} is replaced with a capacitor c_{bs} from base to substrate.

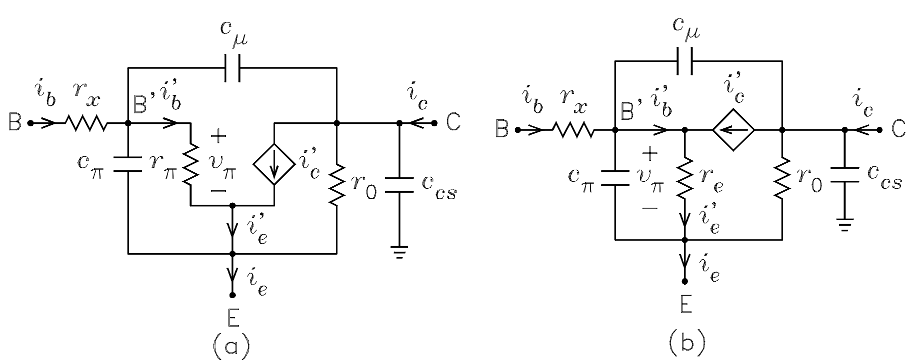

Figure 7.21: High-frequency small-signal models of the BJT. (a) Hybrid-π model. (b) T model.

In these models, the currents are related by

$$i'_c = g_m v_\pi = \beta i'_b = \alpha i'_e \tag{7.96}$$

These relations are the same as those in Eq. (7.51) with i_b replaced with i_b'.

7.8 BJT Noise Model

The principle noise sources in a BJT are thermal noise in the base spreading resistance r_x, shot noise and flicker noise in the base bias current I_B, and shot noise in the collector bias current I_C. Fig. 7.22 shows the BJT symbols with the noise sources added. The base spreading resistance r_x is modeled as an external resistor in series with the base. The polarity of the noise sources is arbitrary. Each has been chosen so as to cause an increase in the collector current.

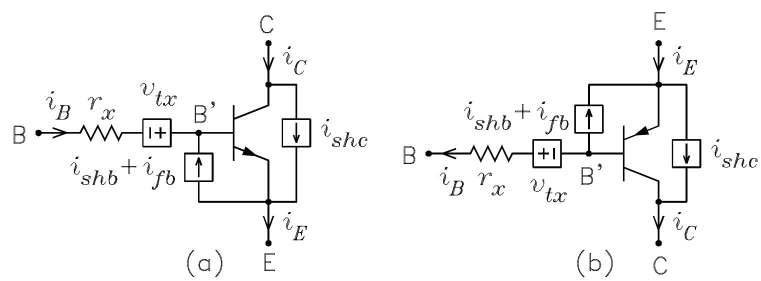

Figure 7.22: BJT circuit symbols with noise sources added.

The source v_{tx} in Fig. 7.22 models the thermal noise in r_x. The shot noise and flicker noise in the base bias current I_B are modeled by $i_{shb} + i_{fb}$. The shot noise in the collector bias current I_C is modeled by i_{shc}. In the band Δf, these have the mean-square values

$$\overline{v_{tx}^2} = 4kTr_x\Delta f \tag{7.97}$$

$$\overline{i_{shb}^2} = 2qI_B\Delta f \tag{7.98}$$

$$\overline{i_{fb}^2} = \frac{K_f I_B \Delta f}{f} \tag{7.99}$$

$$\overline{i_{shc}^2} = 2qI_C\Delta f \tag{7.100}$$

The flicker noise can increase significantly if the base-emitter junction is subjected to reverse breakdown. This might occur during power supply turn-on or by the application of too large an input voltage. A normally reverse-biased diode in parallel with the base-emitter junction is often used to prevent it.

7.8.1 Noise Equivalent Input Voltage

Figure 7.23 shows the npn BJT with the collector connected to signal ground. The external base and emitter circuits are modeled by Thévenin equivalent circuits. With $v_2 = 0$, the circuit models a common-emitter or CE stage. With $v_1 = 0$, it models a common-base or CB stage. The noise sources v_{t1}, and v_{t2}, respectively, model the thermal noise in R_1 and R_2. In the band Δf, these have the mean-square values

$$\overline{v_{t1}^2} = 4kTR_1\Delta f \tag{7.101}$$

$$\overline{v_{t2}^2} = 4kTR_2\Delta f \tag{7.102}$$

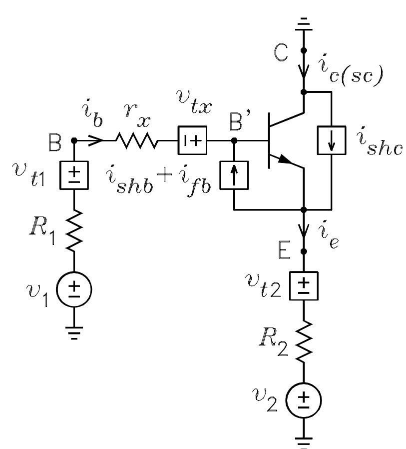

Figure 7.23: BJT noise model with external base and emitter circuits model by Thévenin sources.

The short-circuit collector output current can be written

$$i_{c(sc)} = G_{mb}v_{tb} - G_{me}v_{te} + i_{shc} \tag{7.103}$$

where G_{mb} and G_{me} are given by Eqs.(7.59) and (7.60) with $R_{tb} = R_1$, $R_{te} = R_2$, and

$$v_{tb} = v_1 + v_{t1} + v_{tx} + (i_{shb} + i_{fb})(R_1 + r_x) \tag{7.104}$$

$$v_{te} = v_2 + v_{t2} + (i_{shc} - i_{shb} - i_{fb})R_2 \tag{7.105}$$

The equivalent noise input voltage v_{ni} can be expressed as a voltage in series with either v_1 or v_2. Because the common-emitter amplifier is most often seen, we will express it as a voltage in series with v_1. First, we factor G_{mb} from Eq. (7.103) to obtain

$$\begin{aligned} i_{c(\mathrm{sc})} &= G_{mb}\Bigg\{v_1 + v_{t1} + v_{tx} + (i_{shb} + i_{fb})(R_1 + r_x) \\ &\quad - \frac{G_{me}}{G_{mb}}\left[v_2 + v_{t2} + (i_{shc} - i_{shb} - i_{fb}) R_2\right] \\ &\quad + \frac{i_{shc}}{G_{mb}}\Bigg\} \end{aligned} \tag{7.106}$$

The equivalent noise voltage in series with v_1 is given by all terms in the brackets except the v_1 and v_2 terms. It is given by

$$\begin{aligned} v_{ni} &= v_{t1} + v_{tx} - v_{t2}\frac{G_{me}}{G_{mb}} + (i_{shb} + i_{fb})\left[R_1 + r_x + R_2\frac{G_{me}}{G_{mb}}\right] \\ &\quad + \frac{i_{shc}}{G_{mb}}(1 - G_{me}R_2) \end{aligned} \tag{7.107}$$

After some algebra, this can be reduced to

$$\begin{aligned} v_{ni} &= v_{t1} + v_{tx} - v_{t2}\frac{r_0 + r_e'/\alpha}{r_0 - R_2/\beta} \\ &\quad + (i_{shb} + i_{fb})\left[R_1 + r_x + R_2\frac{r_0 + r_e'/\alpha}{r_0 - R_2/\beta}\right] \\ &\quad + i_{shc}\frac{r_0}{r_0 - R_2/\beta}\left[\frac{R_1 + r_x + R_2}{\beta} + \frac{V_T}{I_C}\right] \end{aligned} \tag{7.108}$$

To express v_{ni} as a voltage in series with v_2, this equation is multiplied by G_{mb}/G_{me}.

The mean-square value of v_{ni} is given by

$$\begin{aligned} \overline{v_{ni}^2} &= \overline{v_{t1}^2} + \overline{v_{tx}^2} + \overline{v_{t2}^2}\left(\frac{r_0 + r_e'/\alpha}{r_0 - R_2/\beta}\right)^2 \\ &\quad + \left(\overline{i_{shb}^2} + \overline{i_{fb}^2}\right)\left(R_1 + r_x + R_2\frac{r_0 + r_e'/\alpha}{r_0 - R_2/\beta}\right)^2 \\ &\quad + \overline{i_{shc}^2}\left(\frac{r_0}{r_0 - R_2/\beta}\right)^2\left(\frac{R_1 + r_x + R_2}{\beta} + \frac{V_T}{I_C}\right)^2 \end{aligned} \tag{7.109}$$

which reduces to

$$\begin{aligned}
\overline{v_{ni}^2} &= 4kT\left[R_1 + r_x + R_2\left(\frac{r_0 + r_e'/\alpha}{r_0 - R_2/\beta}\right)^2\right]\Delta f \\
&+ \left(2qI_B\Delta f + \frac{K_f I_B \Delta f}{f}\right)\left(R_1 + r_x + R_2\frac{r_0 + r_e'/\alpha}{r_0 - R_2/\beta}\right)^2 \\
&+ 2qI_C\Delta f\left(\frac{r_0}{r_0 - R_2/\beta}\right)^2\left(\frac{R_1 + r_x + R_2}{\beta} + \frac{V_T}{I_C}\right)^2 \qquad (7.110)
\end{aligned}$$

This expression gives the mean-square equivalent noise input voltage for the CE amplifier. To obtain $\overline{v_{ni}^2}$ for the CB amplifier, this expression is multiplied by $(G_{mb}/G_{me})^2$.

Equation (7.110) can be simplified if it is assumed that $r_0 \gg R_2/\beta$ and $r_0 \gg r_e'/\alpha$, i.e. we assume the r_0 approximations hold. With these approximations, Eq. (7.110) can be written

$$\begin{aligned}
\overline{v_{ni}^2} &= 4kT\left(R_1 + r_x + R_2\right)\Delta f \\
&+ \left(2qI_B\Delta f + \frac{K_f I_B \Delta f}{f}\right)\left(R_1 + r_x + R_2\right)^2 \\
&+ 2qI_C\Delta f\left(\frac{R_1 + r_x + R_2}{\beta} + \frac{V_T}{I_C}\right)^2 \qquad (7.111)
\end{aligned}$$

This approximation applies to both the CE and the CB amplifiers. It is used in the following to obtain the optimum bias current for the BJT and in comparing the CE and CB amplifiers.

7.8.2 $v_n - i_n$ Noise Models

Two $v_n - i_n$ noise models of the BJT are shown in Fig. 7.24. The model of Fig. 7.24(a) assumes that r_x is an external resistor in series with the base. The asterisk indicates that r_x is to be considered noiseless. The model of Fig. 7.24(b) assumes that r_x is internal to the BJT. Both models are solved for below. We use Eq. (7.108) to solve for the values of v_n and i_n for both models. Because v_{ni} given by this equation is the voltage in series with the base, R_1 must be considered to be the resistance of the signal source. In the $v_n - i_n$ model, the i_n noise source connects between signal ground and the input. Therefore, R_2 must be set to zero in the circuit to solve for i_n. Otherwise, R_2 would appear in the model and i_n would connect from the base to the lower node of R_2.

In this case, Eq. (7.108) becomes

$$v_{ni} = v_{t1} + v_{tx} + (i_{shb} + i_{fb})(R_1 + r_x) + i_{shc}\left(\frac{R_1 + r_x}{\beta} + \frac{V_T}{I_C}\right) \quad (7.112)$$

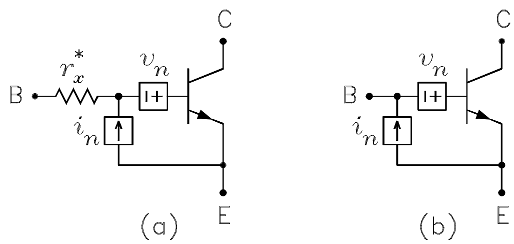

Figure 7.24: Two $v_n - i_n$ amplifier models of the BJT.

First Model

For the first model, we write Eq. (7.112) in the form

$$v_{ni} = v_{ts} + v_n + i_n (R_s + r_x) \quad (7.113)$$

where $v_{ts} = v_{t1}$ and $R_s = R_1$. It follows that v_n and i_n are given by

$$v_n = v_{tx} + i_{shc}\frac{V_T}{I_C} \quad (7.114)$$

$$i_n = i_{shb} + i_{fb} + \frac{i_{shc}}{\beta} \quad (7.115)$$

These expressions can be converted into mean-square sums to obtain

$$\overline{v_n^2} = 4kTr_x\Delta f + 2qI_C\Delta f\left(\frac{V_T}{I_C}\right)^2 = 4kTr_x\Delta f + 2kT\frac{V_T}{I_C}\Delta f \quad (7.116)$$

$$\overline{i_n^2} = 2qI_B\Delta f + \frac{K_f I_B \Delta f}{f} + 2q\frac{I_C}{\beta^2}\Delta f \quad (7.117)$$

Because i_{shc} appears in the expressions for both v_n and i_n, the correlation coefficient is not zero. It is given by

$$\rho = \frac{2kT\Delta f}{\beta\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}} \quad (7.118)$$

The first form of the $v_n - i_n$ BJT noise model is shown in Fig. 7.24(a). The asterisk indicates that the base spreading resistance r_x^* is considered to be a noiseless resistor. It's noise is included in the expression for $\overline{v_n^2}$.

Second Model

For the second model, we write Eq. (7.112) in the form

$$v_{ni} = v_{ts} + v_n + i_n R_s \tag{7.119}$$

where $v_{ts} = v_{t1}$ and $R_s = R_1$. It follows that v_n and i_n are given by

$$v_n = v_{tx} + \left(i_{shb} + i_{fb} + \frac{i_{shc}}{\beta}\right) r_x + i_{shc}\frac{V_T}{I_C} \tag{7.120}$$

$$i_n = i_{shb} + i_{fb} + \frac{i_{shc}}{\beta} \tag{7.121}$$

These expressions can be converted into mean-square sums to obtain

$$\begin{aligned} \overline{v_n^2} \ &= \ 4kTr_x\Delta f + \left(2qI_B\Delta f + \frac{K_f I_B \Delta f}{f}\right) r_x^2 \\ &+2qI_C\Delta f\left(\frac{r_x}{\beta} + \frac{V_T}{I_C}\right)^2 \end{aligned} \tag{7.122}$$

$$\overline{i_n^2} = 2qI_B\Delta f + \frac{K_f I_B}{f}\Delta f + \frac{2qI_C}{\beta^2}\Delta f \tag{7.123}$$

In this case, i_{shb}, i_{fb}, and i_{shc} appear in the expressions for both v_n and i_n. The correlation coefficient is given by

$$\begin{aligned} \rho \ &= \ \frac{1}{\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}}\left[\left(2qI_B\Delta f + \frac{K_f I_B\Delta f}{f}\right) r_x \right. \\ &\left. +2q\frac{I_C}{\beta}\Delta f\left(\frac{r_x}{\beta} + \frac{V_T}{I_C}\right)\right] \end{aligned} \tag{7.124}$$

The second form of the $v_n - i_n$ BJT noise model is shown in Fig. 7.24(b). The first form has the simplest equations.

7.8.3 Parallel BJTs

One method of reducing the thermal noise generated by r_x is to operate several BJTs in parallel. Compared to a single BJT biased at the collector current I_C, N identical BJTs operated in parallel, each with a collector current I_C/N, generate the same shot and flicker noise as the single transistor, but the effective value of r_x is reduced to r_x/N.

7.8.4 Flicker Noise Corner Frequency

The expression for $\overline{i_n^2}$ in each form of the BJT $v_n - i_n$ noise model is the same. The equations predict that a plot of the spectral density $\overline{i_n^2}/\Delta f$ versus frequency would exhibit a slope of -10 dB/decade at low frequencies and a slope of zero at higher frequencies. Such a plot is shown in Fig. 7.25 The flicker noise corner frequency is the frequency at which $\overline{i_n^2}$ is up 3 dB compared to its higher frequency value. This is the frequency for which the center term in Eqs. (7.117) and (7.123) is equal to the sum of the first and last terms. It is given by

$$f_{\text{flk}} = \frac{K_f}{2qI_B\,(1 + 1/\beta)} \tag{7.125}$$

For $f > f_{\text{flk}}$, the thermal noise dominates. For $f < f_{\text{flk}}$, the flicker noise dominates. An experimental method for determining the flicker noise coefficient K_f for a BJT is to measure the flicker noise corner frequency and use Eq. (7.125) to calculate K_f.

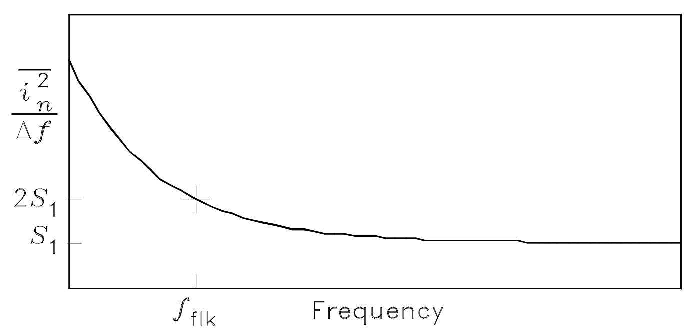

Figure 7.25: Plot of $\overline{i_n^2}/\Delta f$ as a function of frequency.

7.8.5 Methods of Measuring r_x

Three methods for measuring the base spreading resistance are described below. The first two require measuring data at several points. A modification of the second method is based on a single measurement. In addition, a method for measuring $\overline{i_{c(\text{sc})}^2}$ is described.

Method 1

Consider a common-emitter amplifier with a source resistance R_1 in series with the base and with $R_2 = 0$. At very low frequencies, the flicker noise dominates and $\overline{v_{ni}^2}$ is approximately given by

$$\overline{v_{ni}^2} = \frac{K_f I_B \Delta f}{f} (R_1 + r_x)^2 \tag{7.126}$$

The corresponding noise factor is

$$F = \frac{\overline{v_{ni}^2}}{4kTR_1\Delta f} = \frac{K_f I_B}{4kTf}\left(R_1 + 2r_x + \frac{r_x^2}{R_1}\right) \tag{7.127}$$

The value of R_1 which minimizes F is found by setting $\partial F/\partial R_1 = 0$ and is given by $R_1 = r_x$. It follows that r_x can be measured by determining the value of R_1 which minimizes the value of F at low frequencies. The value of R_1 is then equal to r_x.

Method 2

Consider a common-emitter amplifier with $R_1 = R_2 = 0$. Above the flicker noise range, the mean-square, short-circuit collector output current can be written

$$\begin{aligned}
\overline{i_{c(\mathrm{sc})}^2} &= 2qI_C\Delta f + \left[\frac{\alpha}{r_x/(1+\beta) + r_e}\right]^2 \left(4kTr_x\Delta f + 2qI_B\Delta f r_x^2\right) \\
&\simeq 2qI_C\Delta f + \left(\frac{I_C}{V_T}\right)^2 \left(4kTr_x\Delta f + 2qI_B\Delta f r_x^2\right) \\
&= \frac{4q\Delta f}{V_T}\left(\frac{I_C V_T}{2} + I_C^2 r_x + \frac{I_C^3 r_x^2}{2\beta V_T}\right)
\end{aligned} \tag{7.128}$$

where the approximation assumes that $r_e \gg r_x/(1+\beta)$. It follows from Eq. (7.128) that

$$\frac{\overline{i_{c(\mathrm{sc})}^2}}{I_C^2} \times \frac{V_T}{4q\Delta f} = \frac{V_T}{2I_C} + r_x + \frac{I_C r_x^2}{2\beta V_T} \tag{7.129}$$

If this is plotted as a function of I_C, the curve exhibits a minimum at $I_C = V_T\sqrt{\beta}/r_x$. At this current, we have

$$\left.\frac{\overline{i_{c(\mathrm{sc})}^2}}{I_C^2}\right|_{\min} \times \frac{V_T}{4q\Delta f} = r_x\left(1 + \frac{1}{2\sqrt{\beta}}\right) \tag{7.130}$$

This equation can be solved for r_x. For large β, the solution is relatively independent of β. A source of error with this method is the temperature variation of the transistor with its bias current, thus causing V_T to change with bias current.

Figure 7.26 illustrates the theoretical variation of $\overline{i_{c(\mathrm{sc})}^2} V_T/4q\Delta f$ and $\left(\overline{i_{c(\mathrm{sc})}^2}/I_C^2\right)(V_T/4q\Delta f)$ with I_C for $\Delta f = 1$ Hz. The assumed transistor parameters are $\beta = 100$ and $r_x = 60\ \Omega$. It can be seen from Fig. 7.26(b) that the range over which $\left(\overline{i_{c(\mathrm{sc})}^2}/I_C^2\right)(V_T/4q\Delta f)$ is a minimum is fairly broad, thus making it difficult to experimentally determine the value of I_C at the minimum. A possible method of improving the accuracy would be to perform a least squares curve fit of Eq. (7.130) to measured data.

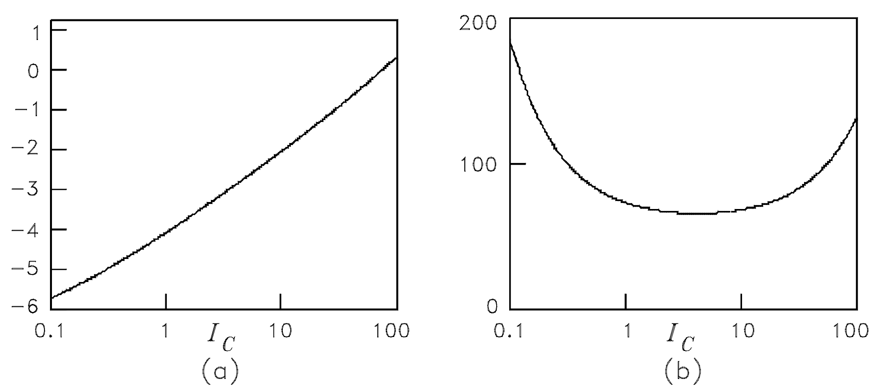

Figure 7.26: (a) $\log\left(\overline{i_{c(\mathrm{sc})}^2} \times V_T/4q\Delta f\right)$ versus I_C in mA. (b) $\left(\overline{i_{c(\mathrm{sc})}^2}/I_C^2\right)(V_T/4q\Delta f)$ versus I_C in mA.

Method 3

A variation of Method 2 is to measure $\overline{i_{c(\mathrm{sc})}^2} \times (V_T/4q\Delta f)$ for a single value of I_C and solve Eq. (7.128) for r_x. The solution is

$$r_x = \frac{\beta V_T}{I_C}\left\{\left[1 + \frac{1}{\beta}\left(\frac{\overline{i_{c(sc)}^2}}{8qI_C\Delta f} - 1\right)\right]^{1/2} - 1\right\} \tag{7.131}$$

This method eliminates the potential error caused by temperature variations for different values of I_C.

Measuring $\overline{i^2_{c(\mathrm{sc})}}$

Figure 7.27 shows a circuit for measuring $\overline{i^2_{c(\mathrm{sc})}}$. An op amp is connected as a current to voltage converter. We assume the capacitors are short circuits for the noise signals. Because the collector of the BJT connects to a virtual ac signal ground, the signal voltage at the collector is zero. Therefore, the small-signal collector current is the Norton current $i_{c(\mathrm{sc})}$. The dc emitter bias current is given by

$$I_E = \frac{-V_{BE} - V^-}{R_E} \tag{7.132}$$

The small-signal voltage at the inverting op amp input is given by

$$v_- = v_o \frac{R_C}{R_F + R_C} - v_{ni} - i_{c(\mathrm{sc})} R_C \| R_F \tag{7.133}$$

where v_{ni} is the equivalent noise input voltage of the op amp, including the thermal noise of R_C and R_F. Because $v_- = 0$, this equation can be solved for v_o to obtain

$$v_o = v_{ni}\left(1 + \frac{R_F}{R_C}\right) + i_{c(\mathrm{sc})} R_F \tag{7.134}$$

Because the noise sources are not correlated, the contribution due to $i_{c(\mathrm{sc})}$ can be obtained by first measuring $\overline{v_o^2}$ with the transistor removed. Denote this measurement by $\overline{v_{o1}^2}$. Let the value with the transistor in the circuit be $\overline{v_{o2}^2}$. It follows that $\overline{i^2_{c(\mathrm{sc})}}$ is given by

$$\overline{i^2_{c(\mathrm{sc})}} = \frac{\overline{v_{o2}^2} - \overline{v_{o1}^2}}{R_F^2} \tag{7.135}$$

We assume here that $\overline{v_{ni}^2}$ is the same with the transistor in or out of the circuit. This is strictly true only for $r_0 \gg R_C \| R_F$, where r_0 is the collector-emitter resistance of the transistor. This is because the op amp i_n noise current flows through $R_C \| R_F$ with the transistor removed and through $r_0 \| R_C \| R_F$ in the circuit. The condition that both capacitors be ac short circuits in the band over which the noise is measured are

$$C_E \gg \frac{1}{2\pi f r_e} \tag{7.136}$$

$$C_C \gg \frac{1}{2\pi f R_C} \tag{7.137}$$

where f is the lowest frequency of interest and $r_e = V_T / I_E$ is the lower bound on the small-signal resistance seen by C_E.

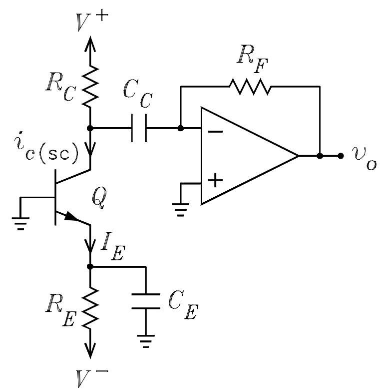

Figure 7.27: Test circuit for measuring $\overline{i^2_{c(\mathrm{sc})}}$.

7.8.6 Optimum Bias Current

Let us assume in Eq. (7.110) that the frequency is high enough so that flicker noise can be neglected. In addition, let us assume that r_0 is large enough so that the approximation $r_0 \to \infty$ can be used. With these approximations, $\overline{v^2_{ni}}$ can be written

$$\begin{aligned}\overline{v^2_{ni}} &= 4kT\left(R_1 + r_x + R_2\right)\Delta f + 2q\frac{I_C}{\beta}\Delta f\left(R_1 + r_x + R_2\right)^2 \\ &\quad +2qI_C\Delta f\left(\frac{R_1 + r_x + R_2}{\beta} + \frac{V_T}{I_C}\right)^2 \qquad (7.138)\end{aligned}$$

It can be seen that $\overline{v^2_{ni}} \to \infty$ if $I_C \to 0$ or if $I_C \to \infty$. It follows that there is a value of I_C which minimizes $\overline{v^2_{ni}}$. This current is called the optimum collector bias current and it is denoted by $I_{C(\mathrm{opt})}$. It is obtained by setting $d\overline{v^2_{ni}}/dI_C = 0$ and solving for I_C. It is given by

$$I_{C(\mathrm{opt})} = \frac{V_T}{R_1 + r_x + R_2} \times \frac{\beta}{\sqrt{1+\beta}} \qquad (7.139)$$

When the BJT is biased at $I_{C(\mathrm{opt})}$, let the equivalent noise input voltage be denoted by $\overline{v^2_{ni(\min)}}$. It is given by

$$\overline{v^2_{ni(\min)}} = 4kT\left(R_1 + r_x + R_2\right)\Delta f \times \frac{\sqrt{1+\beta}}{\sqrt{1+\beta} - 1} \qquad (7.140)$$

For minimum noise, this equation shows that the series resistance in the external base and emitter circuits should be minimized and that the BJT should have a small r_x and a high β.

Although $\overline{v_{ni(\min)}^2}$ decreases as β increases, the sensitivity is not that great for the range of β for most BJTs. Fig. 7.28 shows a plot of the dB change in $\overline{v_{ni(\min)}^2}$ a function of β for $100 \leq \beta \leq 10000$, where the 0 dB reference level corresponds to $\beta = 100$. Most BJTs have a β in the range $100 \leq \beta \leq 1000$. As β increases over this range, $\overline{v_{ni(\min)}^2}$ decreases by 0.32 dB. Superbeta transistors have a β in the range $1000 \leq \beta \leq 10000$. As β increases over this range, $\overline{v_{ni(\min)}^2}$ decreases by only 0.096 dB. It can be concluded that only a slight improvement in noise performance can be expected by using higher β BJTs when the device is biased at $I_{C(\text{opt})}$.

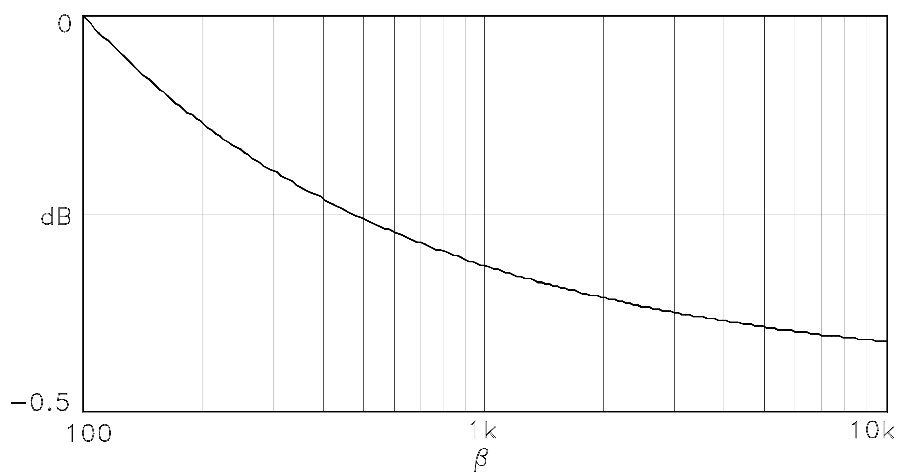

Figure 7.28: dB change in $\overline{v_{ni(min)}^2}$ as a function of β.

If $I_C \neq I_{C(\text{opt})}$, $\overline{v_{ni}^2}$ can be written

$$\overline{v_{ni}^2} = \overline{v_{ni(\min)}^2}\left(1 + \frac{0.5\left(I_C/I_{C(\text{opt})} + I_{C(\text{opt})}/I_C\right) - 1}{1 + \sqrt{1+\beta}}\right) \tag{7.141}$$

Example plots of $\overline{v_{ni}^2}/\overline{v_{ni(\min)}^2}$ versus $I_C/I_{C(\text{opt})}$ are given in Fig. 7.29, where a log scale is used for the horizontal axis. Curve (a) is for $\beta = 10000$, curve (b) is for $\beta = 1000$, and curve (c) is for $\beta = 100$. The plots exhibit even symmetry about the vertical line defined by $I_C/I_{C(\mathrm{opt})} = 1$. This means, for example, that $\overline{v_{ni}^2}$ is the same for $I_C = I_{C(\mathrm{opt})}/2$ as for $I_C = 2I_{C(\mathrm{opt})}$. The figure shows that the sensitivity of $\overline{v_{ni}^2}$ to changes in I_C decreases as β increases. For example, at $I_C = I_{C(\mathrm{opt})}/2$ and $I_C = 2I_{C(\mathrm{opt})}$, $\overline{v_{ni}^2}$ is greater than $\overline{v_{ni(\min)}^2}$ by 0.097 dB for $\beta = 100$, by 0.033 dB for $\beta = 1000$, and by 0.010 dB for $\beta = 10000$.

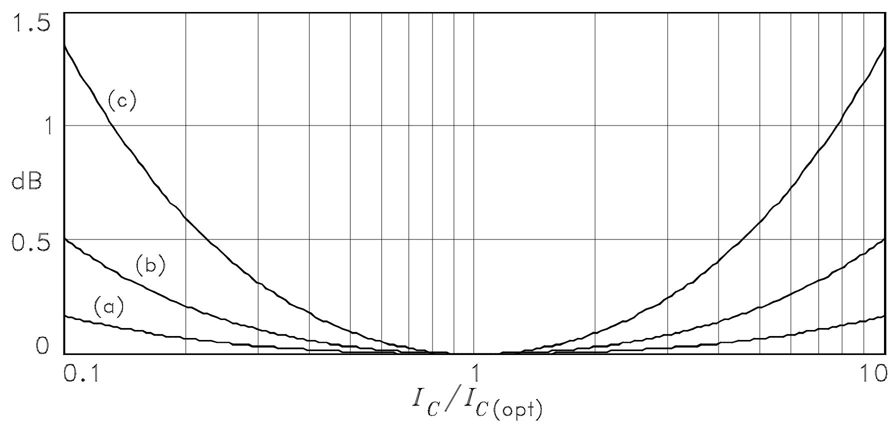

Figure 7.29: dB change in $\overline{v_{ni}^2}$ as a function of $I_C/I_{C(opt)}$.

7.9 Comparison of CE and CB Stages

When $r_0 \gg r_e'/\alpha$ and $r_0 \gg R_2/\beta$, the above analysis shows that the noise performance of the CE amplifier is the same as the CB amplifier. If the noise generated by the following stage cannot be neglected, the CE amplifier can have the lower noise. In this section, we investigate the effect of the noise generated by the following stage. We assume that r_0 is large enough so that the collector current can be calculated with Eqs. (7.63) and (7.64), i.e. we use the r_0 approximations. Let the second stage be modeled by a $v_n - i_n$ amplifier noise model having the noise sources v_{n2} and i_{n2} and the correlation coefficient ρ_2. Let v_{ni} be the new noise equivalent input voltage when the second stage noise is included. Following Eq. (4.49), this is given by

$$v_{ni} = v_{ni1} + \frac{v_{n2}}{G_{m1} r_{ic1}} + \frac{i_{n2}}{G_{m1}} \tag{7.142}$$

where v_{ni1} is the equivalent noise input voltage of the first stage and r_{ic} is its collector output resistance. It follows from this equation that the v_{n2} contribution is the lowest for the first-stage configuration that exhibits the largest $G_{m1}r_{ic1}$ product. The i_{n2} contribution is lowest for the first-stage configuration which exhibits the largest G_{m1}.

For a CE first stage, let $R_1 = R_s$ and $R_2 = 0$, where R_s is the source resistance. For a CB first stage, let $R_1 = 0$ and $R_2 = R_s$. If we use Eq. (7.62) for r_{ic1} and Eq. (7.64) for G_{m1}, it follows that the ratio of the $G_{m1}r_{ic1}$ products for the two configurations is

$$\frac{G_{m1(\mathrm{CE})}r_{ic1(\mathrm{CE})}}{G_{m1(\mathrm{CB})}r_{ic1(\mathrm{CB})}} = \frac{r_0}{r_0 + [r_e + r_x/\,(1+\beta)]\,\|R_s} \simeq 1 \tag{7.143}$$

This result shows that the noise contributed by v_{n2} is approximately the same for the two configurations. The ratio of the G_{m1} terms for the two configurations is

$$\frac{G_{m1(\mathrm{CE})}}{G_{m1(\mathrm{CB})}} = \frac{r_x/\,(1+\beta) + r_e + R_s}{(R_s + r_x)\,/\,(1+\beta) + r_e} \tag{7.144}$$

For $R_s = 0$, this ratio is unity. In this case, the noise contributed by i_{n2} is the same for the two configurations. For R_s large, the ratio approaches $(1+\beta)$. In this case, the effect of the second stage i_n noise on the CB amplifier is greater than on the CE amplifier. Thus the CE amplifier is the preferred topology for low-noise applications when the source resistance is not small.

7.10 CC Stage

Figure 7.30 shows the circuit diagram of a CC amplifier with its output connected to the input of a second stage that is modeled with the $v_n - i_n$ amplifier noise model. For a small-signal ac analysis, the bias sources are not shown. The resistors r_x and r_0 and all BJT noise sources are shown external to the BJT. The source i_{t2} models the thermal noise current in R_2. If the circuit seen looking to the left of Z_{i2} is replaced with a Norton equivalent circuit, it follows that the voltage across R_{i2} is proportional to the Norton current in the circuit. This is the short-circuit current through R_{i2}, i.e. the current i_{i2} evaluated with $R_{i2} = 0$. It is given by

$$\begin{aligned} i_{i2(\mathrm{sc})} &= i_e' - (i_{shb} + i_{fb}) + i_{shc} + i_{t2} + \frac{v_{n2}}{R_2 \| r_0} + i_{n2} \\ &= \frac{v_1 + v_{t1} + v_{tx} + (i_{shb} + i_{fb})(R_1 + r_x) + v_{n2}}{r_e'} \\ &\quad - (i_{shb} + i_{fb}) + i_{shc} + i_{t2} + \frac{v_{n2}}{R_2 \| r_0} + i_{n2} \end{aligned} \tag{7.145}$$

where $i_e' = v_{tb}/r_e'$ has been used and r_e' is given by Eq. (7.53). It follows that the equivalent noise input voltage is given by

$$\begin{aligned} v_{ni} &= v_{t1} + v_{tx} + v_{n2}\left(1 + \frac{r_e'}{R_2 \| r_0}\right) \\ &\quad + (i_{shb} + i_{fb})\left(R_1 + r_x - r_e'\right) \\ &\quad + (i_{shc} + i_{t2} + i_{n2})\, r_e' \end{aligned} \tag{7.146}$$

This can be converted into a mean-square sum to obtain

$$\begin{aligned} \overline{v_{ni}^2} &= 4kT(R_1 + r_x)\Delta f + \overline{v_{n2}^2}\left(1 + \frac{r_e'}{R_2 \| r_0}\right)^2 \\ &\quad + 2\rho_2 \sqrt{\overline{v_{n2}^2}}\sqrt{\overline{i_{n2}^2}}\left(1 + \frac{r_e'}{R_2 \| r_0}\right) r_e' \\ &\quad + \left(2qI_B\Delta f + \frac{K_f I_B \Delta f}{f}\right)\left(R_1 + r_x - r_e'\right)^2 \\ &\quad + \left(2qI_C\Delta f + \frac{4kT\Delta f}{R_2} + \overline{i_{n2}^2}\right) r_e'^2 \end{aligned} \tag{7.147}$$

where ρ_2 is the correlation coefficient between v_{n2} and i_{n2}.

Compared to the second stage by itself, it can be seen from Eq. (7.147) that the CC stage increases the second-stage v_n noise. The second-stage i_n noise is decreased if $r_e' < R_1$. The noise voltage generated by the transistor base shot and flicker noise currents can be cancelled if $r_e' = R_1 + r_x$. For $R_2 >> 2V_T/I_C$, the thermal noise generated by R_2 can be neglected compared to the shot noise in I_C.

7.11 The Diff Amp

Figure 7.31(a) shows a BJT diff amp circuit. For an ac small-signal analysis, the bias sources are not shown. It is assumed that the BJTs

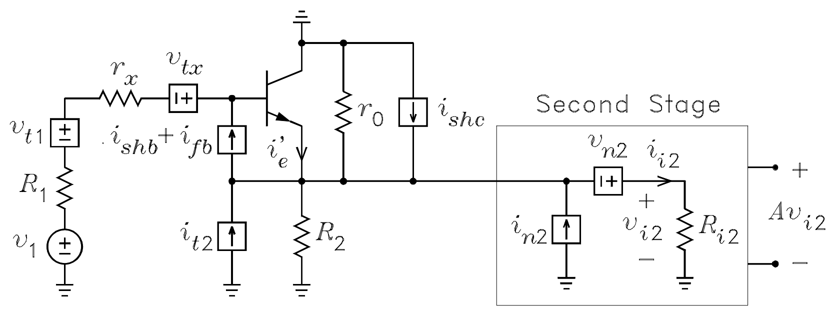

Figure 7.30: Common-collector stage.

are matched and biased at equal currents. The emitter resistors labeled R_2 are included for completeness. For lowest noise, these should be small compared to R_s. The source i_{nq} models the noise current generated by the tail current source and the resistor r_q models its small-signal output resistance.

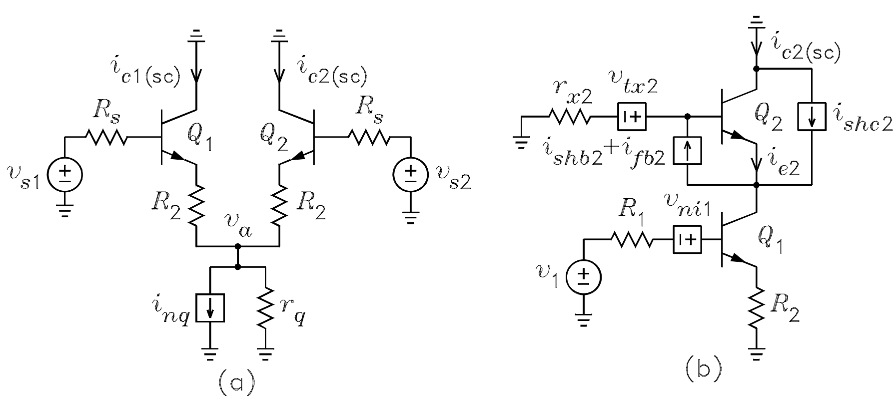

Figure 7.31: (a) Differential amplifier. (b) Cascode amplifier.

For minimum noise output from the diff-amp, the output must be proportional to the differential short-circuit output current $i_{od(\mathrm{sc})} = i_{c1(\mathrm{sc})} - i_{c2(\mathrm{sc})}$. The subtraction cancels the common-mode output noise generated by i_{nq}. Although a current-mirror active load can be used to realize the subtraction, the lowest noise performance is obtained with a resistive load. With a resistive load on each collector, a second diff-amp is required to subtract the output signals. The analysis presented here assumes that the output is taken differentially. In addition, it is assumed that r_q is large enough so that it can be approximated by an open circuit in the small-signal analysis. This is equivalent to the assumption of a high common-mode rejection ratio.

The simplest method to calculate the equivalent noise input voltage of the diff amp is to exploit symmetry by resolving all sources into their differential and common-mode components. The common-mode components are all canceled when the output is taken differentially. Therefore, only the differential components are required. When the sources are replaced by their differential components, the node labeled v_a in Fig. 7.31(a) can be grounded. This decouples the diff amp into two CE stages.

Consider the effect of the base shot noise currents. Define the differential base shot noise current by $i_{shb(\mathrm{d})} = (i_{shb1} - i_{shb2})/2$. For Q_1, replace i_{shb1} with $+i_{shb(\mathrm{d})}$. For Q_2, replace i_{shb2} with $-i_{shb(\mathrm{d})}$. The differential short-circuit collector output current $i_{od(\mathrm{sc})} = i_{c1(\mathrm{sc})} - i_{c2(\mathrm{sc})}$ is proportional to $i_{shb(\mathrm{d})} - \left(-i_{shb(\mathrm{d})}\right) = i_{shb1} - i_{shb2}$. If i_{shb1} and i_{shb2} are not correlated, it follows that $\overline{i^2_{od(\mathrm{sc})}}$ contains a term that is proportional to $\overline{i^2_{shb1}} + \overline{i^2_{shb2}}$. Because $\overline{i^2_{shb1}} = \overline{i^2_{shb2}}$, the base current shot noise is increased by 3 dB compared to a CE amplifier. Likewise, the thermal noise of the base spreading resistance, the thermal noise of R_1 and R_2, the base current flicker noise, and the collector current shot noise are increased by 3 dB compared to the CE amplifier. The mean-square equivalent noise input voltage of the diff amp is given by $2\overline{v^2_{ni}}$, where $\overline{v^2_{ni}}$ is given by Eq. (7.111). Above the flicker noise corner frequency, the noise is minimized when each BJT is biased at a collector current given by Eq. (7.139).

7.12 The Cascode Amplifier

Figure 7.31(a) shows a cascode amplifier. The noise source v_{ni1} models the noise generated by R_1, R_2, and Q_1. Its mean-square value is given by Eqs. (7.110) and (7.111). Although an exact solution can be obtained, the assumption that $r_{01} = r_{02} = \infty$ simplifies solution of the noise contribution of Q_2. With this approximation, we can write

$$\begin{aligned} i_{c2(sc)} &= \alpha_2 i_{e2} + i_{shc2} \\ &= \alpha_2 \left[G_{m1}\left(v_1 + v_{ni1}\right) + i_{shb2} + i_{fb2} - i_{shc2}\right] + i_{shc2} \\ &= \alpha_2 G_{m1}\left(v_1 + v_{ni}\right) \end{aligned} \tag{7.148}$$

where G_{m1} is given by Eq. (7.64) and v_{ni} is given by

$$v_{ni} = v_{ni1} + \frac{1}{\alpha_2 G_{m1}} \left(i_{shb2} + i_{fb2} + \frac{i_{shc2}}{\beta_2} \right) \tag{7.149}$$

This can be converted into a mean-square voltage to obtain

$$\begin{aligned} \overline{v_{ni}^2} &= \overline{v_{ni1}^2} + \frac{1}{\alpha_2^2} \left(\frac{r_{x1} + R_1}{\beta_1} + \frac{V_T}{I_{C1}} + R_2 \right)^2 \Bigg(2qI_{B2}\Delta f \\ &\quad + \frac{K_f I_{B2} \Delta f}{f} + 2q\frac{I_{C2}}{\beta_2^2}\Delta f \Bigg) \end{aligned} \tag{7.150}$$

The thermal noise voltage of r_{x2} does not contribute to v_{ni}. This is because the small-signal resistance seen looking out of the emitter of Q_2 is infinite when the small-signal collector-emitter resistances are assumed to be open circuits. Thus a voltage at the base of Q_2 cannot change its collector current.

7.13 The BJT at High Frequencies

Figure 7.32 shows the high-frequency T model of the BJT with the emitter grounded and the base driven by a voltage source having the output impedance $Z_s = R_s + jX_s$. The base-emitter capacitance c_π and the collector-base capacitance c_μ are given by

$$c_\pi = c_{je} + \frac{\tau_F I_C}{V_T} \tag{7.151}$$

$$c_\mu = \frac{c_{jc}}{[1 + V_{CB}/\phi_C]^{m_c}} \tag{7.152}$$

where c_{je} is the zero-bias junction capacitance of the base-emitter junction, τ_F is the forward transit time of the base-emitter junction, c_{jc} is the zero-bias junction capacitance of the base-collector junction, ϕ_C is the built-in potential, and m_c is the junction exponential factor. All noise sources are shown in the circuit except the base flicker noise current which can be neglected at high frequencies.

For $I_{c(\mathrm{sc})}$ we can write

$$I_{c(\mathrm{sc})} = \alpha I_e' - j\omega c_\mu V_b' + I_{shc} = I_e' (\alpha - j\omega c_\mu r_e) + I_{shc} \tag{7.153}$$

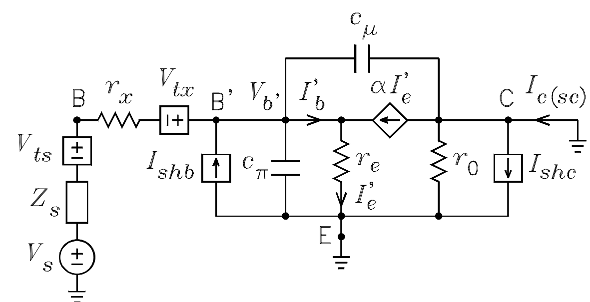

Figure 7.32: Common-emitter amplifier.

where $V_b' = I_e' r_e$ has been used. To solve for I_e', we can write

$$V_b' = I_e' r_e = \left[\frac{V_s + V_{ts} + V_{tx}}{Z_s + r_x} + \alpha I_e' + I_{shb}\right] Z_{p1} \tag{7.154}$$

where Z_{p1} and c_T are given by

$$Z_{p1} = (Z_s + r_x) \| \frac{1}{j\omega c_T} \| r_e \tag{7.155}$$

$$c_T = c_\pi + c_\mu \tag{7.156}$$

Eq. (7.154) can be solved for I_e' to obtain

$$I_e' = \left[\frac{V_s + V_{ts} + V_{tx}}{Z_s + r_x} + I_{shb}\right] \frac{Z_{p1}}{r_e - \alpha Z_{p1}} \tag{7.157}$$

Thus $I_{c(sc)}$ is given by

$$I_{c(\mathrm{sc})} = (\alpha - j\omega c_\mu r_e) \left[\frac{V_s + V_{ts} + V_{tx}}{Z_s + r_x} + I_{shb}\right] \frac{Z_{p1}}{r_e - \alpha Z_{p1}} + I_{shc} \tag{7.158}$$

It is straightforward to show that this reduces to

$$I_{c(\mathrm{sc})} = G_{mb}(\omega) \left(V_s + V_{ts} + V_{tx} + I_{shb}(Z_s + r_x) + \frac{I_{shc}}{G_{mb}(\omega)} \right) \tag{7.159}$$

where $G_{mb}(\omega)$ is given by

$$G_{mb}(\omega) = \frac{\alpha - j\omega c_\mu r_e}{\left(\dfrac{1}{1+\beta} + j\omega c_T r_e\right)(Z_s + r_x) + r_e} \tag{7.160}$$

For $\omega = 0$, this reduces to

$$G_{mb}(0) = \frac{\alpha}{\dfrac{R_s + r_x}{1+\beta} + r_e} \tag{7.161}$$

which agrees with Eq. 7.59 for the case $R_{tb} = R_s$ and $R_{te} = 0$.

7.13.1 Equivalent Noise Input Voltage

The equivalent noise input voltage is given by the sum of the terms in the brackets in Eq. (7.159) with the V_s term omitted. It is given by

$$\begin{aligned} V_{ni} &= V_{ts} + V_{tx} + I_{shb}(Z_s + r_x) + \frac{I_{shc}}{1 - j\omega c_\mu V_T / I_C} \\ &\times \left[\left(\frac{1}{\beta} + j\omega c_T \frac{V_T}{I_C}\right)(Z_s + r_x) + \frac{V_T}{I_C}\right] \end{aligned} \tag{7.162}$$

where $r_e/\alpha = r_\pi/\beta$ and $\alpha = \beta/(1+\beta)$ have been used. The mean-square value is given by

$$\begin{aligned} \overline{v_{ni}^2} &= 4kT(R_s + r_x)\Delta f + 2qI_B\Delta f\left[(R_s + r_x)^2 + X_s^2\right] \\ &+ \frac{2qI_C\Delta f}{1 + (\omega c_\mu V_T/I_C)^2}\left\{\left[\frac{R_s + r_x}{\beta} + \frac{V_T}{I_C}(1 - \omega c_T X_s)\right]^2\right. \\ &\left. + \left[\frac{X_s}{\beta} + \omega c_T \frac{V_T}{I_C}(R_s + r_x)\right]^2\right\} \end{aligned} \tag{7.163}$$

The noise factor is given by

$$F = \frac{\overline{v_{ni}^2}}{4kTR_s\Delta f} \tag{7.164}$$

If we assume that $c_\mu \ll c_\pi$, $c_\pi \simeq \tau_F I_C/V_T$, and $\omega^2 c_\mu^2 r_e^2 << \alpha^2$, the values of X_s and I_C can be solved for to minimize $\overline{v_{ni}^2}$. These are obtained by setting $\partial\overline{v_{ni}^2}/\partial X_s = 0$ and $\partial\overline{v_{ni}^2}/\partial I_C = 0$ and solving the equations simultaneously. It is straightforward to show that X_s and I_C are given by

$$X_s = \omega\tau_F(R_s + r_x)\frac{\beta}{\sqrt{1+\beta}} \tag{7.165}$$

$$I_C = \frac{V_T}{(R_s + r_x)\left(1 + \alpha\beta\omega^2\tau_F^2\right)} \times \frac{\beta}{\sqrt{1+\beta}} \tag{7.166}$$

The corresponding expressions for $\overline{v_{ni}^2}$ and F are

$$\overline{v_{ni}^2} = 4kT\left(R_s + r_x\right)\Delta f \frac{\sqrt{1+\beta}}{\sqrt{1+\beta}-1} \tag{7.167}$$

$$F = \left(1 + \frac{r_x}{R_s}\right)\frac{\sqrt{1+\beta}}{\sqrt{1+\beta}-1} \tag{7.168}$$

7.13.2 $V_n - I_n$ Noise Model

As with the low-frequency analysis, two noise models can be obtained, one with r_x represented by an external noiseless resistor and the other with r_x inside the transistor. These are described below.

First Model

Equation (7.162) is of the form

$$\begin{aligned} V_{ni} &= V_{ts} + V_{tx} + I_{shb}\left(Z_s + r_x\right) \\ &\quad + \frac{I_{shc}}{1 - j\omega c_\mu V_T/I_C}\left[\left(\frac{1}{\beta} + j\omega c_T\frac{V_T}{I_C}\right)\left(Z_s + r_x\right) + \frac{V_T}{I_C}\right] \\ &= V_{ts} + V_n + I_n\left(Z_s + r_x\right) \end{aligned} \tag{7.169}$$

It follows that V_n and I_n are given by

$$V_n = V_{tx} + \frac{I_{shc}}{1 - j\omega c_\mu V_T/I_C}\frac{V_T}{I_C} \tag{7.170}$$

$$I_n = I_{shb} + \frac{I_{shc}}{1 - j\omega c_\mu V_T/I_C}\left(\frac{1}{\beta} + j\omega c_T\frac{V_T}{I_C}\right) \tag{7.171}$$

The mean-square values and the correlation coefficient are given by

$$\overline{v_n^2} = 4kTr_x\Delta f + \frac{1}{1 + \left(\omega c_\mu V_T/I_C\right)^2}\frac{2kTV_T\Delta f}{I_C} \tag{7.172}$$

$$\overline{i_n^2} = 2qI_B\Delta f + \frac{2qI_C\Delta f}{1 + \left(\omega c_\mu V_T/I_C\right)^2}\left[\frac{1}{\beta^2} + \left(\omega c_T\frac{V_T}{I_C}\right)^2\right] \tag{7.173}$$

$$\gamma = \frac{1}{\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}}\frac{2kT\Delta f}{1 + \left(\omega c_\mu V_T/I_C\right)^2}\left(\frac{1}{\beta} - j\omega c_T\frac{V_T}{I_C}\right) \tag{7.174}$$

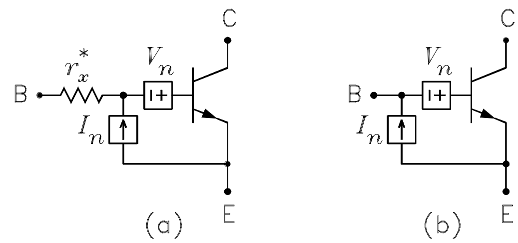

Figure 7.33: $V_n - I_n$ noise models of the CE amplifier.

The model is shown in Fig. 7.33(a). The resistor r_x^* is a noiseless resistor in the model. Its noise is included in V_n.

Because r_x is a noiseless resistor external to the transistor, the optimum source impedance which minimizes the noise factor given by Eq. (6.13) does not apply. It can be shown that the optimum source impedance is given by

$$Z_{so} = \left[\frac{\overline{v_n^2}}{\overline{i_n^2}}\left(1-\gamma_i^2\right) + 2\gamma_r r_x \sqrt{\frac{\overline{v_n^2}}{\overline{i_n^2}}} + r_x^2\right]^{1/2} - j\gamma_i \sqrt{\frac{\overline{v_n^2}}{\overline{i_n^2}}} \tag{7.175}$$

Second Model

Equation (7.159) is of the form

$$I_{c(sc)} = G_m(\omega)\left(V_s + V_{ts} + V_n + I_n Z_s\right) \tag{7.176}$$

It follows that V_n and I_n are given by

$$V_n = V_{tx} + I_{shb} r_x + \frac{I_{shc}}{1 - j\omega c_\mu V_T / I_C}\left[\left(\frac{1}{\beta} + j\omega c_T \frac{V_T}{I_C}\right) r_x + \frac{V_T}{I_C}\right] \tag{7.177}$$

$$I_n = I_{shb} + \frac{I_{shc}}{1 - j\omega c_\mu V_T / I_C}\left(\frac{1}{\beta} + j\omega c_T \frac{V_T}{I_C}\right) \tag{7.178}$$

The mean-square values and the correlation coefficient are given by

$$\begin{aligned}\overline{v_n^2} &= 4kTr_x\Delta f + 2qI_B\Delta f r_x^2 + \frac{2qI_C\Delta f}{1 + \left(\omega c_\mu V_T/I_C\right)^2} \\ &\quad \times \left[\left(\frac{r_x}{\beta} + \frac{V_T}{I_C}\right)^2 + \left(\omega c_T \frac{V_T}{I_C} r_x\right)^2\right]\end{aligned} \tag{7.179}$$

$$\overline{i_n^2} = 2qI_B\Delta f + \frac{2qI_C\Delta f}{1+(\omega c_\mu V_T/I_C)^2}\left[\frac{1}{\beta^2} + \left(\omega c_T \frac{V_T}{I_C}\right)^2\right] \tag{7.180}$$

$$\begin{aligned}\gamma \;&=\; \frac{1}{\sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}}\left[2qI_B\Delta f r_x + \frac{2qI_C\Delta f}{1+(\omega c_\mu V_T/I_C)^2}\right. \\ &\times \left.\left[\left(\frac{1}{\beta} + j\omega c_T \frac{V_T}{I_C}\right) r_x + \frac{V_T}{I_C}\right]\left(\frac{1}{\beta} - j\omega c_T \frac{V_T}{I_C}\right)\right] \end{aligned} \tag{7.181}$$

The model is shown in Fig. 7.33(b).

7.13.3 Amplifier Equivalent Circuit

To solve for the impedance seen looking into the collector, we use the circuit in Fig. 7.32. We can write

$$V_b' = \left[j\omega c_\mu V_t + \alpha I_e'\right] Z_{p1} = \left[j\omega c_\mu V_t + \alpha \frac{V_b'}{r_e}\right] Z_{p1} \tag{7.182}$$

where Z_p is defined in Eq. 7.155. This equation can be solved for V_b' to obtain

$$V_b' = \frac{j\omega c_\mu Z_{p1} V_t}{1 - \alpha Z_{p1}/r_e} \tag{7.183}$$

The current I_c' is given by

$$\begin{aligned} I_c' \;&=\; j\omega c_\mu\left(V_t - V_b'\right) + \alpha I_e' = j\omega c_\mu\left(V_t - V_b'\right) + \alpha\frac{V_b'}{r_e} \\ &=\; V_t\left[j\omega c_\mu + \left(\frac{\alpha}{r_e} - j\omega c_\mu\right)\frac{j\omega c_\mu Z_p}{1-\alpha Z_p/r_e}\right] \\ &=\; V_t j\omega c_\mu \frac{1/Z_{p1} - j\omega c_\mu}{1/Z_{p1} - \alpha/r_e} = \frac{V_t}{Z_{ic}'} \end{aligned} \tag{7.184}$$

The above equation defines the impedance Z_{ic}' given by

$$\begin{aligned} Z_{ic}' \;&=\; \frac{1}{j\omega c_\mu}\frac{1/Z_{p1} - \alpha/r_e}{1/Z_{p1} - j\omega c_\mu} \\ &=\; \frac{1}{j\omega c_\mu}\frac{\dfrac{1}{Z_s + r_x} + \dfrac{1}{r_\pi} + j\omega c_T}{\dfrac{1}{Z_s + r_x} + \dfrac{1}{r_e} + j\omega c_\pi} \end{aligned} \tag{7.185}$$

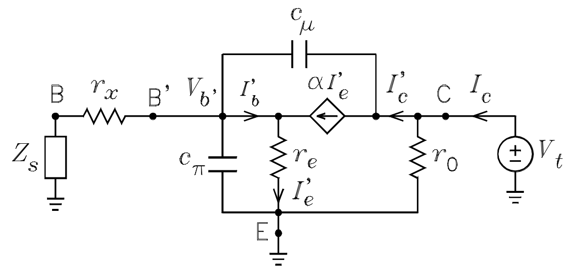

Figure 7.34: Circuit for calculating the collector impedance Z_{ic}.

where $r_\pi = r_e/(1-\alpha)$ has been used. The impedance Z_{ic} seen looking into the collector is given by

$$Z_{ic} = r_0 \| Z'_{ic} \tag{7.186}$$

The Norton equivalent circuit seen looking into the collector is thus the current $I_{c(sc)}$ in parallel with the impedance Z_{ic}.

To solve for the input impedance, we first solve for the input impedance seen looking into the V'_b node. The circuit in Fig. 7.35 shows the V'_b node driven by a test source V_t with a load impedance Z_L from collector to ground. To facilitate the use of superposition in solving for I_b, the $\alpha I'_e$ source is replaced by two series sources of the same value with the common node grounded. By superposition, we can write

$$\begin{aligned} I_b &= \frac{V_t}{Z_{p2}} - \alpha I'_e + \alpha I'_e \frac{r_0\|Z_L}{r_0\|Z_L + 1/j\omega c_\mu} \\ &= V_t\left(\frac{1}{Z_{p2}} - \frac{\alpha}{r_e}\,\frac{1/j\omega c_\mu}{r_0\|Z_L + 1/j\omega c_\mu}\right) \end{aligned} \tag{7.187}$$

where $I'_e = V_t/r_e$ has been used and Z_{p2} is given by

$$Z_{p2} = r_e \| \frac{1}{j\omega c_\pi} \| \left(\frac{1}{j\omega c_\mu} + r_0\|Z_L\right) \tag{7.188}$$

The resistance seen looking into the V'_b node is given by V_t/I_b. When r_x is added to this, the input resistance seen looking into the V_b node is

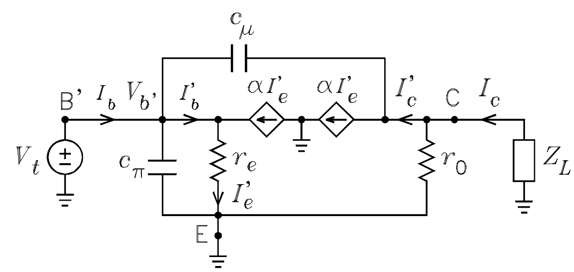

Figure 7.35: Circuit for calculating Z_{ib}.

obtained. It is given by

$$\begin{aligned} Z_{ib} &= r_x + \left[\frac{1}{Z_{p2}} - \frac{\alpha}{r_e}\frac{1}{1 + j\omega\left(r_0 \| Z_L\right) c_\mu}\right]^{-1} \\ &= r_x + \left[\frac{1}{r_\pi} + j\omega c_\pi + j\omega c_\mu \frac{1 + \dfrac{\alpha}{r_e}\left(r_0 \| Z_L\right)}{1 + j\omega c_\mu\left(r_0 \| Z_L\right)}\right]^{-1} \end{aligned} \tag{7.189}$$

For $Z_L = 0$, this reduces to $Z_{ib} = r_x + r_\pi \| \left(1/j\omega c_T\right)$, a result which can be written by inspection of the circuit. The equivalent circuit of the common-emitter amplifier is shown in Fig. 7.36.

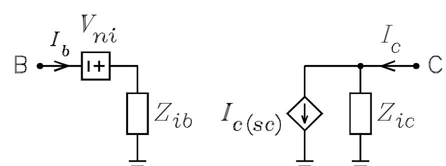

Figure 7.36: Equivalent circuit of the common-emitter amplifier.

Example 2 *Figure 7.37 shows the ac signal circuit of a BJT operated as a common-emitter amplifier. The noise is modeled with the first* $V_n - I_n$ *noise model of Fig. 7.33(a). The BJT is biased at* $I_C = 1$ *mA and* $V_{CB} = 10$ *V and has the parameters* $\beta = 100$, $r_x = 40\ \Omega$, $V_A = 30$ *V*, $c_{jc} = 10$ *pF*, $c_{je} = 15$ *pF, and* $\tau_F = 0.5$ *ns. The operating frequency is* 10 *MHz. (a) Calculate the optimum source impedance which minimizes the noise factor. (b) Calculate the voltage gain if the amplifier is conjugately* *matched to its load impedance. (c) Calculate the input impedance* Z_{ib}*. (d) Calculate the noise figure.*

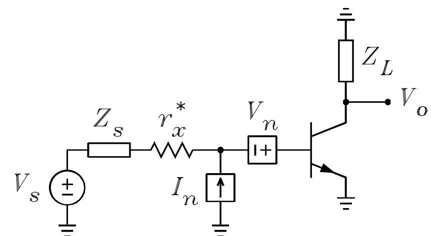

Figure 7.37: Common-emitter amplifier.

Solution. The small-signal parameters are $\alpha = \beta/(1+\beta) = 0.99$, $r_0 = (V_A + V_{CB})/I_C = 40$ kΩ, $r_e = \alpha V_T/I_C = 24.8$ Ω, $r_\pi = \beta V_T/I_C = 2.5$ kΩ, $c_\pi = c_{je} + \tau_F I_C/V_T = 35$ pF, $c_\mu = c_{jc}/(1 + V_{CB}/0.75)^{0.33} = 4.15$ pF, and $c_T = c_\pi + c_\mu = 39.2$ pF. (a) Eqs. (7.172) through (7.174) give $\overline{v_n^2} = 8.4 \times 10^{-19}$ V^2/Hz, $\overline{i_n^2} = 4.44 \times 10^{-24}$ A^2/Hz, and $\gamma = 0.0414 - j0.255$. The optimum source impedance is given by Eq. (7.175) to obtain $Z_{so} = 424 + j111$ Ω. (b) Eqs. (7.160), (7.185), and (7.186) give $G_{mb} = 0.0162 - j0.0213$ S and $Z_{ic} = 170 - j225$ Ω or 242 Ω in parallel with $-j321$ Ω. For a conjugately matched load, the load impedance must be $Z_L = 154 + j116$ Ω or 468 Ω in parallel with $-j353$ Ω. The output voltage is given by $V_o = -I_{c(sc)}(Z_{ic} \| Z_L) = -G_{mb} V_s (Z_{ic} \| Z_L)$. Thus the voltage gain is

$$\frac{V_o}{V_s} = -G_{mb}(Z_{ic} \| Z_L) = 6.26\angle 127^{\text{o}} \text{ (15.9 dB)}$$

(c) Eq. (7.189) gives $Z_{ib} = -42.4 - j187$ Ω. With an input impedance that has a negative real part, the circuit can oscillate if the total series resistance is not positive. The total series input impedance is $Z_{so} + Z_{ib} = 52.2 - j176$, which has a positive real part. Thus the circuit will not oscillate at the design frequency. This does not mean that it will not oscillate at another frequency, for the series resistance can be negative at other frequencies, depending on the source and load impedances. (d) Eq. (6.14) for the optimum noise factor F_0 holds only for the second $V_n - I_n$ noise model where r_x is not external to the BJT. Thus F_0 must be calculated from the "long way." We have

$$F_0 = 1 + \frac{\overline{v_n^2} + \sqrt{\overline{v_n^2}}\sqrt{\overline{i_n^2}}\,\mathrm{Re}\left[\gamma\left(Z_{so}^* + r_x\right)\right] + \overline{i_n^2}\left|Z_{so} + r_x\right|^2}{4kT_0\,\mathrm{Re}\left(Z_{so}\right)\Delta f} = 1.27$$

where the impedance through which I_n flows is $Z_{so}+r_x$. The noise figure is $NF = 10\log F_0 = 1.03$ dB. It is left as an exercise to show that the second $V_n - I_n$ noise model gives the same answers.

Chapter 9

Noise in MOSFETs

Whereas the JFET has a diode junction between the gate and the channel, the metal-oxide semiconductor FET or MOSFET differs primarily in that it has an oxide insulating layer separating the gate and the channel. The circuit symbols are shown in Fig. 9.1. Each device has gate (G), drain (D), and source (S) terminals. Four of the symbols show an additional terminal called the body (B) which is not normally used as an input or an output. It connects to the drain-source channel through a diode junction. In discrete MOSFETs, the body lead is connected internally to the source. When this is the case, it is omitted on the symbol as shown in four of the MOSFET symbols. In integrated-circuit MOSFETs, the body usually connects to a dc power supply rail which reverse biases the body-channel junction. In the latter case, the so-called "body effect" must be accounted for when analyzing the circuit.

The principle noise sources in the MOSFET are thermal noise and flicker noise generated in the channel. Flicker noise in a MOSFET is usually larger than in a JFET because the MOSFET is a surface device in which the fluctuating occupancy of traps in the oxide modulate the conducting surface channel all along the channel. The relations between the flicker noise and the MOSFET geometry and bias conditions depend on the fabrication process. In most cases, the flicker noise, when referred to the input, is independent of the bias voltage and current and is inversely proportional to the product of the active gate area and the gate oxide capacitance per unit area.

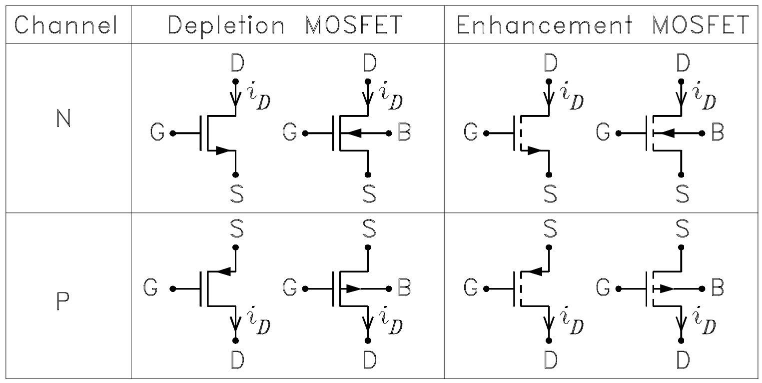

Figure 9.1: MOSFET symbols.

9.1 Device Equations

The discussion here applies to the n-channel MOSFET. The equations apply to the p-channel device if the subscripts for the voltage between any two of the device terminals are reversed, e.g. v_{GS} becomes v_{SG}. The n-channel MOSFET is biased in the active mode or saturation region for $v_{DS} \geq v_{GS} - v_{TH}$, where v_{TH} is the threshold voltage. This voltage is negative for the depletion-mode device and positive for the enhancement-mode device. It is a function of the body-source voltage and is given by

$$v_{TH} = V_{TO} + \gamma \left(\sqrt{\phi - v_{BS}} - \sqrt{\phi} \right) \tag{9.1}$$

where V_{TO} is the value of v_{TH} with $v_{BS} = 0$, γ is the body threshold parameter, ϕ is the surface potential, and v_{BS} is the body-source voltage. In the saturation region, the drain current is given by

$$\begin{aligned} i_D &= \frac{k'}{2} \frac{W}{L} (1 + \lambda v_{DS}) (v_{GS} - v_{TH})^2 \qquad \text{for } v_{GS} \geq V_{TO} \\ &= 0 \qquad \text{for } v_{GS} < V_{TO} \end{aligned} \tag{9.2}$$

where W is the channel width, L is the channel length, λ is the channel-length modulation parameter, and k' is given by

$$k' = \mu_0 C_{ox} = \mu_0 \frac{\epsilon_{ox}}{t_{ox}}$$

In this equation, μ_0 is the average carrier mobility, C_{ox} is the gate oxide capacitance per unity area, ϵ_{ox} is the permittivity of the oxide layer, and t_{ox} is its thickness. It is convenient to define a transconductance coefficient K given by

$$K = K_0 \left(1 + \lambda v_{DS}\right) \tag{9.3}$$

where K_0 is the zero-bias value of K given by

$$K_0 = \frac{k'}{2} \frac{W}{L} \tag{9.4}$$

With this definition, the drain current can be written

$$i_D = K \left(v_{GS} - v_{TH}\right)^2 \tag{9.5}$$

Note that K plays the same role in the MOSFET drain current equation that β plays in the JFET drain current equation given in Eq. (8.1). In some texts, the K in Eq. (9.5) is replaced with $K/2$. In this case, the factor $1/2$ in Eq. (9.4) is omitted.

Figure 9.2 shows the typical variation of drain current with gate-to-source voltage for a constant drain-to-source voltage and zero body-to-source voltage. In this case, the threshold voltage is a constant, i.e. $v_{TH} = V_{TO}$. For $v_{GS} \leq V_{TO}$, the drain current is zero. For $v_{GS} > V_{TO}$, the drain current increases approxiamtely as the square of the gate-to-source voltage. The slope of the curve represents the small-signal transconductance g_m, which is defined in the next section. Fig. 9.3 shows the typical variation of drain current with drain-to-source voltage for a several values of gate-to-source votlage v_{GS} and zero body-to-source voltage v_{BS}. The dashed line divides the triode region from the saturation or active region. In the saturation region, the slope of the curves represents the reciprocal of the small-signal drain-source resistance r_0, which is defined in the next section.

9.2 Bias Equation

Figure 9.4 shows the MOSFET with the external circuits represented by Thévenin dc circuits. If the MOSFET is in the pinch-off region, the following equations for I_D hold:

$$I_D = K \left(V_{GS} - V_{TH}\right)^2 \tag{9.6}$$

$$V_{GS} = V_{GG} - \left(V_{SS} + I_D R_{SS}\right) \tag{9.7}$$

$$K = K_0 \left(1 + \lambda V_{DS}\right) \tag{9.8}$$

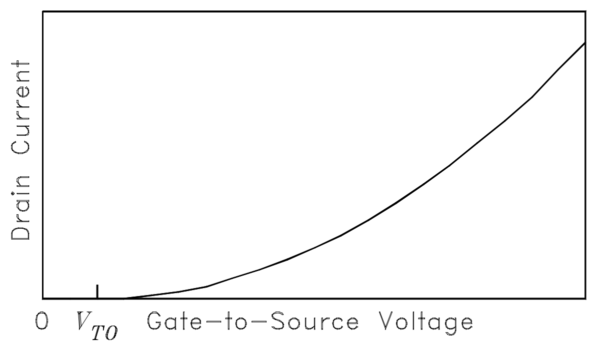

Figure 9.2: Drain current i_D versus gate-to-source voltage v_{GS} for constant drain-to-source voltage v_{DS}.

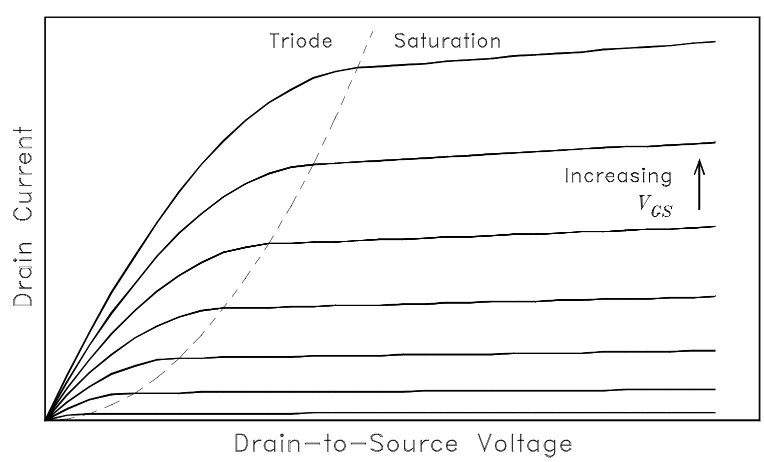

Figure 9.3: Drain current i_D versus drain-to-source voltage v_{DS} for constant gate-to-source voltage v_{GS}.

$$V_{DS} = (V_{DD} - I_D R_{DD}) - (V_{SS} + I_D R_{SS}) \tag{9.9}$$

Because this is a set of nonlinear equations, a closed form solution for I_D cannot be easily written unless it is assumed that K is not a function of V_{DS} and V_{TH} is not a function of V_{BS}. The former assumption requires the condition $\lambda V_{DS} \ll 1$. With these assumptions, the equations can be solved for I_D to obtain

$$I_D = \frac{1}{4KR_{SS}^2}\left[\sqrt{1 + 4KR_{SS}\left(V_{GG} - V_{SS} - V_{TH}\right)} - 1\right]^2 \tag{9.10}$$

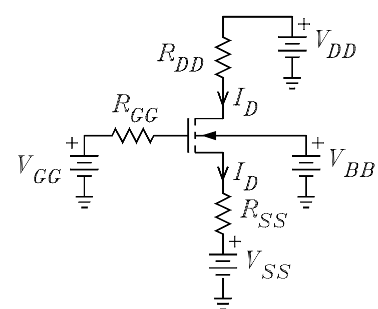

Figure 9.4: MOSFET dc bias circuit.

Unless $\lambda V_{DS} \ll 1$ and the dependence of V_{TH} on V_{BS} is neglected, Eq. (9.10) is only an approximate solution. A numerical procedure for obtaining a more accurate solution is to first calculate I_D with $K = K_0$ and $V_{TH} = V_{TO}$. Then calculate V_{DS} and the new values of K and V_{TH} from which a new value for I_D can be calculated. The procedure can be repeated until the solution for I_D converges. Alternately, computer tools can be used to obtain a numerical solution to the set of nonlinear equations.

9.3 Small-Signal Models

There are two small-signal circuit models which are commonly used to analyze MOSFET circuits. These are the hybrid-π model and the T model. The two models are equivalent and give identical results. They are described below.

9.3.1 Hybrid-π Model

Let the drain current and each voltage be written as the sum of a dc component and a small-signal ac component as follows:

$$i_D = I_D + i_d \tag{9.11}$$

$$v_{GS} = V_{GS} + v_{gs} \tag{9.12}$$

$$v_{BS} = V_{BS} + v_{bs} \tag{9.13}$$

$$v_{DS} = V_{DS} + v_{ds} \tag{9.14}$$

If the ac components are sufficiently small, we can write

$$i_d = \frac{\partial I_D}{\partial V_{GS}} v_{gs} + \frac{\partial I_D}{\partial V_{BS}} v_{bs} + \frac{\partial I_D}{\partial V_{DS}} v_{ds} \tag{9.15}$$

where the derivatives are evaluated at the dc bias values. Let us define

$$g_m = \frac{\partial I_D}{\partial V_{GS}} = 2K\left(V_{GS} - V_{TH}\right) = 2\sqrt{KI_D} \tag{9.16}$$

$$g_{mb} = \frac{\partial I_D}{\partial V_{BS}} = -2K\left(V_{GS} - V_{TH}\right)\frac{\partial V_{TH}}{\partial V_{BS}} = \frac{\gamma\sqrt{KI_D}}{\sqrt{\phi - V_{BS}}} = \chi g_m \tag{9.17}$$

$$\chi = \frac{\gamma}{2\sqrt{\phi - V_{BS}}} \tag{9.18}$$

$$r_0 = \left[\frac{\partial I_D}{\partial V_{DS}}\right]^{-1} = \left[\frac{k'}{2}\frac{W}{L}\lambda\left(V_{GS} - V_{TH}\right)^2\right]^{-1} = \frac{V_{DS} + 1/\lambda}{I_D} \tag{9.19}$$

The small-signal drain current can thus be written

$$i_d = i_{dg} + i_{db} + \frac{v_{ds}}{r_0} \tag{9.20}$$

where

$$i_{dg} = g_m v_{gs} \tag{9.21}$$

$$i_{db} = g_{mb} v_{bs} \tag{9.22}$$

The small-signal circuit which models these equations is given in Fig. 9.5. This is called the hybrid-π model.

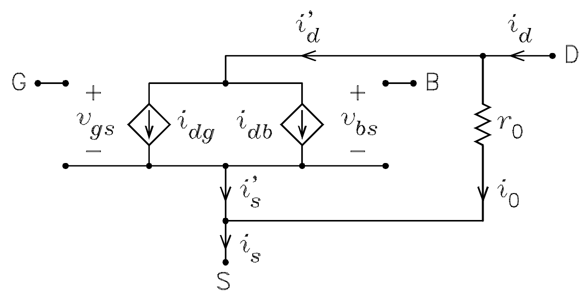

Figure 9.5: Hybrid-π model of the MOSFET.

9.3.2 T Model

The T model of the MOSFET is shown in Fig. 9.6. The resistor r_0 is given by Eq. (9.19). The resistors r_s and r_{sb} are given by

$$r_s = \frac{1}{g_m} \tag{9.23}$$

$$r_{sb} = \frac{1}{g_{mb}} = \frac{1}{\chi g_m} = \frac{r_s}{\chi} \tag{9.24}$$

where g_m and g_{mb} are the transconductances defined in Eqs. (9.16) and (9.17). The currents are given by

$$i_d = i_{sg} + i_{sb} + \frac{v_{ds}}{r_0} \tag{9.25}$$

$$i_{sg} = \frac{v_{gs}}{r_s} = g_m v_{gs} \tag{9.26}$$

$$i_{sb} = \frac{v_{bs}}{r_{sb}} = g_{mb} v_{bs} \tag{9.27}$$

The currents are the same as for the hybrid-π model. Therefore, the two models are equivalent. Note that the gate and body currents are zero because the two controlled sources supply the currents that flow through r_s and r_{sb}.

9.4 Small-Signal Equivalent Circuits

Several equivalent circuits are derived below which facilitate writing small-signal low-frequency equations for the MOSFET. We assume that

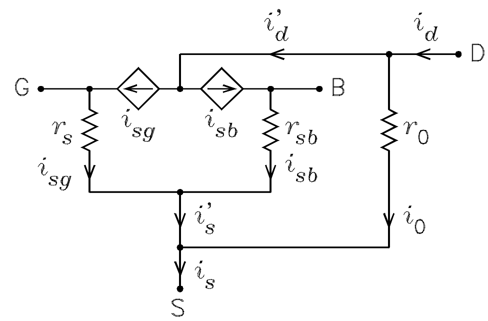

Figure 9.6: T model of the MOSFET.

the circuits external to the device can be represented by Thévenin equivalent circuits. The Norton eqivalent circuit seen looking into the drain and the Thévenin equivalent circuit seen looking into the source are derived. Several examples are given which illustrate use of the equivalent circuits.

9.4.1 Source Equivalent Circuit

Figure 9.7 shows the MOSFET T model with a Thévenin source in series with the gate and the body connected to signal ground. We wish to solve for the equivalent circuit in which the sources i_{sg} and i_{sb} are replaced by a single source which connects from the drain node to ground having the value $i'_d = i'_s$. We call this the source equivalent circuit. The first step is to look up into the branch labeled i'_s and form a Thévenin equivalent circuit. With $i'_s = 0$, we can use voltage division to write

$$v_{s(oc)} = v_{tg}\frac{r_{sb}}{r_s + r_{sb}} = v_{tg}\frac{r_s/\chi}{r_s + r_s/\chi} = \frac{v_{tg}}{1+\chi} \tag{9.28}$$

With $v_{tg} = 0$, the resistance r'_s seen looking up into the branch labeled i'_s is

$$r'_s = r_s \| r_{sb} = \frac{r_s}{1+\chi} = \frac{1}{(1+\chi)\, g_m} \tag{9.29}$$

The source equivalent circuit is shown in Fig.9.8. Compared to the corresponding circuit for the JFET, the MOSFET circuit replaces v_{tg} with $v_{tg}/(1+\chi)$ and r_s with $r'_s = r_s/(1+\chi)$. The circuit is derived with the assumption that the body lead connects to signal ground. In

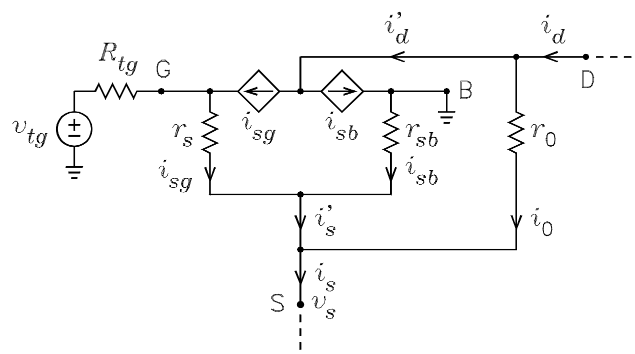

Figure 9.7: T model with Thévenin source connected to the gate and the body connected to signal ground.

the case that the body lead connects to the source lead, it follows from Fig. 9.7 that $i_{sb} = 0$. In this case, the MOSFET circuit is identical to the JFET circuit. Connecting the body to the source is equivalent to setting $\chi = 0$ in the MOSFET equations.

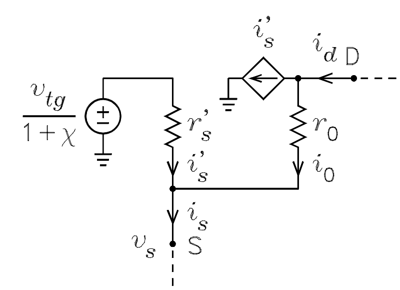

Figure 9.8: MOSFET source equivalent circuit.

9.4.2 Norton Drain Circuit

Figure 9.9(a) shows the MOSFET with Thévenin sources connected to its gate and source leads and the body lead connected to signal ground. The Norton equivalent circuit seen looking into the drain can be obtained from that for the JFET by incorporating the factor $1/(1+\chi)$ into the equation for G_{mg} and using $r_s' = 1/(1+\chi)\, g_m$ in place of $r_s = 1/g_m$.

The circuit is given in Fig. 9.9(b), where $i_{d(sc)}$ and r_{id} are given by

$$i_{d(sc)} = G_{mg} v_{tg} - G_{ms} v_{ts} \tag{9.30}$$

$$r_{id} = r_0 \left(1 + \frac{R_{ts}}{r'_s}\right) + R_{ts} \tag{9.31}$$

The transonductances G_{mg} and G_{ms} are given by

$$\begin{aligned} G_{mg} &= \frac{1}{1+\chi} \frac{1}{r'_s + R_{ts} \| r_0} \frac{r_0}{r_0 + R_{ts}} \\ &= \frac{1}{1+\chi} \frac{1}{R_{ts} + r'_s \| r_0} \frac{r_0}{r_0 + r'_s} \end{aligned} \tag{9.32}$$

$$G_{ms} = \frac{1}{R_{ts} + r'_s \| r_0} \tag{9.33}$$

The equations for the case where the body is connected to the source are obtained by setting $\chi = 0$. In this case, $r'_s = r_s$ and the equations become identical to those for the JFET. For the case $R_{ts} = 0$, the equations for G_{mg} and G_{ms} reduce to

$$G_{mg} = \frac{1}{1+\chi} \frac{1}{r'_s} = g_m \tag{9.34}$$

$$G_{ms} = \frac{1}{r'_s \| r_0} = (1+\chi)\, g_m + \frac{1}{r_0} \tag{9.35}$$

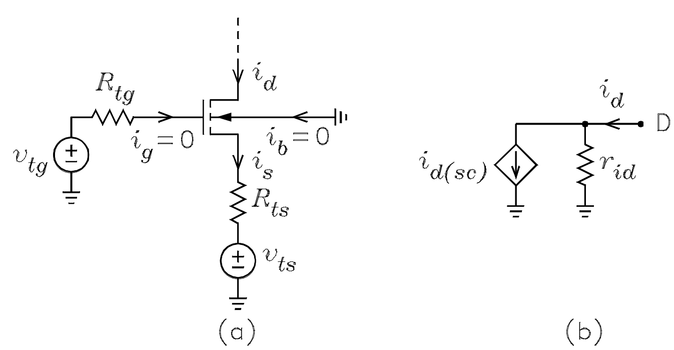

Figure 9.9: (a) MOSFET with Thévenin sources connected to the gate and source. (b) Norton drain circuit.

For the case $r_0 \gg R_{ts}$ and $r_0 \gg r'_s$, we can write

$$G_{mg} = \frac{1}{1+\chi}\frac{1}{r'_s + R_{ts}} \tag{9.36}$$

and

$$G_{ms} = \frac{1}{r'_s + R_{ts}} \tag{9.37}$$

The value of $i_{d(sc)}$ calculated with these approximations is simply the value of i'_s calculated with r_0 considered to be an open circuit. The term "r_0 approximations" is used in the following when r_0 is neglected in calculating $i_{d(sc)}$ but not neglected in calculating r_{id}.

9.4.3 Thévenin Source Circuit

Figure 9.10(a) shows the MOSFET with a Thévenin source connected to its gate, the body lead connected to signal ground, and the external drain load represented by the resistor R_{td}. The Thévenin equivalent circuit seen looking into the source can be obtained from the corresponding circuit for the JFET by replacing v_{tg} with $v_{tg}/(1+\chi)$ and $r_s = 1/g_m$ with $r'_s = 1/(1+\chi)\,g_m$. The circuit is given in Fig. 9.10(b), where $v_{s(oc)}$ and r_{is} are given by

$$v_{s(oc)} = \frac{v_{tg}}{1+\chi}\frac{r_0}{r'_s + r_0} \tag{9.38}$$

$$r_{is} = r'_s \frac{r_0 + R_{td}}{r'_s + r_0} \tag{9.39}$$

The equations for the case where the body is connected to the source are obtained by setting $\chi = 0$. In this case, the equations become identical to those for the JFET.

9.4.4 Summary of Models

Figure 9.11 summarizes the four equivalent circuits derived above. For the case where the body is connected to the source, set $\chi = 0$ in the equations.

9.5 Example Amplifier Circuits

This section describes several examples which illustrate the use of the small-signal equivalent circuits derived above to write by inspection the voltage gain, the input resistance, and the output resistance of several amplifier circuits.

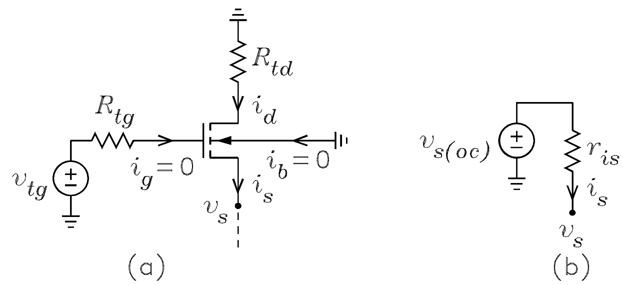

Figure 9.10: (a) MOSFET with Thévenin source connected to gate. (b) Thévenin source circuit.

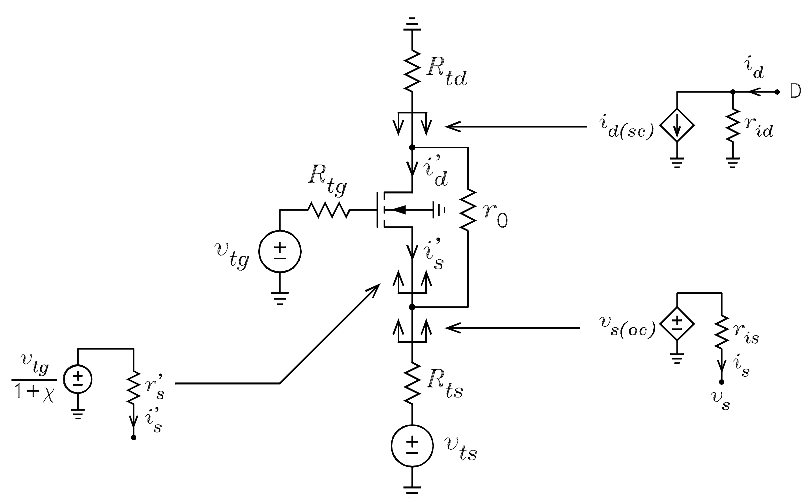

Figure 9.11: Summary of the small-signal equivalent circuits. Set $\chi = 0$ if the body is connected to the source.

9.5.1 Common-Source Amplifier

Figure 9.12(a) shows a common-source amplifier. The active device is M_1. Its load consists of a current-mirror active load consisting of M_2 and M_3. The current source I_Q sets the drain current in M_3 which is mirrored into the drain of M_2. Because the source-to-drain voltage of M_2 is larger than that of M_3, the Early effect causes the dc drain current in M_2 to be slightly larger than I_Q. The input voltage can be written $v_I = V_B + v_i$, where V_B is a dc bias voltage which sets the drain current in M_1. It must be equal to the drain current in M_2 in order for the dc component of the output voltage to be stable. In any application of the circuit, V_B would be set by feedback. Looking out of the drain of M_1, the resistance to ac signal ground is $R_{td1} = r_{02}$.

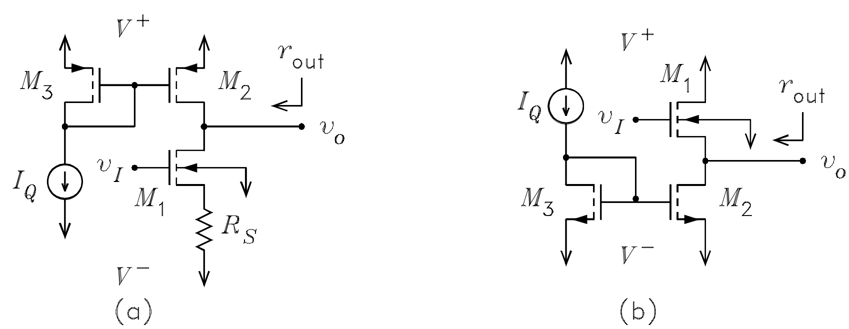

Figure 9.12: (a) CMOS common-source amplifier. (b) Common-drain amplifier.

The voltage gain of the circuit can be written

$$\frac{v_o}{v_i} = \frac{i_{d1(sc)}}{v_i}\frac{v_o}{i_{d1(sc)}} = G_{mg1} \times (-r_{id1} \| r_{02}) \tag{9.40}$$

where G_{mg1} is given by Eq. (9.32), r_{id1} is given by Eq. (9.31), and $R_{ts1} = R_S$. The body effect cancels if $R_S = 0$. The output resistance is given by

$$r_{\text{out}} = r_{id1} \| r_{02} \tag{9.41}$$

9.5.2 Common-Drain Amplifier

Figure 9.12(b) shows a common-drain amplifier. The active device is M_1. Its load consists of a current-mirror active load consisting of M_2

and M_3. The current source I_Q sets the drain current in M_3 which is mirrored into the drain of M_2. As with the common-source amplifier, the Early effect makes the drain current in M_2 slightly larger than that in M_3. The input voltage can be written $v_I = V_B + v_i$, where V_B is a dc bias voltage which sets the dc component of the output voltage. Looking out of the source of M_1, the resistance to ac signal ground is $R_{ts1} = r_{02}$. The voltage gain can be written

$$\frac{v_o}{v_i} = \frac{v_{s1(oc)}}{v_i}\frac{v_o}{v_{s1(oc)}} = \frac{1}{1+\chi_1}\frac{r_{01}}{r'_{s1}+r_{01}}\frac{r_{02}}{r_{02}+r_{is1}} \tag{9.42}$$

where $r_{is1}$is given by Eq. (9.39). The output resistance is given by

$$r_{\text{out}} = r_{is1} \| r_{02} \tag{9.43}$$

9.5.3 Common-Gate Amplifier

Figure 9.13(a) shows a common-gate amplifier. The active device is M_1. Its load consists of a current-mirror active load consisting of M_2 and M_3. The current source I_Q sets the drain current in M_3 which is mirrored into the drain of M_2. As with the common-source amplifier, the Early effect makes the drain current in M_2 slightly larger than that in M_3. The dc voltage V_B is a dc bias voltage which sets the drain current in M_1 which must be equal to the drain current in M_2 in order for the dc component of the output voltage to be stable. In any application of the circuit, V_B would be set by feedback. Looking out of the drain of M_1, the resistance to ac signal ground is $R_{td1} = r_{02}$.

The voltage gain of the circuit can be written

$$\frac{v_o}{v_i} = \frac{i_{d1(sc)}}{v_i}\frac{v_0}{i_{d1(sc)}} = -G_{ms1} \times (-r_{id1} \| r_{02}) \tag{9.44}$$

where G_{ms1} is given by Eq. (9.33) and r_{id1} is given by Eq. (9.31). The input and output resistances are given by

$$r_{\text{in}} = R_i + r_{is1} \tag{9.45}$$

$$r_{\text{out}} = r_{id1} \| r_{02} \tag{9.46}$$

where $r_{is1}$is given by Eq. (9.39).

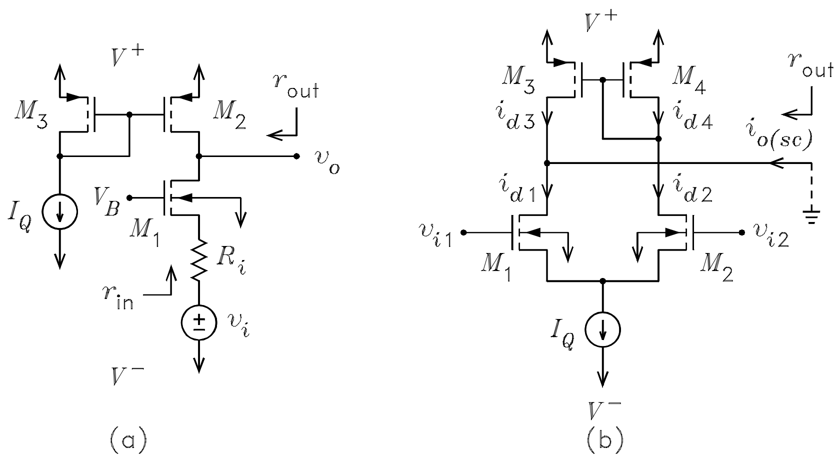

Figure 9.13: (a) CMOS Common-gate amplifier. (b) CMOS differential amplifier.

9.5.4 Differential Amplifier

A MOS differential amplifier with an active current-mirror load is shown in Fig. 9.13(b). The object is to determine the Norton equivalent circuit seen looking into the output To do this, the output is connected to ac signal ground, which is indicated by the dashed line. It will be assumed that the Early effect can be neglected in all devices in calculating $i_{o(sc)}$ but not neglected in calculating r_{out}, i.e. we use the r_0 approximations. We can write

$$i_{o(sc)} = i_{d1(sc)} - i_{d3(sc)} = i_{d1(sc)} - i_{d4(sc)} = i_{d1(sc)} - i_{d2(sc)} \tag{9.47}$$

Because the tail supply is a current source, the currents i_{d1} and i_{d2} can be calculated by replacing v_{i1} and v_{i2} with their differential components. In this case, the ac signal voltage at the sources of M_1 and M_2 is zero. Let $v_{i1} = v_{i(d)}/2$ and $v_{i2} = -v_{i(d)}/2$, where $v_{id} = v_{i1} - v_{i2}$. It follows by symmetry that $i_{d2(sc)} = -i_{d1(sc)}$ so that $i_{o(sc)}$ is given by

$$\begin{aligned} i_{o(sc)} &= 2i_{d1(sc)} = 2G_{mg1}\frac{v_{i(d)}}{2} \\ &= 2\frac{1}{(1+\chi)}\frac{1}{r_s'}\frac{v_{i(d)}}{2} = g_{m1} \times (v_{i1} - v_{i2}) \end{aligned} \tag{9.48}$$

where Eq. (9.36) with $R_{ts} = 0$ is used for G_{mg}. Note that the body effect cancels. This is because the source-to-body ac signal voltage is zero for the differential input signals. The output resistance is given by

$$r_{\text{out}} = r_{01} \| r_{03} \tag{9.49}$$

Note that $r_{id1} = r_{01}$ because $R_{ts1} = R_{ts2} = 0$ for the differential input signals.

9.6 Small-Signal High-Frequency Models

Figures 9.14 and 9.15 show the hybrid-π and T models for the MOSFET with the gate-source capacitance c_{gs}, the source-body capacitance c_{sb}, the drain-body capacitance c_{db}, the drain-gate capacitance c_{dg}, and the gate-body capacitance c_{gb}added. These capacitors model charge storage in the device which affect its high-frequency performance. The first three capacitors are given by

$$c_{gs} = \frac{2}{3} W L C_{ox} \tag{9.50}$$

$$c_{sb} = \frac{c_{sb0}}{(1 + V_{SB}/\psi_0)^{1/2}} \tag{9.51}$$

$$c_{db} = \frac{c_{db0}}{(1 + V_{DB}/\psi_0)^{1/2}} \tag{9.52}$$

where V_{SB} and V_{DB} are dc bias voltages; c_{sb0} and c_{db0} are zero-bias values; and ψ_0 is the built-in potential. Capacitors c_{gd} and c_{gb} model parasitic capacitances. For IC devices, c_{gd} is typically in the range of 1 to 10 fF for small devices and c_{gb} is in the range of 0.04 to 0.15 fF per square micron of interconnect.

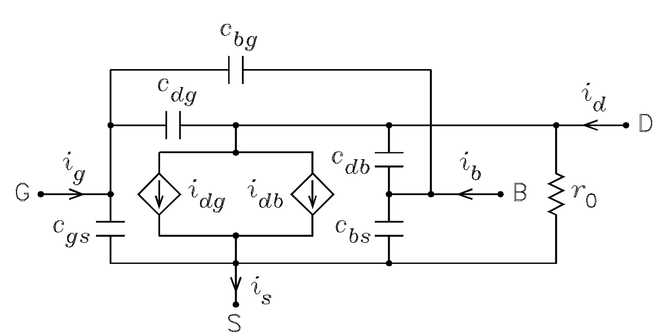

Figure 9.14: High-frequency hybrid-π model.

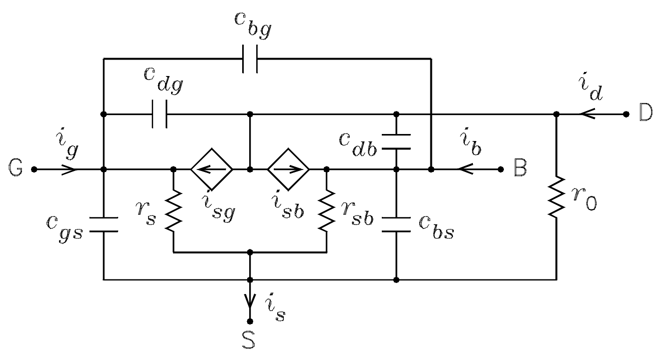

Figure 9.15: High-frequency T model.

9.7 Noise Model

The principle noise sources in a MOSFET are thermal noise and flicker noise in the channel. Fig. 9.16 shows the device symbols with the noise sources added. The polarity of the sources is arbitrary. Each has been chosen so as to cause an increase in the drain current. The thermal noise and flicker noise in the drain bias current I_D are modeled by the current source labeled $i_{td} + i_{fd}$. In the band Δf, i_{td} and i_{fd} have the mean-square values

$$\overline{i_{td}^2} = 4kT\left(\frac{2g_m}{3}\right)\Delta f \tag{9.53}$$

$$\overline{i_{fd}^2} = \frac{K_f I_D \Delta f}{f L^2 C_{ox}} \tag{9.54}$$

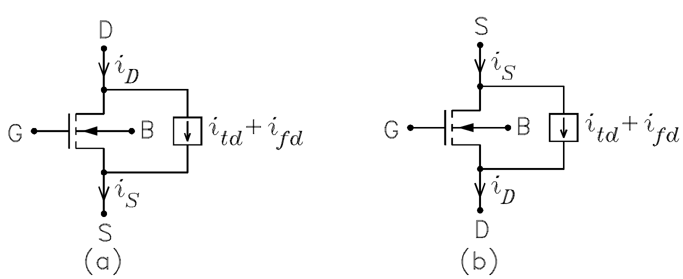

Figure 9.16: MOSFET symbols with noise sources added.

9.7.1 Equivalent Noise Input Voltage

Figure 9.17(a) shows the n-channel MOSFET with the drain and body connected to signal ground. The external gate and source circuits are modeled by Thévenin equivalent circuits. With $v_2 = 0$, the circuit models a common-source or CS stage. With $v_1 = 0$, it models a common-gate or CG stage. The noise sources v_{t1}, and v_{t2}, respectively, model the thermal noise in R_1 and R_2. In the band Δf, these have the mean-square values

$$\overline{v_{t1}^2} = 4kTR_1\Delta f \tag{9.55}$$

$$\overline{v_{t2}^2} = 4kTR_2\Delta f \tag{9.56}$$

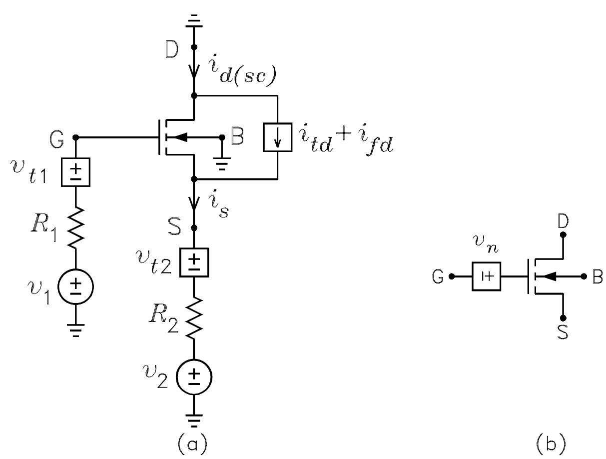

Figure 9.17: (a) MOSFET with Thevenin sources connected to the gate and source. (b) $v_n - i_n$ noise model.

The short-circuit drain output current can be written

$$i_{d(sc)} = G_{mg}v_{tg} - G_{ms}v_{ts} + i_{td} + i_{fd} \tag{9.57}$$

where G_{mg} and G_{ms} are given by Eqs. (9.32) and (9.33) with $R_{ts} = R_2$, and v_{tg} and v_{tg} are given by

$$v_{tg} = v_1 + v_{t1} \tag{9.58}$$

$$v_{ts} = v_2 + v_{t2} + (i_{td} + i_{fd})\,R_2 \tag{9.59}$$

The equivalent noise input voltage v_{ni} can be expressed as a voltage in series with either v_1 or v_2. Because the common-source amplifier is most often seen, we will express it as a voltage in series with v_1. First, we factor G_{mg} from Eq. (9.57) to obtain

$$\begin{aligned} i_{d(\mathrm{sc})} &= G_{mg} \left\{ v_1 + v_{t1} \right. \\ &\left. -\frac{G_{ms}}{G_{mg}} \left[v_2 + v_{t2} + \left(i_{td} + i_{fd} \right) R_2 \right] + \frac{i_{td} + i_{fd}}{G_{mg}} \right\} \end{aligned} \tag{9.60}$$

The equivalent noise voltage in series with v_1 is given by all terms in the brackets except the v_1 and v_2 terms. It is given by

$$v_{ni} = v_{t1} - v_{t2} \frac{G_{ms}}{G_{mg}} + \frac{i_{td} + i_{fd}}{G_{mg}} \left(1 - G_{ms} R_2 \right) \tag{9.61}$$

After some algebra, this can be reduced to

$$v_{ni} = v_{t1} - v_{t2} \left(1 + \chi \right) \left(1 + \frac{r_s'}{r_0} \right) + \left(i_{td} + i_{fd} \right) \left(1 + \chi \right) r_s' \tag{9.62}$$

To express v_{ni} as a voltage in series with v_2, this equation is multiplied by G_{mg}/G_{ms}.

The mean-square value of v_{ni} is given by

$$\overline{v_{ni}^2} = \overline{v_{t1}^2} + \overline{v_{t2}^2} \left(1 + \chi \right)^2 \left(1 + \frac{r_s'}{r_0} \right)^2 + \left(\overline{i_{td}^2} + \overline{i_{fd}^2} \right) \left(1 + \chi \right)^2 r_s'^2 \tag{9.63}$$

which reduces to

$$\begin{aligned} \overline{v_{ni}^2} &= 4kT \left[R_1 + R_2 \left(1 + \chi + \frac{1}{g_m r_0} \right)^2 \right] \Delta f \\ &+ \frac{4kT \Delta f}{3\sqrt{K I_D}} + \frac{K_f \Delta f}{4K f L^2 C_{ox}} \end{aligned} \tag{9.64}$$

where $\chi = 0$ for the case where the body is connected to the source. This expression gives the mean-square equivalent noise input voltage for the CS amplifier. To obtain $\overline{v_{ni}^2}$ for the CG amplifier, this expression is multiplied by $\left(G_{mg}/G_{ms} \right)^2$.

For minimum noise in the MOSFET, it can be concluded from Eq. (9.64) that the series resistance in the external gate and source circuits should be minimized and the device should have a high transconductance parameter K and a low flicker noise coefficient K_f. The component of $\overline{v_{ni}^2}$ due to the channel thermal noise is proportional to $1/\sqrt{I_D}$. This decreases by 1.5 dB each time I_D is doubled.

9.7.2 $v_n - i_n$ Noise Model

For the MOSFET $v_n - i_n$ noise model, the mean-square values of v_n and i_n are given by

$$\overline{v_n^2} = \frac{4kT}{3\sqrt{KI_D}}\Delta f + \frac{K_f}{4KfL^2C_{ox}}\Delta f \tag{9.65}$$

$$\overline{i_n^2} = 0 \tag{9.66}$$

Fig. 9.17(b) shows the model.

9.7.3 Flicker Noise Corner Frequency

The plot of $\overline{v_n^2}/\Delta f$ as a function of frequency would be identical to that shown for the JFET in Figure 8.14. The lower frequency at which the plot is twice its high frequency limit is called the flicker noise corner frequency, which is labeled f_{flk} in the figure. At this frequency, the thermal noise and the flicker noise are equal. The flicker noise corner frequency can be solved for by equating the thermal and flicker noise components in the equation for $\overline{v_n^2}$ in Eq. (9.65). It is given by

$$f_{\text{flk}} = \frac{3K_f}{16kTL^2C_{ox}}\sqrt{\frac{I_D}{K}} \tag{9.67}$$

For $f > f_{\text{flk}}$, the thermal noise dominates. For $f < f_{\text{flk}}$, the flicker noise dominates. An experimental method for determining the flicker noise coefficient is to measure the flicker noise corner frequency and use Eq. (9.67) to calculate K_f.

9.7.4 MOSFET Differential Amplifier

The BJT diff amp noise is 3 dB greater than the noise for a common-emitter stage. Similarly, MOSFET diff amp noise is 3 dB greater than the noise of the common-source stage. This assumes that the signal output from the diff amp is taken differentially. Otherwise, the common-mode noise generated by the tail current bias supply is not canceled.

9.8 Low-Frequency Noise Examples

The equivalent noise input voltage at low frequencies is determined below for four example MOSFET circuits. It is assumed that the frequency is low enough so that the dominant component of the noise is flicker noise. In this case, the mean-square noise voltage is approximately given by

$$\overline{v_n^2} = \frac{K_f}{4KfL^2C_{ox}}\Delta f \tag{9.68}$$

It is straightforward to modify the results for the higher frequency case where the dominant component of the noise is thermal noise or for the more general case where both thermal noise and flicker noise are included. The circuits are shown in Fig. 9.18. The analysis assumes that each transistor is operated in the saturation region and the noise sources are uncorrelated. Because the MOSFET exhibits no current noise, the output resistance of the signal source is omitted with no loss in generality.

9.8.1 CS Amplifier with Enhancement-Mode Load

Figure 9.18(a) shows a single-channel NMOS enhancement-mode CS amplifier with an active NMOS enhancement-mode load. It is assumed that the two MOSFETs have matched model parameters and are biased at the same current. With $v_o = 0$, the short-circuit output current can be written

$$i_{o(\mathrm{sc})} = g_{m1}(v_s + v_{n1}) - g_{m2}v_{n2} \tag{9.69}$$

The equivalent noise input voltage is obtained by factoring g_{m1} from the expression and retaining all terms except the v_s term. It has the mean-square value

$$\overline{v_{ni}^2} = \overline{v_{n1}^2} + \left(\frac{g_{m2}}{g_{m1}}\right)^2 \overline{v_{n2}^2} \tag{9.70}$$

When the low-frequency approximation in Eq. (9.68) is used for each $\overline{v_n^2}$ term, the expression for $\overline{v_{ni}^2}$ reduces to

$$\overline{v_{ni}^2} = \frac{K_f\Delta f}{2\mu_n C_{ox}^2 W_1 L_1 f}\left[1 + \left(\frac{L_1}{L_2}\right)^2\right] \tag{9.71}$$

The value of L_1 which minimizes this is $L_1 = L_2$. The noise can be reduced further by increasing W_1 and L_2. The noise is independent of W_2.

9.8.2 CS Amplifier with Depletion-Mode Load

Figure 9.18(b) shows a single-channel NMOS enhancement-mode CS amplifier with an active NMOS depletion-mode load. It is assumed that the

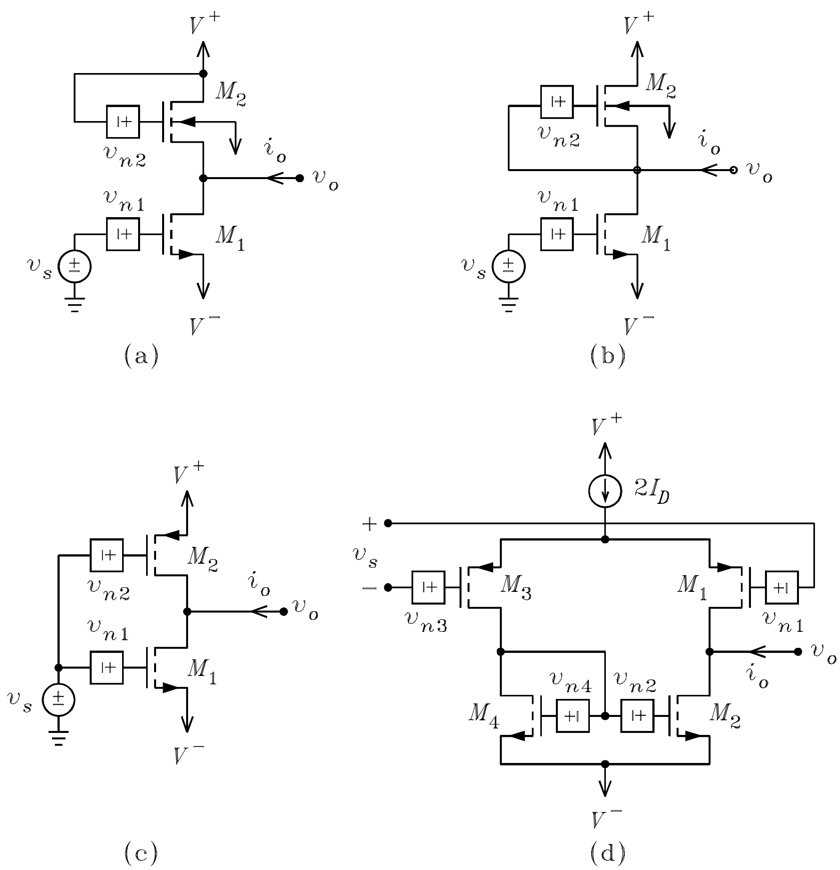

Figure 9.18: MOSFET circuit examples for low-frequency noise calculations.

two MOSFETs are biased at the same current. With $v_o = 0$, the expression for $i_{o(\mathrm{sc})}$ is the same as for the circuit of Fig. 9.18(a). Therefore, the expression for v_{ni} is the same. However, the two MOSFETs cannot be assumed to have the same flicker noise coefficient. The low-frequency expression for $\overline{v_{ni}^2}$ is

$$\overline{v_{ni}^2} = \frac{K_{f1}\Delta f}{2\mu_n C_{ox}^2 W_1 L_1 f}\left[1 + \frac{K_{f2}}{K_{f1}}\left(\frac{L_1}{L_2}\right)^2\right] \tag{9.72}$$

The value of L_1 which minimizes this is $L_1 = L_2\sqrt{K_{f1}/K_{f2}}$. The noise can be reduced further by increasing W_1 and L_2. The noise is independent of W_2.

9.8.3 CMOS Amplifier

Figure 9.18(c) shows a push-pull complementary MOSFET (CMOS) amplifier. It is assumed that the two MOSFETs are biased at the same current. With $v_o = 0$, the short-circuit output current is given by

$$i_{o(\mathrm{sc})} = g_{m1}\left(v_s + v_{n1}\right) + g_{m2}\left(v_s + v_{n2}\right) \tag{9.73}$$

The equivalent noise input voltage is obtained by factoring $(g_{m1} + g_{m2})$ from the equation and retaining all terms except the v_s term. It has the mean-square value

$$\overline{v_{ni}^2} = \frac{g_{m1}^2\overline{v_{n1}^2} + g_{m2}^2\overline{v_{n2}^2}}{\left(g_{m1} + g_{m2}\right)^2} \tag{9.74}$$

In order for the quiescent output voltage to be midway between the rail voltages, the circuit is commonly designed with $g_{m1} = g_{m2}$. When this is true and the low-frequency approximation in Eq. (9.68) is used for each $\overline{v_n^2}$ term, $\overline{v_{ni}^2}$ can be written

$$\overline{v_{ni}^2} = \frac{1}{2}\left[\frac{K_{f1}\Delta f}{2\mu_n C_{ox}^2 W_1 L_1 f} + \frac{K_{f2}\Delta f}{2\mu_p C_{ox}^2 W_2 L_2 f}\right] \tag{9.75}$$

The noise can be decreased by increasing the size of both transistors. For $g_{m1} \neq g_{m2}$, a technique for further reducing $\overline{v_{ni}^2}$ is to increase L for the MOSFET for which K_f/μ is the largest.

9.8.4 Differential Amplifier with Active Load

Figure 9.18(d) shows a diff-amp with a current-mirror active load. It is assumed that M_1 and M_3 have matched model parameters and similarly for M_2 and M_4. In addition, it is assumed that all four transistors are biased at the same current so that $g_{m1} = g_{m3}$ and $g_{m2} = g_{m4}$. Because the noise generated by the tail source is a common-mode signal, it is cancelled at the output by the current mirror load and is not modeled in the circuit. The differential input voltage is given by $v_{id} = v_s + v_{n1} - v_{n3}$. The component of the short-circuit output current due to v_{id} is $i_{o(\mathrm{sc})} = g_{m1} v_{id}$. To solve for the component of $i_{o(\mathrm{sc})}$ due to v_{n2} and v_{n4}, the sources v_s, v_{n1}, and v_{n3} are set to zero. This forces M_4 to have zero drain signal current. Thus the gate of M_4 is a signal ground and $i_{o(\mathrm{sc})} = -g_{m2}(v_{n2} - v_{n4})$. The total short-circuit output current is given by

$$i_{o(\mathrm{sc})} = g_{m1}(v_s + v_{n1} - v_{n3}) + g_{m2}(v_{n2} - v_{n4}) \tag{9.76}$$

The equivalent noise input voltage is obtained by factoring g_{m1} from this equation and retaining all terms except the v_s term. The mean-square value is

$$\overline{v_{ni}^2} = \overline{v_{n1}^2} + \overline{v_{n3}^2} + \left(\frac{g_{m2}}{g_{m1}}\right)^2 \left(\overline{v_{n2}^2} + \overline{v_{n4}^2}\right) \tag{9.77}$$

When the low-frequency approximation in Eq. (9.68) is used for each $\overline{v_n^2}$ term, $\overline{v_{ni}^2}$ can be written

$$\overline{v_{ni}^2} = \frac{K_{f1}\Delta f}{\mu_p C_{ox}^2 W_1 L_1 f}\left[1 + \frac{K_{f2}}{K_{f1}}\left(\frac{L_1}{L_2}\right)^2\right] \tag{9.78}$$

The noise can be reduced by increasing W_1 and L_2 and by making $L_1 = L_2\sqrt{K_{f1}/K_{f2}}$. The expression is independent of W_2.

Chapter 10

Noise in Feedback Amplifiers

10.1 Effect of Feedback on Noise

Figure 10.1 shows the block diagram of a two-stage amplifier with feedback. The equivalent noise input voltage to each stage is modeled as an external input. The feedback network is assumed to be noiseless, for its noise can be included in v_{ni1}. The output voltage is given by

$$v_o = A_2 \left[v_{ni2} + A_1 \left(v_{ni1} + v_i - b v_o\right)\right] \tag{10.1}$$

This equation can be solved for v_o to obtain

$$v_o = \frac{A_1 A_2}{1 + b A_1 A_2} \left(v_i + v_{ni1} + \frac{v_{ni2}}{A_1}\right) \tag{10.2}$$

The voltage gain of the amplifier is thus given by

$$A_v = \frac{v_o}{v_i} = \frac{A_1 A_2}{1 + b A_1 A_2} \tag{10.3}$$

The ratio of the open-loop gain to the closed-loop gain is defined as the amount of feedback. It is given by

$$\text{Amount of Feedback} = 1 + b A_1 A_2 = 1 + bA \tag{10.4}$$

where $A = A_1 A_2$. This is often expressed in dB with the equation $20 \log (1 + bA)$.

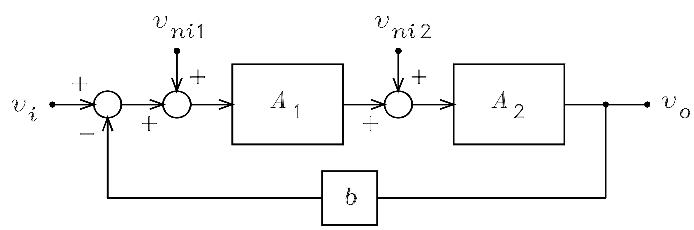

Figure 10.1: Two stage amplifier with feedback.

It follows from Eq. (10.2) that the equivalent noise input voltage is given by

$$v_{ni} = v_{ni1} + \frac{v_{ni2}}{A_1} \tag{10.5}$$

This equation shows that v_{ni} is independent of the feedback. Like noise, non-linear distortion generated in an amplifier circuit can be modeled by an equivalent distortion input voltage. Thus it follows that distortion generated by an amplifier is also independent of the feedback when it is expressed as an equivalent input voltage. This equation implies that the noise generated by the second stage is inversely proportional to the gain of the first stage. This is only true if the gain of the first stage is independent of its output resistance. Otherwise, the i_n noise of the second stage causes v_{ni2} to increase if A_1 is increased by increasing the first stage output resistance. These effects are covered in more detail in the section on multi-stage amplifiers.

It is often said that feedback increases the signal-to-noise ratio and reduces the percent distortion in an amplifier. Eq. (10.5) implies that this is not correct. However, if the amplitude of the signal input voltage v_i is increased when feedback is added so that the amplitude of the signal component of v_o remains constant, then the signal-to-noise ratio is increased by the amount of feedback and the percent distortion is reduced by the amount of feedback. As an example, the open-loop gain of an op amp is typically about 2×10^5 at very low frequencies. To obtain a 1 V signal at its output with no feedback, the input voltage must be 5 μV. If feedback is used to reduce the gain to 10, the noise and distortion at the output would be reduced by a factor of 2×10^5. To obtain the same 1 V signal at the output, the input voltage must be increased to 0.1 V. The signal-to-noise ratio at the output would then be increased by a factor of 2×10^5 or by 86 dB. The percent distortion would be decreased by that factor.

10.2 Series-Shunt Amplifier Example

Figure 10.2(a) shows the simplified diagram of a series-shunt feedback amplifier with a BJT input stage. The following stages are modeled by a current-controlled voltage source, where R_{i2} is its input resistance, R_{o2} is its output resistance, and R_{m2} is its transresistance. The bias sources and networks are omitted for a small-signal ac analysis. In order for the feedback to be negative, the gain from v_s to v_o must be non-inverting. Thus R_{m2} must be positive.

The feedback network is in the form of a voltage divider and consists of resistors R_F and R_E. The feedback ratio b is the ratio v_{e1}/v_o calculated with $i_{e1} = 0$. It is given by

$$b = \left.\frac{v_{e1}}{v_o}\right|_{i_{e1}=0} = \frac{R_E}{R_E + R_F} \tag{10.6}$$

If the loop gain is sufficiently high, $v_{e1} \to v_s$ so that the above equation can be used to obtain

$$\frac{v_o}{v_s} \simeq \frac{1}{b} = 1 + \frac{R_F}{R_E} \tag{10.7}$$

This is the familiar expression for the voltage gain of a non-inverting op amp having the same feedback network.

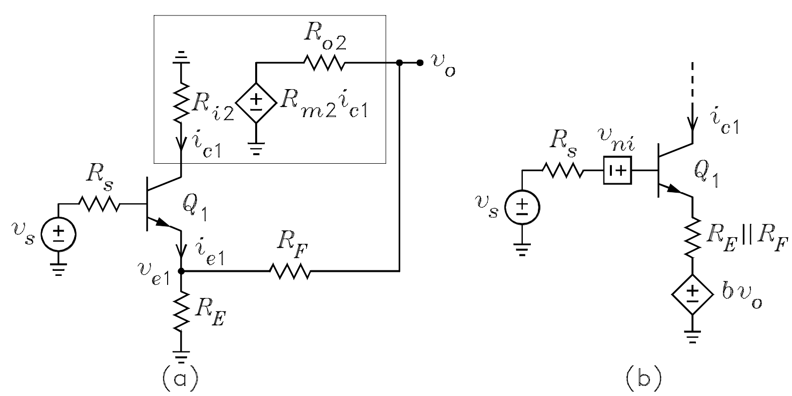

Figure 10.2: Amplifier with series-shunt feedback.

For a small-signal ac analysis, let us assume that the collector-emitter resistance r_0 of Q_1 is large enough so that it can be neglected. In this case, r_{ic} in the collector equivalent circuit of Fig. 7.12(b) is an open circuit, $i_{c1} = i_{c1(sc)}$, and $i_{e1} = \alpha i_{c1}$. The output voltage is given by

$$\begin{aligned} v_o &= R_{m2} i_{c1} \frac{R_F + R_E}{R_{o2} + R_F + R_E} + i_{e1} \frac{R_E R_{o2}}{R_{o2} + R_F + R_E} \\ &\simeq R_{m2} i_{c1} \frac{R_F + R_E}{R_{o2} + R_F + R_E} \end{aligned} \tag{10.8}$$

where the approximation neglects the contribution of i_{e1} to v_o. Because the effect of the negative feedback is to make i_{e1} smaller, this approximation is commonly made in the analysis circuits having series summing at the input.

The circuit in Fig. 10.2(b) can be used to solve for i_{c1}. The figure shows the circuit seen looking out of the emitter of Q_1 replaced by a Thèvenin equivalent circuit with respect to v_o. The equivalent noise input voltage is modeled by the source v_{ni}. We assume that this models the noise generated by R_s, Q_1, $R_E \| R_F$, and the following stages. Following Eq. (7.63), we can write

$$i_{c1} = G_{m1} \left(v_s + v_{ni} - b v_o \right) \tag{10.9}$$

where G_{m1} and r'_e are given by

$$G_{m1} = \frac{\alpha}{r'_e + R_E \| R_F} \tag{10.10}$$

$$r'_e = \frac{R_s + r_x}{1 + \beta} + r_e \tag{10.11}$$

Equations (10.8) and (10.9) can be combined to obtain

$$\begin{aligned} v_o &= G_{m1} R_{m2} \frac{R_F + R_E}{R_{o2} + R_F + R_E} \left(v_s + v_{ni} - b v_o \right) \\ &= A_{vo} \left(v_s + v_{ni} - b v_o \right) \end{aligned} \tag{10.12}$$

where A_{vo} is the voltage gain with feedback removed given by

$$A_{vo} = G_{m1} R_{m2} \frac{R_F + R_E}{R_{o2} + R_F + R_E} \tag{10.13}$$

Eq. (10.12) can be solved for v_o to obtain

$$v_o = \frac{A_{vo}}{1 + b A_{vo}} \left(v_s + v_{ni} \right) \tag{10.14}$$

This equation is in the familiar form for the output voltage of a series-shunt feedback amplifier. When the noise generated by the stages following Q_1 is neglected and it is assumed that r_0 is large, the value of $\overline{v_{ni}^2}$ is obtained from Eq. (7.111) with R_1 replaced with R_s and R_2 replaced with $R_E \| R_F$. It is given by

$$\begin{aligned}\overline{v_{ni}^2} &= 4kT\left(R_s + r_x + R_E \| R_F\right)\Delta f \\ &\quad + \left(2qI_B\Delta f + \frac{K_f I_B \Delta f}{f}\right)\Delta f\left(R_s + r_x + R_E \| R_F\right)^2 \\ &\quad + 2qI_C\Delta f\left[\frac{R_s + r_x + R_E \| R_F}{\beta} + \frac{V_T}{I_C}\right]^2 \end{aligned} \tag{10.15}$$

For minimum noise, $R_E \| R_F$ should be small compared to $R_s + r_x$ and the BJT should be biased at $I_{C(\text{opt})}$. The resistance $R_E \| R_F$ cannot be zero because the amplifier gain is set by the ratio of R_F to R_E.

10.3 Shunt-Shunt Amplifier Example

Figure 10.3(a) shows the simplified diagram of a shunt-shunt feedback amplifier with a BJT input stage. The following stages are modeled by a current-controlled voltage source, where R_{i2} is its input resistance, R_{o2} is its output resistance, and R_{m2} is its transresistance. The bias sources and networks are omitted for a small-signal ac analysis. Note that the polarity of the current-controlled voltage source is reversed compared to that in Fig. 10.2(a). In order for the feedback to be negative, the gain from v_s to v_o must be inverting. Thus R_{m2} must be positive.

The feedback network consists of the resistor R_F connecting the output to the input. The feedback ratio b is the ratio i_f/v_o calculated with $v_{b1} = 0$. It is given by

$$b = \left.\frac{i_f}{v_o}\right|_{v_{b1}=0} = \frac{1}{R_F} \tag{10.16}$$

If the loop gain is sufficiently high, $v_{b1} \to 0$ and $i_f \to -i_s$ so that the above equation can be used to obtain

$$\frac{v_o}{i_s} \simeq -\frac{1}{b} = -R_f \tag{10.17}$$

If the input source is a Thèvenin source having a voltage $v_s = i_s R_s$ in series with R_s, then $i_s = v_s/R_s$ and Eq. (10.17) gives $v_o/v_s \simeq -R_F/R_s$.

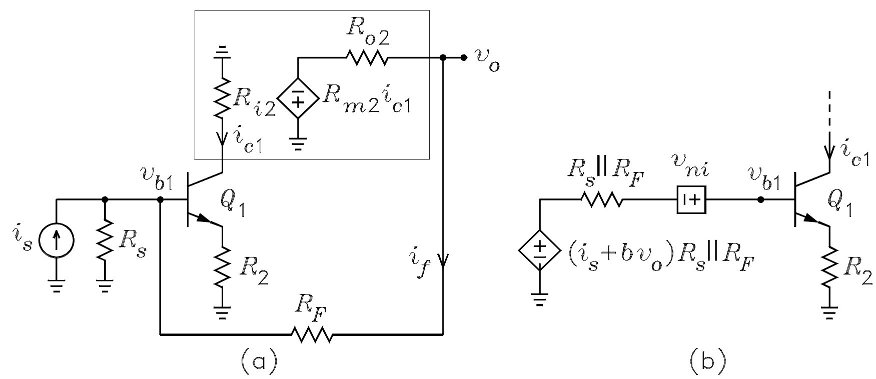

Figure 10.3: Amplifier with shunt-shunt feedback.

This is the familiar expression for the voltage gain of an inverting op amp having the same feedback network.

For a small-signal ac analysis, let us assume that the collector-emitter resistance r_0 of Q_1 is large enough so that it can be neglected. In this case, r_{ic} in the collector equivalent circuit of Fig. 7.12(b) is an open circuit and $i_{c1} = i_{c1(\mathrm{sc})}$. The output voltage is given by

$$\begin{aligned} v_o &= -R_{m2}i_{c1}\frac{R_F}{R_{o2}+R_F} + v_{b1}\frac{R_{o2}}{R_{o2}+R_F} \\ &\simeq -R_{m2}i_{c1}\frac{R_F}{R_{o2}+R_F} \end{aligned} \tag{10.18}$$

where the approximation neglects the contribution of v_{b1} to v_o. Because the effect of the negative feedback is to make v_{b1} smaller, this approximation is commonly made in the analysis of circuits having shunt summing at the input.

The circuit in Fig. 10.3(b) can be used to solve for i_{c1}. The figure shows the circuit seen looking out of the base of Q_1 replaced by a Thèvenin equivalent circuit with respect to i_s and v_o. The equivalent noise input voltage is modeled by the source v_{ni}. We will assume that the noise of the stage following Q_1 can be neglected so that v_{ni} models the noise generated by Q_1 and $R_s\|R_F$. Following Eq. (7.63), we can write

$$\begin{aligned} i_{c1} &= G_{m1}\left[(i_s+bv_o)\,R_s\|R_F + v_{ni}\right] \\ &= G_{m1}R_s\|R_F\left[i_s + bv_o + \frac{v_{ni}}{R_s\|R_F}\right] \end{aligned} \tag{10.19}$$

where G_{m1} and r'_e are given by

$$G_{m1} = \frac{\alpha}{r'_e + R_2} \tag{10.20}$$

$$r'_e = \frac{R_s \| R_F + r_x}{1+\beta} + r_e \tag{10.21}$$

Equations (10.18) and (10.19) can be combined to obtain

$$\begin{aligned} v_o &= -G_{m1} R_{m2} \frac{R_F}{R_{o2} + R_F} R_s \| R_F \left[i_s + b v_o + \frac{v_{ni}}{R_s \| R_F} \right] \\ &= -R_{mo} \left[i_s + b v_o + \frac{v_{ni}}{R_s \| R_F} \right] \end{aligned} \tag{10.22}$$

where R_{mo} is the negative of the transresistance gain with feedback removed given by

$$R_{mo} = G_{m1} R_{m2} \frac{R_F}{R_{o2} + R_F} R_s \| R_F \tag{10.23}$$

Eq. (10.22) can be solved for v_o to obtain

$$v_o = \frac{-R_{mo}}{1 + b R_{mo}} \left(i_s + \frac{v_{ni}}{R_s \| R_F} \right) \tag{10.24}$$

This equation is in the familiar form for the output voltage of a shunt-shunt feedback amplifier. In order to minimize confusion in equations caused by constants that have negative values, b and R_{mo} have been defined here so that both are positive. Some texts define these to be the negative of the definitions used here.

The equivalent noise input current in parallel with i_s is given by $i_{ni} = v_{ni}/R_s \| R_F$. When the noise generated by the stages following Q_1 is neglected and it is assumed that r_0 is large, the value of $\overline{v_{ni}^2}$ is obtained from Eq. (7.111) with R_1 replaced with $R_s \| R_F$. It is given by

$$\begin{aligned} \overline{i_{ni}^2} &= \frac{4kT\Delta f}{R_s \| R_F} \left(1 + \frac{r_x + R_2}{R_s \| R_F} \right) \\ &\quad + \left(2qI_B \Delta f + \frac{K_f I_B \Delta f}{f} \right) \left(1 + \frac{r_x + R_2}{R_s \| R_F} \right)^2 \\ &\quad + 2qI_C \Delta f \left[\frac{1}{\beta} + \frac{1}{R_s \| R_F} \left(\frac{r_x + R_2}{\beta} + \frac{V_T}{I_C} \right) \right]^2 \end{aligned} \tag{10.25}$$

The noise is minimized by making R_2 small and by making R_F large compared to R_s. In addition, the BJT should be biased at $I_{C(\text{opt})}$.

10.4 Design Example

The example design of a low-noise feedback amplifier having a voltage gain of 50 (34 dB) is presented in this section. The circuit is typical of a low-noise microphone preamplifier. The theoretically predicted noise performance of the circuit is compared to that predicted by a SPICE simulation.

Figure 10.4 shows the circuit diagram. The amplifier has a diff amp input stage with a JFET current source for its tail supply. The diff amp is followed by a second diff amp which cancels the common-mode noise generated by the JFET. This topology is commonly used in low-noise monolithic op amps. Resistors R_{C1} and R_{C2} together form what is called a *quiet load* on the input diff amp. A resistor is called a quiet load because it generates less noise than would be generated by an active load such as a current mirror or a dc current source. Active loads generate more noise because they amplify their own internal noise.

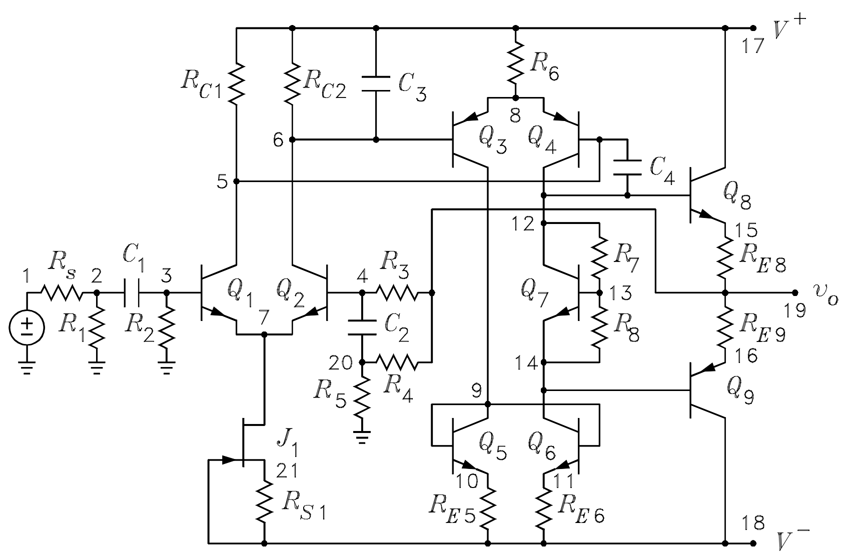

Figure 10.4: Example circuit.

The signal source is assumed to have the resistance $R_s = 200\ \Omega$. This is a typical output resistance for a microphone. If C_1 is considered to be an ac short circuit, the source is connected to the base of Q_1 through an input coupling network that is in the form of a voltage divider having a gain $R_1 \| R_2 / (R_s + R_1 \| R_2)$. For minimum noise, $R_1 \| R_2$ must be large compared to R_s. With $R_1 = R_2 = 40\ \text{k}\Omega$, it follows that $R_1 \| R_2$ is 20 kΩ. This is 100 times larger than R_s. With these resistor values, the effective midband source resistance seen by the base of Q_1 is $R_S \| R_1 \| R_2 = 198\ \Omega$.

The feedback network consists of R_3, R_4, R_5, and C_2. If we assume that the loop gain is large, the midband voltage gain is given by the reciprocal of the feedback ratio. If C_2 is considered to be an ac short circuit, the gain is given by $1 + (R_3 \| R_4)/R_5$ which is specified to be equal to 50. For minimum dc offset at the output, the dc resistances seen looking out of the bases of Q_1 and Q_2 must be the same, where the resistance seen looking out of the base of Q_2 is calculated with $v_o = 0$. Thus we must have $R_2 = R_3$. The noise analysis of the BJT diff amp in Fig. 7.31 assumes equal source resistances at its two inputs. For the source resistances seen by Q_1 and Q_2 in the present circuit to be equal at midband frequencies where C_1 and C_2 are ac short circuits, we must have $R_S \| R_1 \| R_2 = R_3 \| R_4 \| R_5$. The values $R_3 = 40\ \text{k}\Omega$, $R_4 = 13.2\ \text{k}\Omega$, and $R_5 = 202\ \Omega$ satisfy the gain and resistance relations.

The outputs from the input diff amp (Q_1 and Q_2) are connected to the input of a second diff amp (Q_3 and Q_4) which cancels the common-mode noise generated by J_1. The second diff amp has an active current-mirror load (Q_5 and Q_6) which drives the complementary common-collector output stage (Q_8 and Q_9). The base bias voltage for the output stage is provided by a V_{BE} multiplier voltage reference (Q_7). The resistor values given in the SPICE deck for the circuit are calculated to bias Q_3 through Q_6 at 1 mA each, Q_7 at 0.75 mA, and Q_8 and Q_9 at 2 mA each.

Capacitor C_1 prevents the base bias current in Q_1 from flowing in the input source. Capacitor C_2 gives 100% dc feedback for bias stability. To minimize their effect on noise, these capacitors must be large enough so that each is a signal short circuit in the transfer function for the impedance seen looking out of the bases of Q_1 and Q_2 at audio frequencies. This minimizes the small-signal resistance seen by the BJTs. To meet this condition, C_1 and C_2 must be calculated so that the zero frequency in each impedance transfer function is 20 Hz or lower. This requires $(R_S \| R_1) C_1 = (R_4 \| R_5) C_2 \geq 1/(2\pi 20)$. The values $C_1 = C_2 = 40\ \mu\text{F}$ satisfy this condition. The lower -3 dB cutoff frequency is set by the zero in the voltage gain transfer function of the feedback network. For the values given in the SPICE deck, the lower cutoff frequency is $f_\ell = (R_4 + R_5)/[2\pi R_5 (R_3 + R_4) C_2] = 5$ Hz.

Capacitors C_3 and C_4 frequency compensate the two forward paths in the amplifier and set the gain-bandwidth product. It can be shown that the two capacitors must have the same value and that the gain-bandwidth product is given approximately by $f_x = I_Q/(4\pi C V_T)$, where I_Q is the diff-amp tail current and $C = C_3 = C_4$. The upper -3 dB cutoff frequency is given by $f_u = f_x/A$, where A is the magnitude of the voltage gain with feedback. The value of C used in the example is 120 pF. This gives the amplifier an upper cutoff frequency of approximately 1 MHz.

For Q_1 and Q_2, let us assume that $r_x = 40\ \Omega$, $\beta = 500$, $V_T = 0.0259$ V, and $T = 300$ K. For the effective source resistance of 198 Ω, the value of $I_{C(\text{opt})}$ calculated from (7.139) is $I_{C(\text{opt})} = 2.92$ mA. Eq. (7.141) predicts a decrease in $\overline{v_{ni}^2}$ of 0.23 dB if a collector current of 1 mA is used instead of the optimum value. This is the bias current chosen for the example. Thus the drain current in J_1 must be 2 mA. If we assume the JFET parameters $V_{TO} = -2.4$ V and $\beta = 5 \times 10^{-4}$ A/V^2, it follows that $R_{S1} = 200\ \Omega$. For the diff amp, the mean-square equivalent noise input voltage is calculated by doubling the value calculated from (7.138). It follows that $\sqrt{\overline{v_{ni}^2}/\Delta f} = 2.95 \times 10^{-9}\ \text{V}/\sqrt{\text{Hz}}$. This is the equivalent noise input voltage in series with either BJT base. To transform it into the equivalent noise input voltage in series with v_s, it must be divided by $R_1 \| R_2/(R_s + R_1 \| R_2)$. Thus we obtain $\sqrt{\overline{v_{ni}^2}/\Delta f} = 2.98 \times 10^{-9}\ \text{V}/\sqrt{\text{Hz}}$.

The SPICE deck given in Table 10.1 gives the numerical values of all elements and model parameters used in the simulation. Although the model parameters for the JFET and the BJTs are representative, they do not represent the parameters for any specific device. The value for the forward current gain `BF` is chosen to give Q_1 and Q_2 a β_F of 500 for the assumed Early voltage $V_A = 100$ V. To illustrate the calculation of flicker noise, we assume that the flicker noise corner frequency for i_n has the value $f_{\text{flk}} = 150$ Hz. The value of the flicker noise coefficient is calculated from (7.125) to obtain $K_f = 4.81 \times 10^{-17}$.

Figures 10.5 and 10.6. show the results of the SPICE simulation. Fig. 10.5 shows the amplifier gain in dB versus frequency. The gain at 1 kHz is labeled 33.9 dB. Fig. 10.6 shows the equivalent noise input voltage referred to the source and the output noise, where the values are calculated for a bandwidth $\Delta f = 1$ Hz. At 1 kHz, the equivalent input noise referred to the source is 2.93 nV. This is greater than the theoretically predicted value by less than 2%. The low frequency rise in the noise due to flicker noise can be seen in the figure. The frequency below which the flicker noise begins to rise is lower than the flicker noise corner frequency of 150 Hz. This is because the noise sources in the circuit other than the i_n noise raise the midband noise level to effectively lower the frequency

Table 10.1: SPICE Deck for the Design Example

DESIGN EXAMPLE	J1 7 7 18 J1
VPLUS 17 0 DC 15	Q1 5 3 7 N1
VMINUS 18 0 DC -15	Q2 6 4 7 N1
VS 1 0 AC 1V	Q3 9 6 8 P1
RS 1 2 200	Q4 12 5 8 N1
R1 0 2 40K	Q5 9 9 10 N1
R2 0 3 40K	Q6 14 9 11 N1
R3 4 19 40K	Q7 12 13 14 N1
R4 19 20 13.2K	Q8 17 12 15 N1
R5 0 20 202	Q9 18 14 16 P1
R6 8 17 1.5K	.MODEL N1 NPN IS=12.6F
R7 12 13 4.4K	+BF=448 RB=40 VA=100
R8 13 14 2.6K	+TF=240P CJC=6.7P
RC1 5 17 1.51K	+KF=4.81E-17
RC2 6 17 1.51K	.MODEL P1 PNP IS=12.6F
RE5 10 18 1.5K	+BF=448 RB=40 VA=100 TF=240P
RE6 11 18 1.5K	+CJC=6.7P KF=4.81E-17
RE8 15 19 100	.MODEL J1 NJF VTO=-2.4
RE9 16 19 100	+BETA=5E-4
RS1 18 21 200	.OP
C1 2 3 40U	.AC DEC 20 10 1MEG
C2 4 20 40U	.NOISE V(19) VS
C3 6 17 270P	.PROBE
C4 5 12 270P	.END

at which flicker noise appears to rise. SPICE calculates the equivalent noise input voltage referred to the source at any frequency by dividing the output noise by the gain at that frequency.

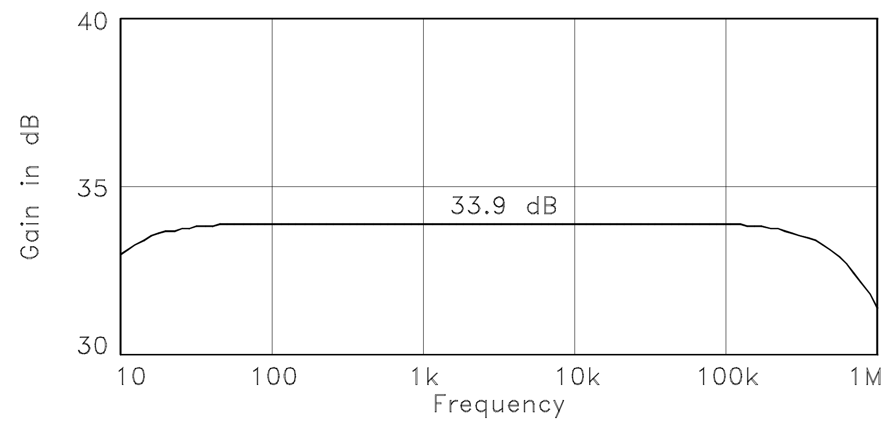

Figure 10.5: Gain versus frequency.

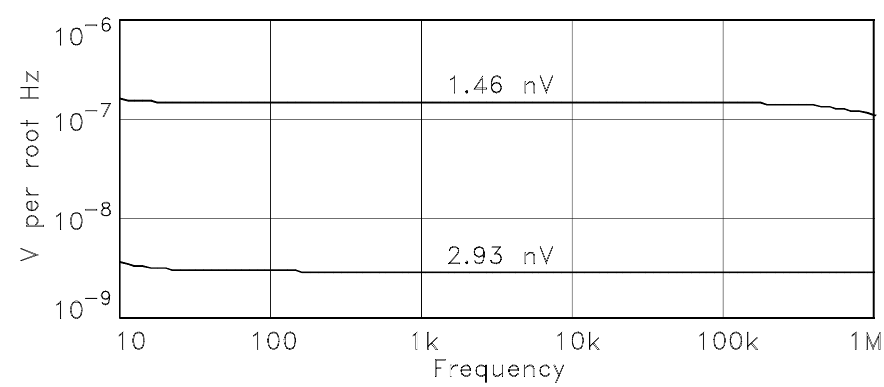

Figure 10.6: Input and output noise versus frequency.