Frequency Response
Overview

- Amplifier Frequency Response
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- Estimation of $\omega_L$ Using the Short-Circuit Time-Constant Method
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- Frequency Response of Multistage Amplifiers
- Single-Pole Operational-Amplifier Compensation
Amplifier Frequency Response (I)

Bode plot for a general amplifier function

\[ A_V(s) = \frac{a_o + a_1s + a_2s^2 \cdots + a_m s^m}{b_o + b_1s + b_2s^2 \cdots + b_n s^n} \]
Amplifier Frequency Response (II)

The numerator and denominator polynomials of the equation on the last slide can be separated into two groups: Those associated with the low-frequency response below the midband region $F_L(\omega)$ and those associated with the high-frequency response above the midband region $F_H(\omega)$.

$$A_V(s) = A_{mid} F_L(s) F_H(s)$$

$$F_L(s) = \frac{(s + \omega_{Z1}^L)(s + \omega_{Z2}^L) \cdots (s + \omega_{Zk}^L)}{(s + \omega_{P1}^L)(s + \omega_{P2}^L) \cdots (s + \omega_{Pk}^L)}$$

$$F_H(s) = \frac{1 + \frac{s}{\omega_{Z1}^H} \left(1 + \frac{s}{\omega_{Z2}^H} \cdots \left(1 + \frac{s}{\omega_{Zl}^H}\right)\right)}{1 + \frac{s}{\omega_{P1}^H} \left(1 + \frac{s}{\omega_{P2}^H} \cdots \left(1 + \frac{s}{\omega_{Pl}^H}\right)\right)}$$

$A_{mid}$ is the midband gain of the amplifier in the region between the lower- and upper cutoff frequencies ($\omega_L$ and $\omega_H$, respectively). Remember:

$$|A_V(j\omega_L)| = |A_V(j\omega_H)| = \frac{A_0}{\sqrt{2}}$$
Amplifier Frequency Response (III)

The representation of $F_H(s)$ is chosen so that its magnitude approaches a value of 1 at frequencies below the upper-cutoff frequency $\omega_H$:

$$|F_H(j\omega)| \to 1 \quad \text{for} \quad \omega << \omega_{z_i}^H, \omega_{p_i}^H \quad \text{for} \quad i = 1 \ldots l$$

Thus, at low frequencies, the transfer function $A(s)$ becomes

$$A_L(s) \approx A_{\text{mid}} F_L(s)$$

The representation of $F_L(s)$ is chosen so that its magnitude approaches a value of 1 at frequencies well above $\omega_L$:

$$|F_L(j\omega)| \to 1 \quad \text{for} \quad \omega >> \omega_{z_j}^L, \omega_{p_j}^L \quad \text{for} \quad i = 1 \ldots k$$

Thus, at high frequencies, the transfer function $A(s)$ can be approximated by

$$A_V(s) \approx A_{\text{mid}} F_H(s)$$
One of the low-frequency poles, say \( \omega_{p2} \), can be designed to be much larger than the others: 
\[
F_L(s) \approx \frac{s}{s + \omega_{p2}}
\]

Pole \( \omega_{p2} \) is referred to as the dominant low-frequency pole and the lower-cutoff frequency \( \omega_L \) is approximately: 

\[
\omega_L \approx \omega_{p2}
\]
Estimating $\omega_L$ in the Absence of a Dominant Pole (I)

If a dominant pole does not exist at low frequencies, then the poles and zeros interact to determine the lower-cutoff frequency.

As an example, consider the case of an amplifier having two zeros and two poles at low frequencies:

$$A_L(s) = A_{mid} F_L(s) = A_{mid} \frac{(s + \omega_{Z1})(s + \omega_{Z2})}{(s + \omega_{P1})(s + \omega_{P2})}$$

For $s=j\omega$:

$$|A_L(j\omega)| = A_{mid}|F_L(j\omega)| = A_{mid} \sqrt{\frac{(\omega^2 + \omega^2_{Z1})(\omega^2 + \omega^2_{Z2})}{(\omega^2 + \omega^2_{P1})(\omega^2 + \omega^2_{P2})}}$$
Estimating $\omega_L$ in the Absence of a Dominant Pole (II)

Squaring both sides:

$$\frac{1}{2} = \frac{\omega_4^2 + \omega_L^2(\omega_{Z1}^2 + \omega_{Z2}^2) + \omega_{Z1}^2 \omega_{Z2}^2}{\omega_4^2 + \omega_L^2(\omega_{p1}^2 + \omega_{p2}^2) + \omega_{p1}^2 \omega_{p2}^2} = \frac{1 + \left(\frac{\omega_{Z1}^2 + \omega_{Z2}^2}{\omega_L^2}\right) + \frac{\omega_{Z1}^2 \omega_{Z2}^2}{\omega_L^4}}{1 + \left(\frac{\omega_{p1}^2 + \omega_{p2}^2}{\omega_L^2}\right) + \frac{\omega_{p1}^2 \omega_{p2}^2}{\omega_L^4}}$$

If we assume that $\omega_L$ is larger than all the individual pole and zero frequencies, then the terms $1/\omega_L^4$ can be neglected and the lower-cutoff frequency can be estimated from

$$\omega_L = \sqrt{\omega_{p1}^2 + \omega_{p2}^2 - 2 \omega_{Z1}^2 - 2 \omega_{Z2}^2}$$

For the more general case of $n$ poles and $n$ zeros, a similar analysis yields:

$$\omega_L = \sqrt{n \omega_{pn}^2 - 2 n \omega_{zn}^2}$$
In the region above midband, $A(s)$ can be represented by its high-frequency approximation:

$$A_H(s) \approx A_{\text{mid}} F_H(s)$$
High Frequency Response (II)

Many of the zeros of $F_H(s)$ are often at infinite frequency, or high enough in frequency that they do not influence the value of $F_H(s)$ near $\omega_H$.

If, in addition, one of the pole frequencies – for example, $\omega_{p3}$ in the figure at slide 3 – is much smaller than all the others, then a **dominant high-frequency pole** exists in the high-frequency response:

$$F_H(s) = \frac{1}{1 + \frac{s}{\omega_{p3}}}$$

For the case of a dominant pole, the upper-cutoff frequency $\omega_H$ is given by $\omega_H \approx \omega_{p3}$
Estimating $\omega_H$ in the Absence of a Dominant Pole (I)

$$A_H(s) = A_{\text{mid}} |F_H(s)| = A_{\text{mid}} \left( \frac{1 + \frac{s}{\omega_{Z1}}}{1 + \frac{s}{\omega_{Z2}}} \right) \left( \frac{1 + \frac{s}{\omega_{P1}}}{1 + \frac{s}{\omega_{P2}}} \right)$$

For $s=j\omega$:

$$|A_H(j\omega)| = A_{\text{mid}} |F_H(j\omega)| = A_{\text{mid}} \sqrt{\left( \frac{1 + \frac{\omega^2}{\omega_{Z1}^2}}{1 + \frac{\omega^2}{\omega_{Z2}^2}} \right) \left( \frac{1 + \frac{\omega^2}{\omega_{P1}^2}}{1 + \frac{\omega^2}{\omega_{P2}^2}} \right)}$$

At the upper-cutoff frequency $\omega=\omega_H$:

$$|A_H(j\omega_H)| = \frac{A_{\text{mid}}}{\sqrt{2}} \quad \text{and} \quad \frac{1}{\sqrt{2}} = \sqrt{\left( \frac{1 + \frac{\omega_H^2}{\omega_{Z1}^2}}{1 + \frac{\omega_H^2}{\omega_{Z2}^2}} \right) \left( \frac{1 + \frac{\omega_H^2}{\omega_{P1}^2}}{1 + \frac{\omega_H^2}{\omega_{P2}^2}} \right)}$$
Estimating $\omega_H$ in the Absence of a Dominant Pole (II)

Squaring both sides:

$$\frac{1}{2} = \left(1 + \frac{\omega_H^2}{\omega_{Z1}^2}\right) \left(1 + \frac{\omega_H^2}{\omega_{Z2}^2}\right) = \frac{1}{2} \left(1 + \frac{\omega_H^2}{\omega_{P1}^2} + \frac{\omega_H^2}{\omega_{P2}^2} \right)$$

If $\omega_H$ is assumed to be smaller than all the individual pole and zero frequencies, then the terms involving $\omega_H^4$ can be neglected and the upper-cutoff frequency can be estimated from

$$\omega_H = \frac{1}{\sqrt{\frac{1}{\omega_{P1}^2} + \frac{1}{\omega_{P2}^2} - \frac{2}{\omega_{Z1}^2} - \frac{2}{\omega_{Z2}^2}}}$$

For the more general case of $n$ poles and $n$ zeros, a similar analysis yields:

$$\omega_H = \sqrt[2n]{\frac{1}{\omega_{Pn}^2} - 2} \frac{1}{\omega_{Zn}^2}$$
Direct Determination of the Low-Frequency Poles and Zeros (I)

In the following, we will analyze a common-source amplifier as an example for calculating all poles and zeros of a circuit.

Subsequently, we will develop approximation techniques for estimating $\omega_L$ and $\omega_H$. 

Frequency Response
Direct Determination of the Low-Frequency Poles and Zeros (II)

AC equivalent circuit:

Small-signal equivalent circuit (neglecting $r_o$):

Frequency Response
Direct Determination of the Low-Frequency Poles and Zeros (III)

\[ V_o(s) = I_o(s)R_3 \]

where

\[ I_o(s) = -g_m V_{gs}(s) \frac{R_D}{R_D + \frac{1}{sC_2} + R_3} \]

(appearing current division)

\[ V_o(s) = -g_m V_{gs}(s) \frac{R_D}{R_D + \frac{1}{sC_2} + R_3} R_3 \]

\[ V_o(s) = -g_m \left( R_3 \parallel R_D \right) \frac{s}{s + \frac{1}{C_2(R_D + R_3)}} V_{gs}(s) \]
Direct Determination of the Low-Frequency Poles and Zeros (IV)

\[ V_g(s) = V_i(s) \frac{R_G}{R_s + \frac{1}{sC_1} + R_G} = V_i(s) \frac{sC_1R_G}{sC_1(R_s + R_G) + 1} \]

(applying voltage division)

\[ g_m(V_g - V_s) - G_4V_s - sC_3V_s = 0 \]

\[ V_s = \frac{g_m}{sC_3 + g_m + G_4}V_g \]

\[ V_{gs}(s) = (V_g - V_s) = V_g \left[ 1 - \frac{g_m}{sC_3 + g_m + G_4} \right] = \frac{sC_3 + G_4}{sC_3 + g_m + G_4}V_g \]

\[ V_{gs}(s) = \frac{s + \frac{1}{C_3R_4}}{s + C_3 \left( \frac{1}{g_m \| R_4} \right)} V_g(s) \]
Direct Determination of the Low-Frequency Poles and Zeros (V)

\[ A_v(s) = \frac{V_o(s)}{V_i(s)} = A_{mid} F_L(s) \]

\[ A_v(s) = \left[ -g_m \left( R_3 \parallel R_D \right) \frac{R_G}{(R_S + R_G)} \right] \cdot \frac{s^2 \left[ s + \frac{1}{C_3 R_4} \right]}{s + \frac{1}{C_1 (R_S + R_G)}} \cdot \frac{s + \frac{1}{C_3 \left( \frac{1}{g_m} \parallel R_4 \right)}}{s + \frac{1}{C_2 (R_D + R_3)}} \]

\[ A(s) = A_{mid} F_L(s) \quad \text{where} \quad A_{mid} = -g_m \left( R_D \parallel R_4 \right) \frac{R_G}{R_D + R_3} \]

\( A_{mid} \) should be recognized as the voltage gain of the circuit, with the capacitors all replaced by short circuits.
Let us now explore the origin of the poles and zeros of the voltage transfer function.

It has three poles and three zeros, one pole and one zero for each independent capacitor in the circuit.

Two of the zeros are $s=0$ (DC), corresponding to series capacitors $C_1$ and $C_3$, each of which blocks the propagation of DC signals through the amplifier. The third zero occurs at the frequency for which the impedance of the parallel combination of $R_4$ and $C_3$ becomes infinite:

$$s = 0, 0, -\frac{1}{R_4C_3}$$
Direct Determination of the Low-Frequency Poles and Zeros (VII)

The three poles are located at frequencies of

\[
\begin{align*}
    s &= -\frac{1}{(R_S + R_G)C_1}, -\frac{1}{(R_D + R_3)C_2}, -\frac{1}{R_4 \parallel \frac{1}{g_m} C_3}
\end{align*}
\]

These pole frequencies are determined by the time constants associated with the three individual capacitors.

Because the input resistance of the MOSFET is infinite, the resistance present at the terminal of capacitor \(C_1\) is simply the series combination of \(R_S\) and \(R_G\), and because the output resistance \(r_o\) of the MOSFET has been neglected, the resistance associated with capacitor \(C_2\) is the series combination of \(R_3\) and \(R_D\). The effective resistance in parallel with capacitor \(C_3\) is the parallel combination of resistor \(R_4\) and \(1/g_m\).
Estimation of $\omega_L$ Using the Short-Circuit Time-Constant Method (I)

The midband region is of primary interest in most amplifier transfer functions. Knowledge of the exact position of all the poles and zeros is not necessary.

Two techniques, the short-circuit time-constant (SCTC) method and the open-circuit time-constant (OCTC) method, have been developed; these produce good estimates of $\omega_L$ and $\omega_H$, respectively, without having to evaluate the complete transfer function.
Estimation of $\omega_L$ Using the Short-Circuit Time-Constant Method (II)

It can be shown theoretically that the lower-cutoff frequency for a network having $n$ coupling and bypass capacitors can be estimated from

$$\omega_L \approx \frac{1}{\prod_{i=1}^{n} \frac{1}{R_{is}C_i}}$$

in which $R_{is}$ represents the resistance at the terminals of the $i$-th capacitor $C_i$ with all the other capacitors replaced by short circuits.

The product $R_{is}C_i$ represents the short-circuit time constant (SCTC) associated with capacitor $C_i$.

In the following, we use the SCTC method to find $\omega_L$ for the three classes of single-stage amplifiers.
Estimation of $\omega_L$ for the CE Amplifier (I)

Common-emitter amplifier including finite capacitor values
Estimation of $\omega_L$ for the CE Amplifier (II)

AC model for the common-emitter amplifier
Estimation of $\omega_L$ for the CE Amplifier: $R_{1S}$

For $C_1$, $R_{1S}$ is found by replacing $C_2$ and $C_3$ by short circuits, yielding the network shown above.

\[ R_{1S} = R_S + R_B \parallel R_{IN}^{CE} = R_S + R_B \parallel r_\pi \]
For $C_2$, $R_{2s}$ is found be replacing $C_1$ and $C_3$ by short circuits, yielding the network shown above.

\[ R_{2s} = R_3 + R_C \parallel R_{OUT}^{CE} = R_3 + R_C \parallel r_o \approx R_3 + R_C \]
For $C_3$, $R_{3S}$ is found by replacing $C_1$ and $C_2$ by short circuits, yielding the network shown above.

\[
R_{3S} = R_E \parallel R_{OUT}^{CC} = R_E \parallel \frac{r_\pi + R_{th}}{\beta_o + 1} \quad \text{where} \quad R_{th} = R_S \parallel R_B
\]
Estimation of $\omega_L$ for the CE Amplifier: Result

\[
\frac{1}{R_1 s C_1} = \frac{1}{(2.22 \, k\Omega)(2.00 \, \mu F)} = 225 \text{rad/} s
\]

\[
\frac{1}{R_2 s C_2} = \frac{1}{(104 \, k\Omega)(0.100 \, \mu F)} = 96.1 \text{rad/} s
\]

\[
\frac{1}{R_3 s C_3} = \frac{1}{(22.7 \, \Omega)(10 \, \mu F)} = 4410 \text{rad/} s
\]

\[
\omega_L \approx \left( \sum_{i=1}^{3} \frac{1}{R_i s C_i} \right) = 225 \text{rad/} s + 96.1 \text{rad/} s + 4410 \text{rad/} s = 4730 \text{rad/} s
\]

\[
f_L = \frac{\omega_L}{2\pi} = 753 \text{Hz}
\]

Note in this example that the time constant associated with emitter bypass capacitor $C_3$ is dominant; that is, the value of $R_3 s C_3$ is more than an order of magnitude larger than the two other time constants.

\[
f_L \approx \frac{1}{R_3 s C_3 \, 2\pi} = 702 \text{Hz}
\]
Estimation of $\omega_L$ for the CS Amplifier

AC model for the common-source amplifier

Same procedure as for the CE amplifier:

$$R_{1S} = R_S + R_G \parallel R_{IN}^{CS} = R_S + R_G$$
$$R_{2S} = R_3 + R_D \parallel R_{OUT}^{CS} = R_3 + R_D \parallel r_o \approx R_3 + R_D$$
$$R_{3S} = R_4 \parallel R_{OUT}^{CG} = R_4 \parallel \frac{1}{g_m}$$

Frequency Response
Estimation of $\omega_L$ for the CB Amplifier (I)

In this particular circuit, coupling capacitors $C_1$ and $C_2$ are the only capacitors present, and expressions for $R_{1S}$ and $R_{2S}$ are needed.
Estimation of $\omega_L$ for the CB Amplifier (II): $R_{1S}$

Equivalent circuit for determining $R_{1S}$

\[
R_{1S} = R_S + R_E \parallel R_{IN}^{CB} \approx R_S + R_E \parallel \frac{1}{g_m}
\]
Estimation of $\omega_L$ for the CB Amplifier (III): $R_{2S}$

Equivalent circuit for determining $R_{1S}$

\[ R_{th} = R_S || R_E \]

\[ R_{2S} = R_3 + R_C \parallel R_{OUT}^C \approx R_3 + R_C \]

because \( R_{OUT}^C \approx r_o (1 + g_m R_{th}) \) is large
Estimation of $\omega_L$ for the CG Amplifier: $R_{1S}$ and $R_{2S}$

AC circuit for the common-gate amplifier

The expressions for $R_{1S}$ and $R_{2S}$ for the common-gate amplifier are virtually identical to those of the common-base stage:

$R_{1S} = R_S + R_4 \parallel R_{IN}^{CG} = R_S + R_4 \parallel \frac{1}{g_m}$

$R_{2S} = R_3 + R_D \parallel R_{OUT}^{CG} \approx R_3 + R_D$

because $R_{OUT}^{CG} \approx \mu_f (R_4 \parallel R_S)$
Estimation of $\omega_L$ for the CC Amplifier (I)

Emitter-follower

AC circuit for the common-collector amplifier

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Frequency Response
Estimation of $\omega_L$ for the CC Amplifier (II): $R_{1S}$

Circuit for finding $R_{1S}$

$$R_{1S} = R_S + R_B \parallel R^{CC}_{IN} = R_S + R_B \parallel \left[r_\pi + (\beta_o + 1)\parallel (R_E \parallel R_3)\right]$$
Estimation of $\omega_L$ for the CC Amplifier (III): $R_{2S}$

Circuit for finding $R_{2S}$

\[
R_{th} = R_S \parallel R_B
\]

\[
R_{2S} = R_3 + R_E \parallel \frac{R_{CC}^{\text{OUT}}}{R_3 + R_E} \parallel \frac{r_\pi + R_{th}}{\beta_o + 1}
\]

where $R_{th} = R_S \parallel R_B$

Frequency Response
Estimation of $\omega_L$ for the CD Amplifier

Low-frequency AC equivalent circuit for common-drain amplifier

Same procedure as in the C-C case ($\beta_o$ and $r_\pi$ approach infinity):

$$R_{1S} = R_s + R_G \| R_{IN}^{CD} = R_s + R_G \quad \text{because} \quad R_{IN}^{CD} = \infty$$

$$R_{2S} = R_3 + R_4 \| R_{OUT}^{CD} = R_3 + R_4 \| \frac{1}{g_m}$$
Transistor Models at High Frequencies

To explore the upper limits of amplifier frequency response, the high-frequency limitations of the transistors must be taken into account.

All electronic devices have capacitances between their various terminals, and these capacitances limit the range of frequencies for which the devices can provide useful voltage, current, or power gain.
Frequency-Dependent Hybrid-Pi Model for the Bipolar Transistor

\[ C_\mu = \frac{C_{\mu o}}{\sqrt{1 + \frac{V_{CB}}{\Phi_{jc}}}} \]

- \( C_{\mu o} \) is the total collector-base junction capacitance at zero bias
- \( \Phi_{jc} \) is the built-in potential of the collector-base junction

\[ C_\pi = g_m \tau_F \]

- \( g_m \) is the transconductance
- \( \tau_F \) is the forward transit time

\( C_\mu \) represents the capacitance of the reverse-biased collector-base junction of the bipolar transistor and is related to the Q-point.

\( C_\pi \) represents the diffusion capacitance associated with the forward-biased base-emitter junction.
A General Observation

Capacitances such as $C_\mu$ are always present in electronic devices and circuits. At low frequencies, the impedance of these capacitances is usually very large and so has negligible effect relative to the resistances such as $r_\pi$. However, as frequency increases, the impedance of $C_\mu$ becomes smaller and smaller, and $v_{be}$ eventually approaches zero.

Thus, transistors cannot provide amplification at arbitrarily high frequencies.
A quantitative description of the behavior of the transistor at high frequencies can be found by calculating the frequency-dependent short-circuit current gain $\beta(s)$. 
Unity-Gain Frequency $f_T$ (II)

\[ I_c(s) = g_m V_{be}(s) - I_\mu(s) \]
\[ I_c(s) = \left( g_m - sC_\mu \right) V_{be}(s) \]

\[ V_{be}(s) = I_b(s) \frac{1}{s(C_\pi + C_\mu)} = I_b(s) \frac{r_\pi}{s(C_\pi + C_\mu) r_\pi + 1} \]

\[ I_c(s) = \left( g_m - sC_\mu \right) I_b(s) \frac{1}{s(C_\pi + C_\mu)} = I_b(s) \frac{r_\pi (g_m - sC_\mu)}{s(C_\pi + C_\mu) r_\pi + 1} \]

\[ \Rightarrow \beta(s) = \frac{I_c(s)}{I_b(s)} = \frac{\beta_o \left( 1 - \frac{sC_\mu}{g_m} \right)}{s(C_\pi + C_\mu) r_\pi + 1} \]
Unity-Gain Frequency $f_T$ (III)

A right-half-plane transmission zero occurs in the current gain at an extremely high frequency:

$$s = + \omega_z = + \frac{g_m}{C_\mu}$$

Neglecting $\omega_z$ results in the following simplified expression:

$$\beta(s) \approx \frac{\beta_o}{s(C_\pi + C_\mu)r_\pi + 1} = \frac{\beta_o}{s + 1}$$

in which $\omega_\beta$ represents the beta-cutoff frequency, defined by

$$\omega_\beta \approx \frac{1}{(C_\pi + C_\mu)r_\pi} \text{ and } f_\beta = \frac{\omega_\beta}{2\pi}$$
The current gain has the value of $\beta_o = g_m r_\pi$ at low frequencies and exhibits a single-pole roll-off at frequencies above $f_\beta$, decreasing at a rate of 20 dB/decade and crossing through unity gain at $\omega = \omega_T = 2\pi f_T$.

The parameter $f_T$ is referred to as the **unity gain-bandwidth product**.
For the MOSFET, the two added capacitors represent the gate oxid and overlap capacitance. At high frequencies, currents through these two capacitors combine to form a current in the gate terminal, and the signal current $i_g$ can no longer be assumed to be zero.

Thus, even the MOSFET has a finite current gain at high frequencies.
High-Frequency Model for the MOSFET (II)

The short-circuit current gain for the MOSFET can be calculated in the same manner as for the BJT:

\[
I_d(s) = (g_m - sC_{GD}) V_{gs}(s) = I_g(s) \frac{g_m - sC_{GD}}{s(C_{GS} + C_{GD})}
\]

\[
\Rightarrow \beta(s) = \frac{I_d(s)}{I_g(s)} = \frac{g_m \left(1 - \frac{sC_{GD}}{g_m}\right)}{s(C_{GS} + C_{GD})} = \frac{\omega_T}{s} \left(1 - \frac{s}{\omega_T \left(1 + \frac{C_{GS}}{C_{GD}}\right)}\right)
\]

At DC, the current gain is infinite but falls at a rate of 20dB/decade as frequency increases.

The unity gain-bandwidth product \(\omega_T\) of the MOSFET is defined in a manner identical to that of the BJT:

\[
\omega_T = \frac{g_m}{C_{GS} + C_{GD}}
\]
Limitations of the High-Frequency Models

The discussed models are good representations of the characteristics of the transistors for frequencies up to approximately 0.3 $f_T$. In addition, $f_T$ depends on the operating current of the device.

In the following, we assume that the specified value of $f_T$ corresponds to the operating point being used.
Base Resistance in the Hybrid-Pi Model (I)
Circuit element $r_x$ models the voltage drop between the base contact and the active region of the transistor and is included between the internal and external base nodes, B’ and B, respectively.

Resistance $r_x$ can represent an important limitation to the frequency response of the transistor in low-source resistance applications.

Typical values of $r_x$ range from a few 100 to a few 1000 ohms.
Effect of Base Resistance on Midband Amplifiers

\[ g'_m = g_m \frac{r_x}{r_x + r_\pi} = \frac{\beta_o}{r_x + r_\pi} \]

\[ r'_\pi = r_x + r_\pi \]
### Single-Stage Bipolar Amplifiers Including Base Resistance

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<th>Common-Emitter Amplifier</th>
<th>Common-Collector Amplifier</th>
<th>Common-Base Amplifier</th>
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<tr>
<td><strong>Voltage Gain</strong></td>
<td>$A_{vh} = \frac{v_v}{v_{th}}$</td>
<td>$\frac{-\beta_o R_t}{R_{th} + r_\pi + (\beta_o +1) R_s}$</td>
<td>$\frac{(\beta_o +1) R_L}{R_{th} + r_\pi + (\beta_o +1) R_L}$</td>
</tr>
<tr>
<td>$g_m^' = \frac{\beta_o}{r_x + r_\pi}$</td>
<td>$- \frac{g_m^' R_L}{1 + g_m^' R_s + \frac{R_{th} + R_s}{r_\pi}}$</td>
<td>$\frac{</td>
<td>\frac{1}{\alpha_o}</td>
</tr>
<tr>
<td></td>
<td>$r_x^' = r_x + r_\pi$</td>
<td>$r_x^' + (\beta_o +1) R_s$</td>
<td>$r_x^' + (\beta_o +1) R_L$</td>
</tr>
<tr>
<td><strong>Input Resistance</strong></td>
<td>$r_x^' + (\beta_o +1) R_s$</td>
<td>$r_x^' + (\beta_o +1) R_L$</td>
<td>$r_o + \mu_F R_s$</td>
</tr>
<tr>
<td><strong>Output Resistance</strong></td>
<td>$r_o + \mu_F R_s$</td>
<td>$\frac{\alpha_o}{g_m^'} + \frac{R_{th}}{\beta_o +1}$</td>
<td>$r_o + \mu_F R_{th}$</td>
</tr>
<tr>
<td><strong>Input Signal Range</strong></td>
<td>$\equiv 0.005 \left(1 + g_m^' R_s \right)$</td>
<td>$\equiv 0.005 \left(1 + g_m^' R_L \right)$</td>
<td>$\equiv 0.005 \left(1 + g_m^' R_{th} \right)$</td>
</tr>
<tr>
<td><strong>Current Gain</strong></td>
<td>$- \beta_o$</td>
<td>$\beta_o +1$</td>
<td>$\alpha_o \equiv +1$</td>
</tr>
</tbody>
</table>
High-Frequency Limitations of Single-Stage Amplifiers

We begin with an example of direct analysis of the common-emitter amplifier. The approximation technique for estimating $\omega_H$, the open-circuit time-constant method is presented afterwards.
Example of Direct High-Frequency Analysis (I)

At high frequencies, the impedance of the coupling and bypass capacitors are negligible small, and the three capacitors can once again be considered short circuits.
Example of Direct High-Frequency Analysis (II)

- $R_L$ represents the parallel combination of $R_3$ and $R_C$
- The emitter is connected directly to ground through bypass capacitor $C_3$
- Base resistor $R_B$ equals the parallel combination of $R_1$ and $R_2$
Example of Direct High-Frequency Analysis (III)

AC model for a common-emitter amplifier at high frequencies

\[ v_{th} = v_s \frac{R_B}{R_S + R_B} \]

\[ R_{th} = \frac{R_S R_B}{R_S + R_B} \]

Direct analysis proceeds using nodal equations. We apply a Norton transformation of the voltage sources and resistors attached to node \( v_1 \).
Example of Direct High-Frequency Analysis (IV)

\[ i_s = \frac{v_{th}}{R_{th} + r_x} \]

\[ r_{\pi o} = r_{\pi} \parallel (R_{th} + r_x) \]

\[ v_1 \]

\[ v_2 \]

\[ R_L \]

\[ C_{\mu} \]

\[ C_{\pi} \]
Example of Direct High-Frequency Analysis (V)

\[
\begin{bmatrix}
  I_s(s) \\
  0
\end{bmatrix} =
\begin{bmatrix}
  s(C_\pi + C_\mu) + g_\pi \omega_0 & -sC_\mu \\
  -(sC_\mu - g_m) & sC_\mu + g_L
\end{bmatrix}
\begin{bmatrix}
  V_1(s) \\
  V_2(s)
\end{bmatrix}
\]

An expression for the output voltage, node voltage \( V_2(s) \), can be found using Cramer’s rule:

\[
V_2(s) = I_s(s) \frac{(sC_\mu - g_m)}{\Delta}
\]

In which \( \Delta \) represents the determinant if the system of equations:

\[
\Delta = s^2C_\pi C_\mu + s[C_\pi g_L + C_\mu (g_m + g_\pi \omega_0 + g_L)] + g_L g_\pi \omega_0
\]

We see that the the high-frequency response is characterized by two poles, one finite zero, and one zero at infinity. The finite zero appears in the right-half of the s-plane at a frequency

\[
s = +\omega_z = +\frac{g_m}{C_\mu} > \omega_T \quad \text{(can usually be neglected)}
\]
Approximate Polynomial Factorization

Unfortunately, the denominator appears in unfactored polynomial form. We estimate the pole locations based on a technique for approximate factorization of polynomials. Let us assume that the polynomial has two real roots $a$ and $b$:

$$(s + a)(s + b) = s^2 + (a + b)s + ab = s^2 + A_1 s + A_0$$

If we assume that a dominant root exists ($a >> b$), then:

$$A_1 = a + b \approx a \quad \text{and} \quad A_0 = \frac{ab}{A_1} = \frac{ab}{a + b} \approx \frac{ab}{a} = b$$

The approximate factorization technique can be extended to polynomials having any number of widely spaced roots.
Dominant Pole of the C-E Amplifier –
The $C_T$ Approximation (I)

For the case of the common-emitter amplifier, the smallest root is the most important, because it is the one that limits the frequency response of the amplifier and determines $\omega_H$.

$$\omega_{p1} \approx \frac{A_0}{A_1} = \frac{g_L s \pi_o}{C_\pi g_L + C_\mu (g_m + g_{\pi o} + g_L)} \left( \frac{C_\pi C_\mu}{C_\pi C_\mu} \right)$$

$$\omega_{p1} \approx \frac{1}{r_{\pi o} \left[ C_\pi + C_\mu \left( 1 + g_m R_L + \frac{R_L}{r_{\pi o}} \right) \right]} = \left( \frac{1}{r_{\pi o} C_T} \right)$$

The lower-frequency pole is determined by resistor $r_{\pi o}$ and the total effective capacitance $C_T$:

$$C_T = C_\pi + C_\mu \left( 1 + g_m R_L + \frac{R_L}{r_{\pi o}} \right)$$
Dominant Pole of the C-E Amplifier –
The $C_T$ Approximation (II)

An overall expression for the gain of the common-emitter amplifier can be obtained like in the following:

$$V_o(s) = \frac{V_{th}(s)}{R_{th} + r_x} \frac{(sC_\mu - g_m)}{g_L g_{\pi o}} \left(1 + \frac{s}{\omega_{p1}}\right) \left(1 + \frac{s}{\omega_{p2}}\right)$$

$$V_o(s) = \frac{V_{th}(s)}{R_{th} + r_x} \left(- g_m R_L r_{\pi o}\right) \left(1 - \frac{s}{\omega_Z}\right) \left(1 + \frac{s}{\omega_{p1}}\right) \left(1 + \frac{s}{\omega_{p2}}\right)$$

Because $\omega_Z$ and $\omega_{p2}$ are both greater than $\omega_T$:

$$V_o(s) \approx \frac{V_{th}(s)}{R_{th} + r_x} \left(- g_m R_L r_{\pi o}\right) \left(1 + \frac{s}{\omega_{p1}}\right)$$
Recognizing that \( r_{\pi 0} = r_{\pi} (R_{th} + r_x) / (R_{th} + r_x + r_{\pi}) \) yields

\[
A_{Vth}(s) = \frac{V_o(s)}{V_{th}(s)} = -\frac{\beta_0 R_L}{R_{th} + r_x + r_{\pi}} \left( 1 + \frac{s}{\omega_{p1}} \right)
\]

or

\[
A_{Vth}(s) \approx \frac{A_{mid}}{1 + \frac{s}{\omega_{p1}}}
\]

with

\[
A_{mid} = -\frac{\beta_0 R_L}{R_{th} + r_x + r_{\pi}}
\]

\[
\omega_{p1} = \frac{1}{r_{\pi 0} C_T}
\]

The high-frequency behavior of the common-emitter amplifier can be modeled by a single dominant pole.
Gain-Bandwidth Product Limitations of the C-E Amplifier

The important role of \( r_x \) in ultimately limiting the frequency response of the C-E amplifier has also to be considered. Let us look at the asymptotic behavior of the gain-bandwidth product of the C-E amplifier:

\[
GBW = |A_v \omega_H| \leq \left( \frac{\beta_0 R_L}{R_{th} + r_x + r_\pi} \right) \left( \frac{1}{r_{\pi_0} C_T} \right)
\]

Using the following approximations:

\[
R_{th} = 0, \quad r_x << r_\pi \quad \text{so that} \quad r_{\pi_0} \approx r_x \quad \text{and} \quad C_T \approx C_\mu (g_m R_L)
\]

we get:

\[
GBW \leq \frac{1}{r_x C_\mu}
\]

This product of the base resistance \( r_x \) and the collector-base capacitance \( C_\mu \) places an upper bound on the gain-bandwidth product of the C-E amplifier.
Dominant Pole for the C-S Amplifier

Analysis of the C-S amplifier mirrors that of the C-E amplifier, except that $r_x$ and $r_\pi$ are absent from the small-signal model.

Frequency Response
Miller originally derived an expression for the input admittance of a general amplifier, in which capacitor $C$ is connected between the input and output terminals of an inverting amplifier. Let us calculate the input admittance of this amplifier: $V_o(s) = -AV_s(s)$ and $I_s(s) = sC[V_s(s) - V_o(s)]$

\[
Y(s) = \frac{I_s(s)}{V_s(s)} = sC(1 + A)
\]
Based on the equation on the last slide, the input admittance of the circuit can be represented by the equivalent circuit shown below.

The amplification of $C$ by the factor $(1+A)$ is often referred to as **Miller multiplication**.
Equivalent Representation of the C-E Amplifier Stage Based on Miller Multiplication

Capacitor $C_\pi$ appears across the input of an amplifier with gain $A=-g_m R_L$, and $C_\mu$ is connected as a feedback element between the input and output of the amplifier. Using Miller’s result, the total input capacitance will be

$$C_T = C_\pi + C_\mu (1 + A) = C_\pi + C_\mu (1 + g_m R_L)$$

(Miller analysis does not directly include the effect of finite input and output resistances of the amplifier.)
Miller Integrator

For frequencies well above $\omega_o$, ($\omega \gg \omega_o$), and assuming $A \gg 1$, we get

$$A_V(s) \approx -\frac{A\omega_o}{s} \approx -\frac{1}{sRC}$$

which should be recognized as the transfer function for an integrator. The Miller multiplication of capacitor $C$ is apparent.
Estimation of $\omega_H$ Using the Open-Circuit Time-Constant Method (I)

The upper-cutoff frequency $\omega_H$ is found by calculating the open-circuit time-constants (OCTC) associated with the various device capacitances rather than the short-circuit time constants associated with the coupling and bypass capacitors.

It can be shown theoretically that the mathematical estimate for $\omega_H$ for a circuit having $m$ capacitors is

$$\omega_H \approx \frac{1}{\sum_{i=1}^{m} R_{io} C_i}$$

in which $R_{io}$ represents the resistance measured at the terminals of capacitor $C_i$ with the other capacitors open circuited.
Estimation of $\omega_H$ Using the Open-Circuit Time-Constant Method (II)

We continue our analysis with the already analyzed C-E amplifier as an example:

Two capacitors, $C_{\pi}$ and $C_{\mu}$, are present, and $R_{\pi o}$ and $R_{\mu o}$ will be needed to evaluate the equation from the last slide.
Estimation of $\omega_H$ Using the Open-Circuit Time-Constant Method (III)

$R_{\pi \omega}$ can easily be determined from this circuit, in which $C_\mu$ is replaced by an open circuit, and we see that

$$R_{\pi \omega} = r_{\pi \omega}$$
Estimation of $\omega_H$ Using the Open-Circuit Time-Constant Method (IV)

\[ v_x = i_x r_{\pi o} + i_L R_L = i_x r_{\pi o} + (i_x + g_m v) R_L \]

\[ R_{\mu o} = \frac{v_x}{i_x} = r_{\pi o} + (1 + g_m r_{\pi o}) R_L = r_{\pi o} \left[ 1 + g_m R_L + \frac{R_L}{r_{\pi o}} \right] \]
Estimation of $\omega_H$ Using the Open-Circuit Time-Constant Method (V)

\[
\omega_H \approx \frac{1}{R_{\pi0}C_{\pi} + R_{\mu0}C_{\mu}} = \frac{1}{r_{\pi0}C_{\pi} + r_{\pi0}C_{\mu} \left(1 + \frac{g_m R_L}{r_{\pi0}}\right)} = \frac{1}{r_{\pi0}C_T}
\]

This is exactly the same result achieved with the $C_T$ approximation some slides ago but with far less effort.
Gain-Bandwidth Tradeoff
Using an Emitter Resistor (I)

We know that the gain of the C-E stage can be decreased if an unbypassed emitter resistor is added to the high-frequency equivalent circuit. For this stage, the midband gain is

$$A_{\text{mid}} = -\frac{\beta_0 R_L}{R_{th} + r_x + r_\pi + (\beta_0 + 1) R_E} \approx -\frac{g_m R_L}{1 + g_m R_E} \approx -\frac{R_L}{R_E}$$

For $r_\pi \gg (R_{th} + r_x)$ and $g_m R_E \gg 1$. The gain decreases as the value of $R_E$ increases. We expect that the bandwidth will increase as $A_{\text{mid}}$ decreases.
Gain-Bandwidth Tradeoff Using an Emitter Resistor (II)

High frequency small-signal model for the common-emitter amplifier

The two resistances, $R_{πo}$ and $R_{μo}$, must now be calculated for this new circuit configuration.
Gain-Bandwidth Tradeoff
Using an Emitter Resistor (III): $R_{\pi\alpha}$

Circuit for determining $R_{\pi\alpha}$

Simplified circuit with $r_{\pi}$ removed

Frequency Response
Gain-Bandwidth Tradeoff
Using an Emitter Resistor (IV): $R_{\pi o}$

\[ v_x = i_x (R_{th} + r_x) + (i_x - g_m v_x)R_E \]

\[ R_{EQ} = \frac{v_x}{i_x} = \frac{R_{th} + r_x + R_E}{1 + g_m R_E} \]

\[ R_{\pi o} = r_\pi \parallel R_{EQ} = r_\pi \parallel \frac{R_{th} + r_x + R_E}{1 + g_m R_E} \approx \frac{R_{th} + r_x + R_E}{1 + g_m R_E} \]
Gain-Bandwidth Tradeoff
Using an Emitter Resistor (V): $R_{\mu o}$

Circuit for finding $R_{\mu o}$

Equivalent two-source transformation
Gain-Bandwidth Tradeoff
Using an Emitter Resistor (VI): $R_{\mu o}$

Two circuits for finding $v_x = (v_b - v_c)$ by superposition

$v_c = -i_x R_L$ and $v_b = 0$

\[ v_b = i_x [(R_{th} + r_x) || (r_\pi + (\beta_0 + 1) R_E)] \]

\[ v_c = -\frac{\beta_0 R_L}{r_\pi + (\beta_0 + 1) R_E} v_b \]
Gain-Bandwidth Tradeoff
Using an Emitter Resistor (VII): $R_{\mu o}$

Combining the results yields $R_{\mu o}$:

$$R_{\mu o} = \frac{v_x}{i_x} = \frac{v_b - v_c}{i_x} = R_L + \left( (R_{th} + r_x) \parallel \left[ (r_p + (\beta_0 + 1)R_E) \right] \right) \left[ 1 + \frac{\beta_0 R_L}{r_p + (\beta_0 + 1)R_E} \right]$$

Assuming that $\beta_0 >> 1$ and $R_{th} + r_x << r_p + (\beta_0 + 1)R_E$:

$$R_{\mu o} \approx (R_{th} + r_x) \left[ 1 + \frac{g_m R_L}{1 + g_m R_E} + \frac{R_L}{R_{th} + r_x} \right]$$

Again combining the results yields the estimate for the upper-cutoff frequency of the generalized C-E stage as

$$\omega_H \approx \frac{1}{(R_{th} + r_x) \left[ \frac{C_p}{1 + g_m R_E} \left( 1 + \frac{R_E}{R_{th} + r_x} \right) + C_\mu \left( 1 + \frac{g_m R_L}{1 + g_m R_E} + \frac{R_L}{R_{th} + r_x} \right) \right]}$$

The upper-cutoff frequency is increased by approximately the same factor by that the midband gain has been reduced.
Dominant Pole for the C-B Amplifier (I)

\[
R_{th} = R_E \parallel R_S \quad \text{and} \quad R_L = R_C \parallel R_3
\]
Dominant Pole for the C-B Amplifier (II): $R_{\pi \varnothing}$

$$v_x = (i_x - g_m v_x) R_{th} + i_x r_x$$

$$R_{\pi \varnothing} = r_{\pi} \parallel \frac{v_x}{i_x} = r_{\pi} \parallel \frac{R_{th} + r_x}{1 + g_m R_{th}} \approx \frac{R_{th} + r_x}{1 + g_m R_{th}}$$
Dominant Pole for the C-B Amplifier (III): $R_{\mu_0}$
Dominant Pole for the C-B Amplifier (IV): $R_{\mu o}$

\[ v_c = -i_x R_L \quad \text{and} \quad v_b = 0 \]

\[ v_b = i_x [r_x \parallel (r_{\pi} + (\beta_0 + 1)R_{th})] \]

\[ R_L \quad v_c = -\frac{\beta_0 R_L}{r_{\pi} + (\beta_0 + 1)R_{th}} v_b \]
Dominant Pole for the C-B Amplifier (V)

Combining the results yields \( R_{\mu o} \):

\[
R_{\mu o} = \frac{v_x}{i_x} = \frac{v_b - v_c}{i_x} = R_L + \left\{ r_x || \left[ r_\pi + (\beta_0 + 1)R_{th} \right] \right\} \left( 1 + \frac{\beta_0 R_L}{r_\pi + (\beta_0 + 1)R_{th}} \right)
\]

Assuming that \( \beta_0 >> 1 \) and \( r_x << r_\pi \)

\[
R_{\mu o} \approx r_x \left( 1 + \frac{g_m R_L}{1 + g_m R_{th}} \right) + R_L
\]

Again combining the results yields the estimate for the upper-cutoff frequency of the generalized C-B stage as

\[
\omega_H \approx \frac{1}{r_x \left[ \frac{C_\pi}{1 + g_m R_{th}} \left( 1 + \frac{R_{th}}{r_x} \right) + C_\mu \left( 1 + \frac{g_m R_L}{1 + g_m R_{th}} \right) \right] + C_\mu R_L}
\]

And with further neglections:

\[
\omega_H \approx \frac{1}{r_x C_\mu \left( 1 + \frac{g_m R_L}{1 + g_m R_{th}} \right) + C_\mu R_L} \approx \frac{1}{R_L C_\mu}
\]
Dominant Pole for the C-G Amplifier (I)

High-frequency AC equivalent circuit for C-G amplifier

Corresponding small-signal model
Dominant Pole for the C-G Amplifier (II)

The results are similar to those for the C-B circuit except $r_x=0$ and some changed symbols.

\[
R_{GSo} = \frac{R_{th}}{1 + g_m R_{th}} = \frac{1}{G_{th} + g_m}
\]

\[R_{GDo} = R_L\]

\[
\omega_H \approx \frac{1}{\frac{C_{GS}}{G_{th} + g_m} + R_L C_{GD}} \leq \frac{1}{R_L C_{GD}}
\]
Dominant Pole for the C-C Amplifier (I)

High-frequency AC equivalent circuit for C-C amplifier
Dominant Pole for the C-C Amplifier (II)

Small-signal model

Rearrangement of the small-signal model ($C_\mu$ actually appears in parallel with the input of the transistor)

$$R_{th} = R_B \parallel R_S \quad \text{and} \quad R_L = R_E \parallel R_3$$
Dominant Pole for the C-C Amplifier (III)

\[ R_{\mu o} = (R_{th} + r_x) \parallel R_{IN}^{CC} = (R_{th} + r_x) \parallel \left[ r_{\pi} + (\beta_0 + 1)R_L \right] \approx (R_{th} + r_x) \]
Dominant Pole for the C-C Amplifier (IV)

Circuit for finding $R_{\pi 0}$

$$R_{\pi 0} = r_\pi || R_{EQ}$$

$R_{EQ}$ is the resistance presented to the test current source $i_x$ in the bottom figure on this slide.

$$v_x = i_x (R_{th} + r_x) + i_L R_L$$

$$v_x = i_x (R_{th} + r_x) + (i_x - g_m v_x) R_L$$

$$R_{EQ} = \frac{v_x}{i_x} = \frac{R_{th} + r_x + R_L}{1 + g_m R_L}$$

$$R_{\pi 0} = r_\pi \| \frac{R_{th} + r_x + R_L}{1 + g_m R_L}$$

$$R_{\pi 0} \approx \frac{R_{th} + r_x + R_L}{1 + g_m R_L}$$
Dominant Pole for the C-C Amplifier (V)

\[
\omega_H \approx \frac{1}{(R_{th} + r_x + R_L) \frac{C_\pi}{1 + g_m R_L} + (R_{th} + r_x) C_\mu}
\]

In the case of the emitter-follower, the OCTC technique underestimates \( \omega_H \). A better estimate is obtained if \( R_L \) is set to zero in the calculation of \( R_{EQ} \):

\[
\omega_H \approx \frac{1}{(R_{th} + r_x) \left[ \frac{C_\pi}{1 + g_m R_L} + C_\mu \right]}
\]

The upper-cutoff frequency of the emitter follower is very high. Thus the voltage gain of the emitter follower is essentially constant at a value of unity for all frequencies for which the hybrid-pi model is valid. Note that the last equation also represents the GBW product for \( R_{th}=0 \) and large \( R_L \): 

\[
\text{GBW} = 1 \cdot (\omega_H) \leq \frac{1}{r_x C_\mu}
\]
Dominant Pole for the C-D Amplifier (I)

The source follower offers behavior very similar to that of the emitter follower. Substituting $r_\pi=\infty$ and $r_x=0$:

\[ R_{th} = R_S \parallel R_G \]
\[ R_L = R_4 \parallel R_3 \]

\[ \omega_H = \frac{1}{R_{th} \left( \frac{C_{GS}}{1 + g_m R_L} + R_{th} C_{GD} \right)} \]

\[ \omega_H = \frac{1}{R_{th} \left[ \frac{C_{GS}}{1 + g_m R_L} + C_{GD} \right]} \]
Single-Stage Amplifier
High-Frequency Response Summary

- The inverting amplifiers provide high voltage gain but with the most limited bandwidth.

- The noninverting stages offer improved bandwidth with voltage gains similar to those of the inverting amplifiers. Remember, however, that the input resistance of the noninverting amplifiers is relatively low.

- The followers provide unity gain with very wide bandwidth.

It is also worth noting that both the C-E and C-B (or C-S and C-G) stages have a bandwidth that is always less than that set by the time constant of $R_L$ and $C_\mu$ (or $C_{GD}$ and $R_L$) at the output node:

$$\omega_H < \frac{1}{R_L C_\mu} \quad \text{or} \quad \omega_H < \frac{1}{R_L C_{GD}}$$
## Upper Cutoff Frequencies for the Single-Stage Amplifiers

<table>
<thead>
<tr>
<th>Common-</th>
<th>( \omega_H )</th>
</tr>
</thead>
</table>
| Emitter        | \[
\frac{1}{r_\pi C_T} = \frac{1}{r_\pi \left[ C_x + C_\mu \left| 1 + g_m R_L + \frac{R_L}{r_\pi} \right| \right]}
\] ; \( r_\pi = \left( R_{th} + r_x \right) \]
|                | \[
\frac{1}{R_{th} C_T} = \frac{1}{R_{th}} \left[ C_{GS} + C_{GD} \left| 1 + g_m R_L + \frac{R_L}{R_{th}} \right| \right]
\]
| Common-        | \[
C_{TB} = \frac{C_x}{1 + g_m R_E} \left( 1 + \frac{R_L}{R_{th} + r_x} \right) + C_\mu \left( 1 + \frac{g_m R_L}{1 + g_m R_E} + \frac{R_L}{R_{th} + r_x} \right)
\]
| Common-        | \[
\frac{1}{R_{th}} \left[ \frac{C_x}{1 + g_m R_{th}} \left( 1 + \frac{r_x}{R_{th}} \right) + C_\mu \left( 1 + \frac{g_m R_l}{1 + g_m R_{th}} \right) + C_l R_L \right]
\]
| Base           | \[
\omega_H \equiv \frac{1}{C_{GS} \left( G_{th} + g_m \right) + R_L C_{GD}} \equiv \frac{1}{R_L C_{GD}}
\]
| Common-        | \[
\left( R_{th} + r_x \right) \frac{C_x}{1 + g_m R_L + C_\mu}
\]
| Common-        | \[
R_{th} \left[ C_{GS} \left( 1 + g_m R_L \right) + C_{GD} \right]
\]

### Institute of Microelectronic Systems

Frequency Response
Frequency Response of Multistage Amplifiers

The open- and short-circuit time-constant methods are not limited to single-transistor amplifiers but are directly applicable to multistage circuits as well.

In the following, we will use the OCTC techniques to estimate the frequency response of several important two-stage DC-coupled amplifiers:

- differential amplifier
- cascode stage
- current mirror

Because these amplifiers are direct-coupled, they have low-pass characteristics and only the OCTC method is needed to determine $f_H$. 
An important element, $C_{EE}$, has been included in the differential amplifier circuit. $C_{EE}$ represents the total capacitance at the emitter node of the differential pair.

Analysis of the frequency response of the symmetrical amplifier is greatly simplified through the use of the half-circuits shown on the next slides.
Differential Amplifier (II): Differential-Mode Signals

We recognize the differential-mode half-circuit as being equivalent to the standard C-E stage.

Thus the bandwidth for differential-mode signals is determined by the already presented $r_{\pi}C_T$ product.

Because the emitter node is a virtual ground, $C_{EE}$ has no effect on differential-mode signals.
Differential Amplifier (II): Common-Mode Frequency Response

At very low frequencies, we know that the common-mode gain to either collector is small, given approximately by

$$|A_{cc}(0)| \approx \frac{R_C}{2R_{EE}} \ll 1$$

Capacitance $C_{EE}$ introduces a transmission zero:

$$s = -\omega_z = -\frac{1}{R_{EE} C_{EE}}$$

The common-mode half circuit is equivalent to a common-emitter stage with emitter resistor $2R_{EE}$; the OCTC for $C_{EE}/2$ is just the resistance looking back into the emitter

$$R_{EE0} = 2R_{EE} \parallel \frac{r_x + r_\pi}{\beta_0 + 1} \approx \frac{1}{g_m}$$
Differential Amplifier (III):
Common-Mode Frequency Response, cont’d

\[ \omega_p \approx \frac{1}{r_x \left[ \frac{C_\pi}{1+2g_mR_{EE}} \left( 1+\frac{2R_{EE}}{r_x} \right) + C_\mu \left( 1+\frac{g_mR_C}{1+2g_mR_{EE}} + \frac{R_C}{r_x} \right) \right] + \frac{C_{EE}}{2g_m}} \]

\[ \omega_p \approx \frac{1}{\frac{C_\pi + C_{EE}}{2g_m} + C_\mu (r_x + R_C)} \approx \frac{1}{C_\mu (r_x + R_C)} \]
The C-C/C-B Cascade (I)

The circuit to the left shows an unbalanced version of the differential amplifier. This circuit can also be represented as the cascade of a common-collector and common-base amplifier (see below).
The C-C/C-B Cascade (II)

The cutoff-frequency of this two-stage amplifier is found by applying the OCTC approach, using the results of the previous single-stage amplifier analyses.

We assume $R_{EE} = \infty$

$$R_{CB2}^{IN} = \frac{r_{x2} + r_{\pi2}}{\beta_{02} + 1} \approx \frac{1}{g_{m2}}$$

$$R_{OUT}^{CC1} = \frac{r_{x1} + r_{\pi1}}{\beta_{01} + 1} \approx \frac{1}{g_{m1}}$$

Frequency Response
The C-C/C-B Cascade (III)

When using the OCTC approach, the capacitances of $Q_1$ and $Q_2$ can be considered individually or grouped in pairs. By grouping the capacitors in pairs, we can use the already derived single-stage amplifier results:

$$r_{x1} \left[ \frac{C_{\pi 1}}{1 + g_m \frac{1}{g_m}} + C_{\mu 1} \right] = r_{x1} \left[ \frac{C_{\pi 1}}{2} + C_{\mu 1} \right]$$

(Emitter Follower with a load resistance equal to $1/g_{m2}$)

$$r_{x2} \left[ \frac{C_{\pi 2}}{1 + g_m \frac{1}{g_m r_{x2}}} \right] + C_{\mu 2} \left[ 1 + \frac{g_m R_L}{1 + \frac{g_m}{g_m}} \right] + C_{\mu 2} R_L \approx r_{x2} \left[ \frac{C_{\pi 2}}{2} + C_{\mu 2} \left( 1 + \frac{g_m R_L}{2} + \frac{R_L}{r_{x2}} \right) \right]$$

(Common-base stage with a source resistance equal to $1/g_{m1}$)
The C-C/C-B Cascade (IV)

\[
\omega_H = \frac{1}{r_{x1} \left[ \frac{C_{\pi 1}}{2} + C_{\mu 1} \right] + r_{x2} \left[ \frac{C_{\pi 2}}{2} + C_{\mu 2} \left( 1 + \frac{g_m R_L}{2} + \frac{R_L}{r_{x2}} \right) \right]}
\]

Assuming that the transistors are matched:

\[
\omega_H = \frac{1}{r_x \left[ C_{\pi} + C_{\mu} \left( 2 + \frac{g_m R_L}{2} + \frac{R_L}{r_x} \right) \right]}
\]
The cascode stage offers a midband gain and input resistance equal to that of the common-emitter amplifier but with a much improved upper-cutoff frequency $f_H$, as will be demonstrated by the forthcoming analysis.

OCTC approach: Time constants associated with $C_{\pi 1}$, $C_{\mu 1}$, $C_{\pi 2}$, and $C_{\mu 2}$ must be found.
High-Frequency Response of the Cascode Amplifier (II)

The open-circuit time constants associated with $C_{\pi_1}$ and $C_{\mu_1}$ are calculated with $C_{\pi_2}$, $C_{\mu_2}$ removed from the circuit. $Q_2$ can simply be replaced by its midband input resistance $1/g_{m2}$.

$$R_{\pi_01}C_{\pi_1} + R_{\mu_01}C_{\mu_1} = r_{\pi_01}C_{T1} = r_{\pi_01}\left[C_{\pi_1} + C_{\mu_1}\left(1 + \frac{g_{m1}}{g_{m2}} + \frac{1}{g_{m2}r_{\pi_01}}\right)\right]$$

Because $I_{C2} \approx I_{C2}$, $g_{m1} \approx g_{m2}$, and the intrinsic gain of the first stage is unity:

$$A_{V1} \approx -g_{m1}R_L \approx -g_{m1}\frac{1}{g_{m2}} = -1$$

If we assume $g_{m2}r_{\pi_01} \gg 1$, then

$$r_{\pi_01}C_{T1} \approx r_{\pi_01}(C_{\pi_1} + 2C_{\mu_1})$$
The open-circuit time constants associated with $C_{\pi 2}$ and $C_{\mu 2}$ are calculated with $C_{\pi 1}$, $C_{\mu 1}$ removed from the circuit. Thus, the first stage can be replaced by $r_{o1}$.

$$R_{\pi 02} C_{\pi 2} + R_{\mu 02} C_{\mu 2} = \frac{C_{\pi 2}}{1 + g_{m2} r_{o1}} (r_{o1} + x) + r_{x2} C_{\mu 2} \left( 1 + \frac{g_{m2} R_L}{1 + g_{m2} r_{o1}} + \frac{R_L}{r_{x2}} \right)$$

$$R_{\pi 02} C_{\pi 2} + R_{\mu 02} C_{\mu 2} \approx \frac{C_{\pi 2}}{g_{m2}} + r_{x2} C_{\mu 2} \left( 1 + \frac{R_L}{r_{o1}} + \frac{R_L}{r_{x2}} \right)$$

$$R_{\pi 02} C_{\pi 2} + R_{\mu 02} C_{\mu 2} \approx \frac{C_{\pi 2}}{g_{m2}} + (r_{x2} + R_L) C_{\mu 2} \text{ for } r_{o1} >> R_L \text{ and } \mu_F >> 1$$
High-Frequency Response of the Cascode Amplifier (IV)

Combining the results yields:

\[
\omega_H = \frac{1}{R_{\pi 1} C_\pi + R_{\mu 1} C_\mu + R_{\pi 2} C_\pi + R_{\mu 2} C_\mu}
\]

Assuming matched devices:

\[
\omega_H \approx \frac{1}{r_{\pi 1} C_\pi + C_\mu \left(2r_{\pi 1} + r_x + R_L\right)}
\]

\[
\omega_H \approx \frac{1}{r_{\pi 1} \left(C_\pi + 2C_\mu\right) + (r_x + R_L) C_\mu}
\]
Cutoff Frequency for the Current Mirror (I)

As a final example of the analysis of direct-coupled amplifiers, let us find $\omega_H$ for the current mirror configuration shown above.

Small-signal model for the current mirror (this circuit should be recognized as identical to the simplified model of the C-E stage analyzed above)
Cutoff Frequency for the Current Mirror (II)

Setting \( r_{\pi o} \rightarrow \frac{1}{g_{m1}} \) \( R_L \rightarrow r_{o2} \) \( C_\pi \rightarrow C_{GS1} + C_{GS2} \) \( C_\mu \rightarrow C_{GD2} \)

\[ \omega_{p1} \approx \frac{1}{r_{\pi 0} C_T} = \frac{1}{\frac{1}{g_{m1}} C_{GS1} + C_{GS2} + C_{GD2} \left( 1 + g_{m2} r_{o2} + \frac{r_{o2}}{1/g_{m1}} \right)} \]

Assuming matched devices:

\[ \omega_{p1} \approx \frac{1}{2 C_{GS1} + 2 C_{GD2} r_{o2}} \approx \frac{1}{2 C_{GD2} r_{o2}} \]
Single-Pole Operational-Amplifier Compensation (I)

General-purpose operational amplifier often use internal **frequency compensation**, which forces the overall amplifier to have a single-pole frequency response.

The voltage transfer functions of these amplifiers can be represented by

\[
A_v(s) = \frac{A_o \omega_B}{s + \omega_B} = \frac{\omega_T}{s + \omega_B}
\]

This form of transfer function can be obtained by connecting a compensation capacitor \( C_C \) around the second gain stage of the basic operational amplifier, as depicted in the circuit on the next slide.
This capacitor transforms the second gain stage, transistor $M_5$, into a Miller integrator.
Single-Pole Operational-Amplifier Compensation (III)

Simplified model for three-stage op amp

- The input stage is modeled by its Norton equivalent circuit (represented by current source $G_m v_{dm}$ and output resistance $R_o$)
- The second stage forms a Miller integrator with voltage gain $A_{V2} = g_{m5} r_{o5} = \mu_{f5}$
- The follower output stage is represented as a unity-gain buffer
Single-Pole Operational-Amplifier Compensation (IV)

The circuit can be further simplified using the **Miller effect** relations. Feedback capacitor $C_C$ is multiplied $(1+A_{V2})$ and placed in parallel with the input of the second-stage amplifier.

The output voltage $V_o(s)$ must equal $V_b(s)$ because the output buffer has a gain of 1. Also, $V_b(s)$ equals $-A_{V2}V_a(s)$.

The Derivation of $\omega_B$ and $\omega_T$ follows on the next slides.
Single-Pole Operational-Amplifier Compensation (V)

\[-G_m V_{dm}(s) = V_a(s)[sC_C(1 + A_{V2}) + G_o]\]

\[\frac{V_a(s)}{V_{dm}(s)} = \frac{-G_m R_o}{sR_o C_C(1 + A_{V2}) + 1}\]

\[A_V(s) = \frac{V_b(s)}{V_{dm}(s)} = \frac{-A_{V2}V_a(s)}{V_{dm}(s)} = \frac{G_m R_o A_{V2}}{1 + sR_o C_C(1 + A_{V2})}\]

\[A_V(s) = \frac{G_m A_{V2}}{C_C(1 + A_{V2})} = \frac{\omega_T}{s + \omega_B} = \frac{A_o \omega_B}{s + \omega_B}\]

\[\omega_B = \frac{1}{R_o C_C(1 + A_{V2})}\]

\[\omega_T = \frac{G_m A_{V2}}{C_C(1 + A_{V2})} \approx \frac{G_m}{C_C}\] for large \(A_{V2}\)
The single pole of the amplifier is at a relatively low frequency, as determined by the large values of the output resistance of the first stage and the Miller input capacitance of the second stage.
Transmission Zeros in FET Op Amps (I)

The simplified Miller approach applied on the last slides does not take into account the finite transconductance of the second-stage amplifier.

Using the complete small-signal model for the transistor $M_5$ (like depicted below) shows the zero that is determined by $g_{m5}$ and the total feedback capacitance between the drain and gate of $M_5$. (The circuit below has the same topology as the circuit for the simplified C-E amplifier).
Transmission Zeros in FET Op Amps (II)

Setting \( r_{\pi o} \to R_o \), \( R_L \to r_{o5} \), \( C_\pi \to C_{GS5} \), \( C_\mu \to C_C + C_{GD5} \)

\[
A_{th}(s) = \left(-g_{m5}r_{o5}\right) \left(1 - \frac{s}{\omega_Z}\right)
\]

in which \( \omega_Z = \frac{g_{m5}}{C_C + C_{GD5}} = \omega_r \frac{g_{m5}}{g_{m2}} \)

\[\omega_{p1} = \frac{1}{R_o C_T} \quad \text{with} \quad C_T = C_{GS5} + (C_C + C_{GD5}) \left(1 + \mu_f r_{o5} + \frac{r_{o5}}{R_o}\right)\]
Transmission Zeros in FET Op Amps (III)

The addition of resistor $R_Z$ cancels the zero in the voltage transfer function.

If we assume that $C_C >> C_{GD}$, then the location of $\omega_Z$ in the numerator becomes

$$\omega_Z = \frac{g_{m5}}{C_C} \left( \frac{1}{R_R} \right)$$

and the zero can be eliminated by setting $R_Z = 1/g_{m5}$. 
Bipolar Amplifier Compensation (I)
Bipolar Amplifier Compensation (II)

The bipolar op amp is compensated in the same manner as the MOSFET amplifier. However, because the transconductance of the BJT is generally much higher than that of a FET for a given operating point, the transmission zero occurs at such a high frequency that it does not cause a problem.

\[
\omega_T = \frac{g_m}{C_C} = \frac{40I_{C2}}{C_C} = \frac{20I_1}{C_C} \quad \text{and} \quad \omega_Z = \frac{g_m}{C_C} = \omega_T \left( \frac{I_{C5}}{I_{C2}} \right)
\]

Because \( I_{C5} \) is 5 to 10 times \( I_{C2} \) in most designs, \( \omega_Z \) is typically at a frequency of 5 to 10 times the unity gain frequency \( \omega_T \).
Slew Rate of an Operational Amplifier (I)

Slew-rate limiting of the output voltage of amplifiers occurs because there is a limited amount of current available to charge and discharge the internal capacitors of the amplifiers.

For an internally compensated amplifier, $C_C$ typically determines the slew rate.
Example of a CMOS amplifier with a large input signal (no longer a small signal). In this case, the voltages applied to the differential input stage cause current $I_1$ to switch completely to one side of the differential pair.
The figure above shows a simplified model for the amplifier in the condition described on the last slide.

Because of the unity-gain output buffer, output voltage $v_o$ follows voltage $v_B$. Current $I_1$ must be supplied through compensation capacitor $C_C$. 
Slew Rate of an Operational Amplifier (IV)

The rate of change of $v_B$, and hence $v_o$, must satisfy

$$I_1 = C_C \frac{d(v_B(t) - v_A(t))}{dt} = C_C \frac{d(v_B(t) - v_B(t))}{dt}$$

If $A_{v2}$ is assumed to be very large, then the amplifier will behave in a manner similar to an ideal integrator; that is, node voltage $v_A$ represents a virtual ground and we yield:

$$I_1 \approx C_C \frac{dv_B(t)}{dt} = C_C \frac{dv_o(t)}{dt}$$

The slew rate is the maximum rate of change of the output signal:

$$SR = \frac{dv_o(t)}{dt}_{\text{max}} = \frac{I_1}{C_C}$$
Relationships Between Slew Rate and Gain-Bandwidth Product

\[
SR = \frac{I_1}{C_C} = \frac{I_1}{G_m} = \frac{\omega_T}{I_1}
\]

For the simple CMOS amplifier on slide 110, the input stage transconductance is equal to that of transistors M_1 and M_2:

\[
\left( \frac{G_m}{I_1} \right) = \frac{1}{I_1} \sqrt{\frac{2K_{n2}}{2}} = \frac{2K_{n2}}{I_1}
\]

\[
SR = \omega_T \sqrt{\frac{I_1}{K_{n2}}}
\]

For the BJT amplifier on slide 118:

\[
\left( \frac{G_m}{I_1} \right) = \frac{40I_1}{2I_1} = 20
\]

\[
SR = \frac{\omega_T}{20}
\]