Markov Chains-Based Derivation of the Phase Detector Gain in Bang-Bang PLLs

Nicola Da Dalt, *Member, IEEE*

***Abstract*—Due to the presence of a binary phase detector (BPD) in the loop, bang-bang phase-locked loops (BBPLLs) are hard nonlinear systems. Since the BPD is usually also the only nonlinear element in the loop, in practical applications, BBPLLs are commonly analyzed by first linearizing the BPD and then using the traditional mathematical techniques for linear systems. To the author's knowledge, in the literature, the gain of the linearized BPD (K_{bpd}) is determined neglecting the effect of the BBPLL dynamics on the effective jitter seen by the BPD. In this brief, we develop an approach to the determination of K_{bpd} which takes into consideration also this effect. The approach is based on modeling the dynamics of a BBPLL as a Markov chain. This approach gives new insights into the behavior of the BBPLL and leads to an expression for the K_{bpd}, which is more general than the one currently known in literature.**

***Index Terms*—Bang-bang phase-locked loops (BBPLLs), binary phase detector (BPD), Markov chains.**

I. Introduction

The use of bang-bang phase-locked loops (BBPLLs) has become increasingly common in a lot of communications system, in particular, but not only, in the area of clock-and-data recovery (see, for instance, [1]–[4]). These systems are characterized by the the presence of a binary phase detector (BPD) in the loop, which quantizes the phase error between reference and feedback clock with 1-bit resolution.

Since the BPD is usually also the only nonlinear element in the loop, for practical applications BBPLLs are commonly analyzed by linearizing the BPD and using the traditional mathematical techniques for linear systems. It is known that the gain of the BPD K_{bpd} depends on the jitter between the reference clock and the feedback clock, also sometimes known as *untracked* jitter. To the author's knowledge, the expressions for K_{bpd}, which can be found in literature (see e.g., [5]) are determined by considering only the jitter on the reference clock. In this way, the BPD is characterized in principle as a stand-alone component, neglecting the effect of the BBPLL dynamics on the untracked jitter seen by the BPD. When the jitter on the reference clock is of the same order of magnitude of the jitter generated by the PLL, this assumption does not hold any longer.

By modeling the behavior of the BBPLL with the help of basic Markov chain theory, we were able to derive a more general analytical expression for K_{bpd}, which takes into consideration also the effect of the dynamics of the BBPLL. The analysis here is limited to the case of a first-order BBPLL or of a second-order one where the loop filter proportional constant is much bigger than the integral one.

Manuscript received August 10, 2005; revised June 9, 2006. This paper was recommended by Associate Editor E. Rogers.

The author is with Infineon Technologies Austria, Villach 9500, Austria (e-mail: nicola.dadalt@infineon.com).

Digital Object Identifier 10.1109/TCSII.2006.883197

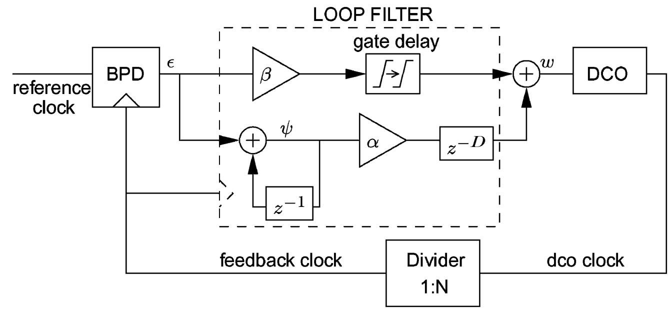

Fig. 1. Block diagram of the Type II architecture for the digital BBPLL.

It is interesting to note that a similar approach was considered a long time ago in [6], but after writing the basic equations, the authors resorted to extensive simulation to derive useful results.

The brief is organized as follows. Section II describes the BBPLL architecture under study. Although this is a digital one, all considerations in this brief apply to an analog architecture as well. In Section III, it is explained how and why the K_{bpd} depends on the untracked jitter, while in Section IV, we will apply Markov chain theory to derive the expression for the linearized gain.

II. Second-Order BBPLL Architecture

The block diagram of the architecture of the second-order BBPLL is reported in Fig. 1.

The PLL consists of a BPD, a digital loop filter, a digitally controlled oscillator (DCO), and a feedback divider. The function of the BPD is to provide an indication of the phase difference between the reference clock and the feedback clock in a binary form. Its operation is logically identical to the operation of an ideal sampling register, with the reference clock as data input and the divided clock as sampling clock. The binary phase information is fed to the digital loop filter, which consists of a proportional and integral path. The constants in the two paths will be indicated with β and α for the proportional and integral path, respectively.

Differently from the architecture analyzed in [7], the proportional path is not added to the integral path inside the loop filter. Indeed, this summation can be electrically performed rather easily inside the DCO. In this way, the latency of the loop filter for the proportional path is reduced to the delay of a few digital gates, which should be negligible compared to the reference clock period. This kind of architecture has already been implemented successfully on silicon (see [4]).

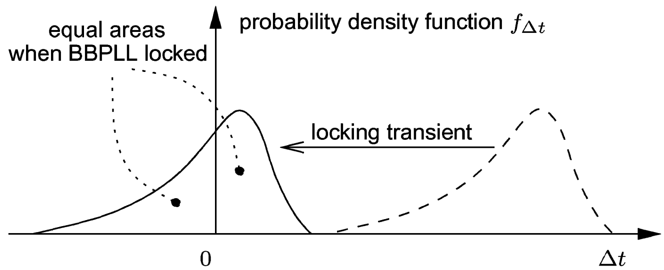

Fig. 2. Centering of the pdf of Δt.

Following the same approach as exposed in [7], the nonlinear map describing the function of the BBPLL can be written as

$$\begin{cases} \Delta t_{k+1} = \Delta t_k - N\alpha K_T \cdot \psi_{k-D} - N\beta K_T \mathrm{sgn}(\Delta t_{k-D}) \\ \psi_{k+1} = \psi_k + \mathrm{sgn}(\Delta t_{k+1}) \end{cases} \tag{1}$$

where $\Delta t \triangleq t_r - t_d$ is the difference between the rising edges instants of the reference (t_r) and feedback clock (t_d), α and β are the loop filter integral and proportional constants, K_T is the period gain of the DCO, N is the feedback divider factor, D is the latency of the integral path in reference clock cycles and finally ψ is the numerical value of the loop filter integrator output. The subscript k denotes the value of these quantities at the k-th iteration (or clock cycle).

III. Linearization of the BPD

Consider the time difference between the rising edges of the reference and of the feedback clock Δt. Due to the noise generated by the blocks of the BBPLL and present on the reference clock, Δt can be described as a random variable with a given cumulative distribution function (cdf) $F_{\Delta t}(x) = P[\Delta t < x]$ and probability density function (pdf) $f_{\Delta t}(x) = \partial F_{\Delta t}(x)/\partial x$, where the operator $P[A]$ indicates the probability of event A [8].

When the BBPLL is not locked, $f_{\Delta t}$ has most of its area concentrated far away from zero (see Fig. 2). During the locking process, the value of Δt converges to zero, meaning that the pdf is gradually centered around the origin. In locked condition, the average value of the BPD output ϵ must be zero, $E[\epsilon] = 0$ (the operator $E[x]$ indicates the expectation of the random variable x). If this condition was not satisfied, the output of the integral path of the loop filter would gradually drift in one direction, modifying the output frequency and contradicting the assumption that the BBPLL is locked. Since $\epsilon = \mathrm{sgn}(\Delta t)$ can have only value ± 1, the condition $E[\epsilon] = 0$ translates into

$$1 \cdot P[\epsilon = 1] + (-1) \cdot P[\epsilon = -1] = 0$$

which can be rewritten as

$$1 - 2P[\Delta t < 0] = 1 - 2F_{\Delta t}(0) = 0$$

so that in locked condition the distribution of Δt is such that

$$F_{\Delta t}(0) = \frac{1}{2}. \tag{2}$$

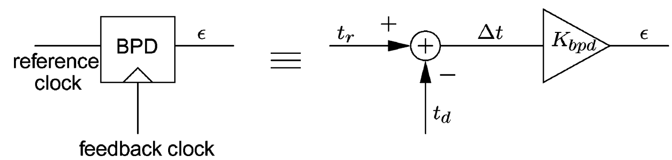

Fig. 3. BPD linearized model.

Assume that for any reason (for instance due to the intrinsic BBPLL dynamics) the pdf of Δt is shifted away from its equilibrium point by a small amount a in the positive direction. In this case there will be more iterations with $\epsilon = 1$ and less with $\epsilon = -1$, so that the average value of ϵ will be slightly positive. The BBPLL will react to this situation and center the pdf again.

Following this considerations, it possible to define a gain of the BPD (K_{bpd}) around the locked condition as rate of change in $E[\epsilon]$ due to a small shift a of the pdf around the locked condition. In formulas

$$K_{\mathrm{bpd}} \triangleq \frac{\partial}{\partial a} \left(E[\epsilon|\mathrm{shift} = a] \right)\big|_{a=0} \tag{3}$$

where $E[x|A]$ is the conditional expectation of x given the event A. In a first-order approximation, if the shift a is small enough, the probability that $\epsilon = 1$ is increased by the quantity $a \cdot f_{\Delta t}(0)$, while by the probability that $\epsilon = -1$ is decreased by the same quantity. The net change in $E[\epsilon]$ is thus $2af_{\Delta t}(0)$, so that $K_{\mathrm{bpd}} = 2f_{\Delta t}(0)$.

This result can be derived more formally in the following way. Expression (3) can be rewritten as

$$K_{\mathrm{bpd}} = \frac{\partial}{\partial a} \left(1 - 2P[\Delta t < 0|\mathrm{shift} = a] \right)\big|_{a=0}.$$

The probability that $\Delta t < 0$ given that the pdf has been shifted by some quantity a is equal to the probability that $\Delta t < -a$ without any shift. Thus

$$K_{\mathrm{bpd}} = \frac{\partial}{\partial a} \left(1 - 2P[\Delta t < -a] \right)\big|_{a=0} = -2 \frac{\partial}{\partial a} F_{\Delta t}(-a) \bigg|_{a=0}$$

and finally

$$K_{\mathrm{bpd}} = 2f_{\Delta t}(0). \tag{4}$$

From this expression, the gain of the BPD depends only on the value of the pdf of the time difference Δt at the point where the BBPLL is locked, which is defined by (2). This result agrees with what reported in [5], though derived in a different way. Fig. 3 illustrates the BPD and the resulting linearized model. It has to be noted that this derivation is valid under the assumption that the dynamic of the BBPLL during the re-centering of the pdf is slow enough, so that the probabilistic approach based on expectations makes sense.

In the next section, the value of the gain K_{bpd} will be computed.

IV. Expression for K_{bpd}

The equivalent gain of the BPD K_{bpd} cannot be fixed independently from the other parameters of the BBPLL (as it is nor-

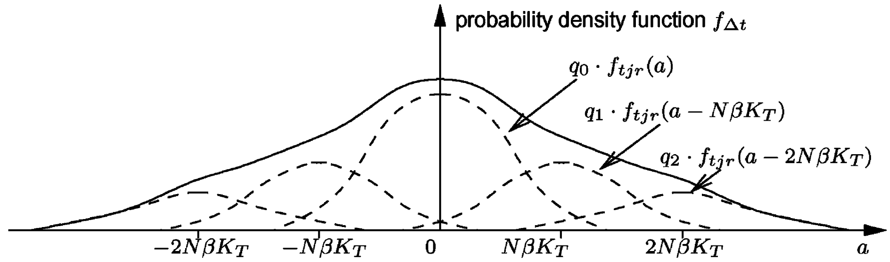

Fig. 4. The pdf of Δt as weighted superposition of the pdfs of t_{jr}.

mally done in the literature [5]), since it depends on $f_{\Delta t}$, which in turn depends on the dynamics of the BBPLL itself. In this derivation, we will assume $\alpha \ll \beta$. The nonlinear map (1) in presence of jitter t_{jr} on the reference clock can be written as

$$\Delta t_{k+1} = \Delta t_k + t_{jr} - N\beta K_T \mathrm{sgn}(\Delta t_k). \tag{5}$$

Indicating with Δt^* the value of Δt in case of unjittered reference, Δt^* can assume values only on discrete states: $nN\beta K_T + \Delta t_0^*$, with $n \in \mathbb{Z}$. When the BBPLL is locked, the integral path will have centered the dynamics so that we can assume $\Delta t_0^* = 0$ and $\Delta t^* = nN\beta K_T, n \in \mathbb{Z}$.

Every time Δt^* is in a given state $nN\beta K_T$, the jitter on the reference distributes the actual Δt probabilistically on a range around $nN\beta K_T$, replicating the pdf of t_{jr}.

Therefore, the pdf of Δt, $f_{\Delta t}$, will be given by the superposition of the pdfs of t_{jr}, $f_{t_{jr}}$, shifted by an amount equal to each occupied state and weighted by the probability that in steady-state Δt^* will occupy that state (see Fig. 4). Indicating with n the state $nN\beta K_T$ and with q_n the stationary probability of occupancy of the state n

$$q_n \triangleq P[\Delta t^* \in n]$$

then

$$f_{\Delta t}(a) = \sum_{n=-\infty}^{n=+\infty} q_n f_{t_{jr}}(a - nN\beta K_T)$$

so that

$$K_{\mathrm{bpd}} = 2 \sum_{n=-\infty}^{n=+\infty} q_n f_{t_{jr}}(-nN\beta K_T). \tag{6}$$

In order to find the stationary probabilities q_n, the BBPLL can be modeled as a Markov chain (see [8]). From a given state n, Δt^* must go either to state $n+1$ or to state $n-1$. The transition probabilities are

$$P\left[\Delta t_{k+1}^* \in n+1 | \Delta t_k^* \in n\right] = P[t_{jr} + nN\beta KT \leq 0]$$

which can be expressed in terms of the cdf of t_{jr}, $F_{t_{jr}}$, as

$$P\left[\Delta t_{k+1}^* \in n+1 | \Delta t_k^* \in n\right] = F_{t_{jr}}(-nN\beta K_T)$$

and, similarly

$$P\left[\Delta t_{k+1}^* \in n-1 | \Delta t_k^* \in n\right] = 1 - F_{t_{jr}}(-nN\beta K_T)$$

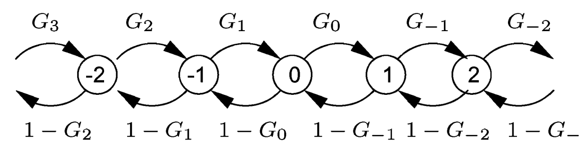

Fig. 5. Markov chain representing the BBPLL.

The Markov chain representing the BBPLL is illustrated in Fig. 5, where $G_n \triangleq F_{t_{jr}}(nN\beta K_t)$. It is actually a random walk with nonuniform transition probabilities. Define the transition probability matrix $\mathbf{P}$, such that its element (i, j) is the probability to go from state i to state j in one step

$$\mathbf{P} = \begin{bmatrix} \ddots & & & \vdots & & & \\ & 0 & G_2 & 0 & 0 & 0 & \\ & 1-G_1 & 0 & G_1 & 0 & 0 & \\ \cdots & 0 & 1-G_0 & 0^* & G_0 & 0 & \cdots \\ & 0 & 0 & 1-G_{-1} & 0 & G_{-1} & \\ & 0 & 0 & 0 & 1-G_{-2} & 0 & \\ & & & \vdots & & & \ddots \end{bmatrix}.$$

In this brief we ignore the problem of cycle slips, so that the number of possible states is in principle infinite and $\mathbf{P}$ is a square matrix with infinite dimension. The superscript * in the matrix elements denotes the element (0, 0).

If $f_{t_{jr}}$ is symmetrical around 0 then $G_{-n} = 1 - G_n$ (in particular $G_0 = 1/2$) and the transition matrix has center symmetry around the (0, 0) element

$$\mathbf{P} = \begin{bmatrix} \ddots & & & & \vdots & & & & \\ & G_{-2} & 0 & G_2 & 0 & 0 & 0 & 0 & \\ & 0 & G_{-1} & 0 & G_1 & 0 & 0 & 0 & \\ \cdots & 0 & 0 & G_0 & 0^* & G_0 & 0 & 0 & \cdots \\ & 0 & 0 & 0 & G_1 & 0 & G_{-1} & 0 & \\ & 0 & 0 & 0 & 0 & G_2 & 0 & G_{-2} & \\ & & & & \vdots & & & & \ddots \end{bmatrix}.$$

The stationary probabilities q_n are defined by the stationary Chapman–Kolmogorov equation

$$\mathbf{q} = \mathbf{q} \cdot \mathbf{P} \tag{7}$$

where $\mathbf{q}$ is the row vector $[\ldots, q_{-2}, q_{-1}, q_0, q_1, q_2, \ldots]$.

Although, technically speaking, this kind of Markov chain has period 2 and the theory would request to calculate first $\mathbf{P}^2$ and then solve $\mathbf{q} = \mathbf{q} \cdot \mathbf{P}^2$ (see [8]), we take the short way and solve (7). Indeed if $\mathbf{q}$ satisfies (7), then it is also a solution for $\mathbf{P}^2$.

Since the states describe all possible events and they are disjoint, q_n must satisfy the normalization equation

$$\sum_{n=-\infty}^{+\infty} q_n = 1. \tag{8}$$

Equation (7) translates into a infinite set of linear equations

$$q_n = G_{-n+1}q_{n-1} + G_{n+1}q_{n+1}, \qquad n \in \mathbb{Z}. \tag{9}$$

Since $\mathbf{P}$ is symmetrical, $q_{-n} = q_n$ and it is enough to consider $n = 0, 1, \ldots$. Rewriting (9) for $n = 0, q_0 = G_1 q_{-1} + G_1 q_1$ so that;

$$q_1 = \frac{1}{2G_1} q_0. \tag{10}$$

Taking the sum of the first $n+1$ terms of (9) and using the result (10)

$$q_{n+1} = \frac{1 - G_n}{G_{n+1}} q_n$$

so that

$$q_n = q_{-n} = \prod_{m=1}^{n} \left(\frac{1 - G_{m-1}}{G_m} \right) q_0. \tag{11}$$

In order to find q_0 the normalization (8) is used

$$\sum_n q_n = q_0 + 2\sum_{n=1}^{+\infty} q_n = 1$$

and finally

$$q_0 = \left[1 + 2\sum_{n=1}^{+\infty} \prod_{m=1}^{n} \left(\frac{1 - G_{m-1}}{G_m} \right) \right]^{-1}. \tag{12}$$

Equations (6), (11), and (12) allow the exact computation of the BPD linearized gain K_{bpd}.

If t_{jr} is Gaussian with variance $\sigma_{t_{jr}}^2$ then

$$G_n = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{nN\beta K_T}{\sigma_{t_{jr}}\sqrt{2}} \right)$$

where

$$\mathrm{erf}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_0^x e^{-b^2} \mathrm{d}b.$$

Fig. 6 reports the plot of K_{bpd} versus σ_{tjr} for the case of a Gaussian reference jitter using (6) (11) and (12) (thick solid line). For this computation the Markov chain has been limited to 101 states and $N\beta K_T$ has been normalized to 1.

The asymptote for small values of $\sigma_{t_{jr}}$ can be derived as follows. On the assumption that $\sigma_{t_{jr}} \ll N\beta K_T$, in the Markov chain the states n with $|n| \geq 2$ occur with a probability which is negligibly small. The infinite Markov chain simplifies then to

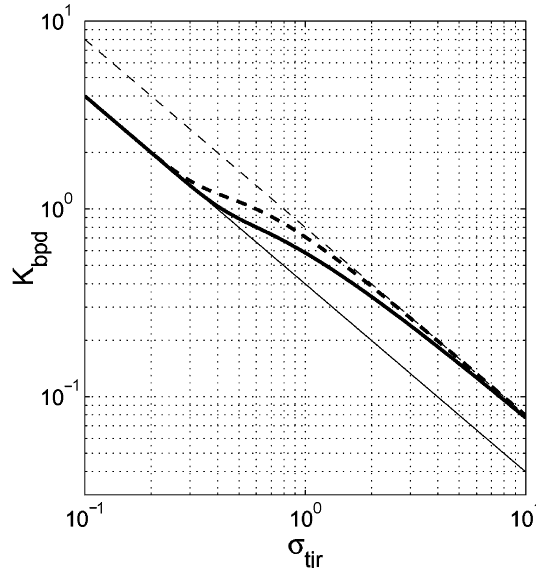

Fig. 6. Plot of K_{bpd} versus σ_{tjr} (Gaussian jitter, $N\beta K_T$ normalized to 1): K_{bpd} computed with a 101 states Markov chain (thick solid), K_{bpd} approximated with a 3-state Markov chain (thick dashed), asymptote $K_{\mathrm{bpd}} = 1/(\sqrt{2\pi}\sigma_{t_{jr}})$ (thin solid), asymptote $K_{\mathrm{bpd}} = 2/(\sqrt{2\pi}\sigma_{t_{jr}})$ (thin dashed).

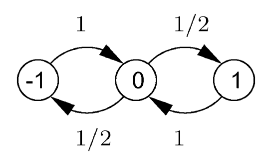

Fig. 7. Markov chain approximating the BBPLL.

a three state chain with perfectly reflecting barriers (see Fig. 7) and it is easy to show that $q_0 = 1/2$, $q_1 = q_{-1} = 1/4$.

Applying (6) an approximate expression for K_{bpd} is found

$$K_{\mathrm{bpd}} \approx \frac{1}{\sqrt{2\pi}\sigma_{t_{jr}}} \left[1 + e^{-\frac{1}{2}\left(\frac{N\beta K_T}{\sigma_{t_{jr}}}\right)^2} \right] \tag{13}$$

and for $\sigma_{t_{jr}} \ll N\beta K_T$ we find $K_{\mathrm{bpd}} = 1/(\sqrt{2\pi}\sigma_{t_{jr}})$.

The asymptote for large values of $\sigma_{t_{jr}}$ can be understood by considering that in this case the dynamic of the BBPLL has little influence on the pdf of Δt, since the displacement of the sampling point at multiples of $N\beta K_T$ is negligible compared to the jitter on the reference clock. Therefore, going back to (4), we can say $K_{\mathrm{bpd}} \approx 2f_{tjr} = 2/(\sqrt{2\pi}\sigma_{t_{jr}})$, which is the expression usually found in literature (see [5]).

Note that this result, if used in the case of small $\sigma_{t_{jr}}$, would be wrong by a factor of 2.

Fig. 6 reports the plots of (13) and of the two asympotes. Interestingly, although derived under the assumption of small $\sigma_{t_{jr}}$, (13) gives a good approximation (error is smaller than 25%) of the real K_{bpd} on the whole axis.

V. Verification

The validity of the linearized expression for K_{bpd} has been verified through behavioral simulations of the BBPLL described in Fig. 1. The reference clock has been jittered with a random Gaussian white noise and the power-spectral density (PSD) of

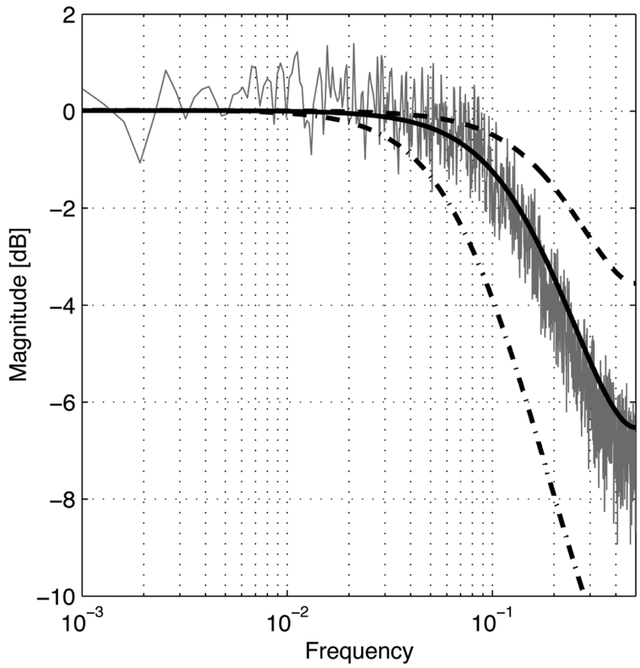

Fig. 8. PSD of the simulated feedback clock jitter compared to the linear model for three differen values of K_{bpd}: expression (13) (solid), $K_{\mathrm{bpd}} = 1/(\sqrt{2\pi}\sigma_{t_{jr}})$ (dash-dotted) and $K_{\mathrm{bpd}} = 2/(\sqrt{2\pi}\sigma_{t_{jr}})$ (dashed).

the jitter on the feedback clock has been computed and compared with the results of the BBPLL linear model. Considering as input and output signals the time instants of the rising edges of the reference and feedback clock respectively, the z-domain open-loop transfer function of the BBPLL is

$$G(z) = K_{\mathrm{bpd}} \left(\beta + \frac{\alpha z^{-D}}{1 - z^{-1}} \right) K_T \frac{z^{-1}}{1 - z^{-1}}$$

where $z^{-1} = \exp(-j\omega T)$, with T the period of the reference clock. The closed-loop transfer function from the reference to the feedback clock is

$$H(z) = \frac{G(z)}{1 + G(z)}.$$

The closed-loop transfer function has a low-pass characteristic, the bandwith of which is determined also by K_{bpd}. A comparison between the bandwidths of the PSD of the simulated feedback clock jitter and of the linear model provides evidence of the accuracy of the linearization of K_{bpd}. In all performed simulations a very good agreement has been found between simulation results and the linearized model, where the expression (13) has been used. In Fig. 8 an example is reported. In this case the BBPLL has been simulated with the following normalized parameters : period of the reference clock $= 1$, $N = 1$, $K_T = 1$, $D = 0$, $\alpha = 10^{-5}$, $\beta = 10^{-2}$ and $\sigma_{tjr} = 9 \cdot 10^{-2}$. The figure shows the PSD of the feedback clock jitter together with the results of the linear model using three different values for K_{bpd}: expression (13), $K_{\mathrm{bpd}} = 1/(\sqrt{2\pi}\sigma_{t_{jr}})$ and $K_{\mathrm{bpd}} = 2/(\sqrt{2\pi}\sigma_{t_{jr}})$. Note that, the PSD curve has been vertically shifted to allow a better comparison of the bandwidths. It can be clearly seen that expression (13) predicts very well the behavior of the system, while the other two expressions are quite inaccurate.

VI. Conclusion

In this brief, the basic theory of Markov chains has been used to model the behavior of a first-order BBPLL. This approach has proven to be fruitful, since it allowed us to derive easily an analytical expression for the linearized gain of the BPD, taking into account the dynamics of the PLL in presence of Gaussian input jitter. This expression reduces to that found in literature for the case that the jitter of the reference clock is much bigger than the BB correction step. The accuracy of the derivation has been verified with behavioral simulations.

References

[1] Y. M. Greshishchev, P. Schvan, J. L. Showell, M. Xu, J. J. Ojha, and J. E. Rogers, "A fully integrated SiGe receiver IC for 10-Gb/s data rate," *IEEE J. Solid-State Circuits*, vol. 35, no. 12, pp. 1949–1957, Dec. 2000.

[2] R. C. Walker, "Designing bang-bang PLLs clock-and-data recovery in serial data transmission systems," in *Phase Locking in High-Performance Systems*. New York: IEEE Press, 2003.

[3] R. B. Staszewski, C. Hung, K. Maggio, J. Wallberg, D. Leipold, and P. T. Balsara, "All digital phase domain TX frequency synthesizer for bluetooth radios in 0.13-um CMOS," in *Dig. Tech. Papers ISSCC* , San Francisco, CA, 2004.

[4] N. D. Dalt, E. Thaller, P. Gregorius, and L. Gazsi, "A compact triple-band low-jitter digital *LC* PLL with programmable coil in 130-nm CMOS," *IEEE J. Solid-State Circuits*, vol. 40, no. 7, pp. 1482–1490, Jul. 2005.

[5] Y. Choi, D. K. Jeong, and W. Kim, "Jitter transfer analysis of tracked oversampling techniques for multigigabit clock-and-data recovery," *IEEE Trans. Circuits Syst. II, Analog Digit. Signal Process.*, vol. 50, no. 11, pp. 775–783, Nov. 2003.

[6] J. R. Cessna and D. M. Levy, "Phase noise and transient times for a binary quantized digital phase-locked loop in white Gaussian noise," *IEEE Trans. Commun.*, vol. 20, no. 2, pp. 94–104, Apr. 1972.

[7] N. D. Dalt, "A design-oriented study of the nonlinear dynamics of digital bang-bang PLLs," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 52, no. 1, pp. 21–31, Jan. 2005.

[8] A. Papoulis and S. U. Pillai, *Probability, Random Variables and Stochastic Processes*, 4th ed. New York: McGraw-Hill, 2002.