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Preface

Wireless networks and services are vital elements of daily life around the world. Presently, there are close to 4.8 billion mobile subscribers and more than 8 billion mobile connections (including machine-to-machine) in the world, and the numbers are increasing at a very rapid pace. The revenue generated worldwide in 2015 for providing mobile services exceeded US$1 trillion. It is expected that by 2020, there will be more than 50 billion mobile connections; hence, there is a need to provide the required resources for this anticipated phenomenal growth. However, the resources (e.g., frequency spectrum, energy) are limited, which calls for innovative approaches to improving efficiencies, managing complexities, providing quality, and ensuring availability and security. It is of the utmost importance to meet the requirements of new services in a cost-effective and efficient manner; their proliferation depends on it.

The conventional approach is to formulate resource allocation in wireless networks as optimization problems that aim to maximize the goodput (e.g., throughput) of networks or users while minimizing their badput (e.g., energy consumption, interference). Many existing resource allocation schemes assume exact parameter values and side information to achieve their objectives. However, their performance is sensitive to the accuracy and availability of parameter values and other ancillary information.

Such assumptions in many instances are unrealistic due to the ever increasing number of wireless devices in the neighborhood and their mobility, as well as the nonlinear and time-varying nature of propagation of electromagnetic waves, and in some cases result in significant and unacceptable degradation of the quality of service experienced by users. To deal with the undesirable side effects of such simplifying and unrealistic assumptions, there is a need to introduce robustness into resource allocation schemes that would be efficient and fair, with acceptable complexity, cost, and overhead.

The nominal (i.e., no uncertainty) optimization problems in many cases are not easy to solve in a straightforward manner because they are nondeterministic polynomial-time and nonconvex. Introducing robustness in such problems is normally done by way of introducing additional constraints that at times are stochastic and nonlinear, which aggravates the problem even further. It is in this light that developing practical schemes for the robust allocation of resources in wireless networks is a formidable challenge.

This book represents an attempt to present the state of the art in current research on this topic and to show that, in general, many existing techniques and methods in robust allocation will be usable in future wireless networks. Another objective of this book is to demonstrate that there is an urgent need to devise alternative schemes to improve performance and avoid some of the very serious obstacles and limitations pointed out in the literature. The book contains five chapters, which are described briefly in what follows.

In Chapter 1, we explain why robust optimization theory is important in wireless networks and how, in general, it can be applied for allocating resources in such networks. We begin the chapter by presenting fundamental notions of wireless communications relevant to the book and describe different types of resource allocation problems, namely, network-centric (cooperative) and user-centric (competitive) approaches. The objective of the cooperative approach is to maximize the total network utility, whereas in the competitive approach, the goal of each user is to maximize its own utility. We also show how to map a nominal (i.e., nonrobust) optimization problem into its robust counterpart. Finally, we explain the implementation issues pertaining to robust optimization problems.

In Chapter 2, we cover robust cooperative transmit power allocation in wireless networks, where the uncertain parameters are channel gains between secondary users and primary access points. The objective of secondary users is to use the frequency spectrum that belongs to the primary network subject to keep their interference to the primary network below a given threshold while maximizing their own social utility. We present the system model and formulate the robust resource allocation problem using the concept of uncertainty region within which all instances of uncertain parameter values are assumed to be confined. We also show that by properly defining the uncertainty region, the computational complexity of solving robust problems can be reduced to the level of nonrobust problems, develop algorithms for trading off between throughput reduction and robustness, and devise schemes to reduce signaling in robust solutions for distributed approaches.

In Chapter 3, we study robust noncooperative resource allocation where each user competes with other users over utilizing the resources to maximize its own utility. In doing so, we present a game-theoretic formulation of the problem and discuss its solution where greedy, noncooperative, and rational users utilize the available but noisy and uncertain side information to achieve their objectives. To tackle uncertainty and improve the utility of each user, we apply worst-case robust optimization in noncooperative games and present their analysis for allocating resources in wireless networks. Specifically, via variational inequalities, we present a systematic approach to securing the conditions for the existence and uniqueness of the games' equilibria, derive the gap between the utility values of nonrobust and robust games, and present distributed algorithms for solving such games.

In Chapter 4, we present a taxonomy of relaxation methods for solving nonconvex and intractable robust optimization problems for allocating resources in wireless networks and provide several examples of such problems in existing and future wireless networks. We also show how such problems can be solved. In particular, we present cases in which uncertainty in channel state information (CSI) is assumed to be within a given region, when only statistics of uncertainty in CSI are available, or when no CSI is available to users, and we discuss the relevance of each case in future wireless networks.

In Chapter 5, we present a brief overview of important features of future wireless networks that will affect resource allocation; then we identify important problems in robust resource allocation in such networks that can be tackled using the material in Chapters 1, 2, 3, and 4.

We would like to acknowledge the contributions of the coauthors of our joint papers on the topics discussed in this book. Our gratitude goes to Professor Mihaela van der Schaar, Professor Ekram Hossain, Professor Paeiz Azmi, Dr. Hamid Saeedi, Dr. Mehdi Rasti, and Dr. Mohammad Reza Javan for their invaluable help.

We would also like to express our deepest appreciation to our families, for their understanding, support, encouragement, and sacrifices while this book was being written. To them we dedicate our book.

Tehran, Iran

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Contents

Notations and Symbols

Table 1 Notations and Symbols

Notation	Description
$\mathbb{R}$	Set of real numbers
$\mathbb{R}_+$	Set of nonnegative real numbers
$\mathbb{R}_{++}$	Set of positive real numbers
$\mathbb{R}^n$	Set of n-dimensional real vectors
$\mathbb{R}^{m\times n}$	Set of real $m \times n$ matrices
$\mathbb{C}$	Set of n-dimensional complex vectors
$\mathbb{C}^n$	Set of complex numbers
$\mathbb{H}^n$	Set of $n \times n$ complex Hermitian matrices
$\mathbb{S}^n$	Set of $n \times n$ symmetric matrices
$\mathbb{S}^n_+$	Set of symmetric positive semidefinite matrices
$\mathbb{S}^n_{++}$	Set of symmetric positive definite matrices
$(\cdot)^{\mathrm{T}}$	Matrix or vector transpose
$(\cdot)^*$	Matrix or vector conjugate
$\succeq$	Element-wise greater than or equal to for vectors
$\succ$	Element-wise greater than for vectors
$\preceq$	Element-wise less than or equal to for vectors
$\prec$	Element-wise less than for vectors
$(\cdot)^{\mathrm{H}}$	Complex Hermitian conjugate
$\mathrm{tr}(\mathbf{A})$	Trace of $\mathbf{A}$
$\lambda_{\min}(\mathbf{A})$	Minimum eigenvalue of $\mathbf{A}$
$\lambda_{\max}(\mathbf{A})$	Maximum eigenvalue of $\mathbf{A}$
$\lambda^+(\mathbf{A})$	$\max\{\lambda_{\max}(\mathbf{A}), 0\}$
$vec(\mathbf{A})$	Vector obtained by stacking the column vector of $\mathbf{A}$
$\mathbf{I}_n$	$n \times n$ identity matrix
$\mathrm{diag}\{a_1, \cdots, a_n\}$	$n \times n$ diagonal matrix whose ith diagonal entry is a_z
$\|\cdot\|_2$ and $\|\cdot\|_1$	Euclidean norm and vector 1-norm, respectively
$\|\cdot\|_F$	Matrix Frobenius norm
$\mathbb{E}\{\cdot\}$	Statistical expectation function
$\Pr\{\}$	Probability function
$\exp(\cdot)$	Exponential function
$\lvert\cdot\rvert^2$	Magnitude squared for scalars or element-wise magnitude squared for vectors
∇	Vector differential operator
$\mathrm{rank}(\cdot)$	Matrix rank
$\mathcal{R}(\cdot)$	Range of matrix
$\mathbf{x} \sim CN(\mathbf{y}, \mathbf{Z})$	Circularly symmetric complex Gaussian random vector with mean $\mathbf{y}$ and covariance matrix $\mathbf{Z}$
$\mathrm{Re}\{\mathbf{A}\}$ and $\mathrm{Im}\{\mathbf{A}\}$	Real and imaginary parts of complex matrix $\mathbf{A}$
$\det(\mathbf{A})$	Determinant of $\mathbf{A}$
$\otimes$	Kronecker product
$\odot$	Element-wise or dot product

Abbreviations

Table 2 Abbreviations

Abbreviation	Definition
A/D	Analog to digital
AF	Amplify and forward
AGMA	Arithmetic-geometric mean approximation
AP	Access point
BA	Bernstein approximation
BS	Base station
BT	Bounding techniques
BTI	Bernstein-type inequality
CCT	Charnes-Cooper transformation
CDF	Cumulative distribution function
CDI	Channel distribution information
CDMA	Code division multiple access
CGP	Complementary geometric programming
CoMP	Coordinated multipoint
CRAN	Cloud radio access networks
CRN	Cognitive radio network
CSI	Channel state information
CSMA/CA	Carrier sense multiple access with collision avoidance
C-RAN	Cloud radio access network
CoMP	Coordinated multipoint
CVaR	Conditional value at risk
D/A	Digital to analog
D2D	Device to device
DCA	Difference of two concave functions approximation
DCP	Difference of two concave functions programming
DF	Decode and forward
DI	Determinant inequality
DL	Downlink
DSLAM	Digital subscriber line access multiplexer
EF	Epigraph form
EPEC	Equilibrium program with equilibrium constraints
FDMA	Frequency division multiple access
FSO	Free space optics
5G	Fifth generation
GNE	Generalized Nash equilibrium
GP	Geometric programming
GVI	Generalized variational inequality
HetNet	Heterogeneous wireless networks
IoE	Internet of everything

(continued)

Table 2 (continued)

Abbreviation	Definition
IoT	Internet of things
IT	Interference threshold
KKT	Karush–Kuhn–Tucker
LDI-CGQF	Large deviation inequality for complex Gaussian quadratic forms
LMIs	Linear matrix inequalities
LMMSE	Linear minimum mean square error
LR	Lagrangian relaxation
LTE-A	Long term evolution-advanced
M2M	Machine-to-machine communications
MI	Markov's inequality
MINLP	Mixed-integer nonlinear program
MIMO	Multiple-input multiple-output
MISO	Multiple-input single-output
MOP	Multiobjective optimization problem
MPEC	Mathematical programs with equilibrium constraints
MRC	Maximal ratio combining
MTC	Machine-type communications
NA	Norm approximation
NCP	Nonlinear complementarity problem
NE	Nash equilibrium
NFV	Network functions virtualization
NMIMO	Network multiple input multiple output
NLFP	Nonlinear fractional programming
NNE	Nominal Nash equilibrium
NOMA	Nonorthogonal multiple access
NSE	Nominal Stackelberg equilibrium
NSG	Nominal Stackelberg game
NUM	Network utility maximization
OFDMA	Orthogonal frequency division multiple access
PBS	Primary base station
pdf	Probability density function
PU	Primary user
QCQP	Quadratically constrained quadratic program
QoS	Quality of service
QVI	Quasi-variational inequality
RA	Relaxation algorithms
RF	Radio frequency
RNE	Robust Nash equilibrium
RSE	Robust Stackelberg equilibrium
RSG	Robust Stackelberg game

(continued)

Table 2 (continued)

Abbreviation	Definition
SBS	Secondary base station
SC	Schur complement
SCA	Successive convex approximation
SCALE	Successive convex approximation for low complexity
SDMA	Space division multiple access
SCMA	Sparse code multiple access
SDN	Software defined networking
SDP	Semidefinite programming
SDR	Semidefinite relaxation
SI	System information
SIMO	Single-input multiple-output
SINR	Signal-to-interference-plus-noise ratio
SISO	Single-input single-output
SON	Self-organizing networks
SP	S-procedure
SPCA	Sequential parametric convex approximation
SU	Secondary user
TB	Taylor bounding
TDMA	Time division multiple access
TI	Triangle inequality
TTMI	Trace of two matrices inequality
UL	Uplink
VI	Variational inequality
VNTI	Von Neumann's trace inequality
VPI	Vysochanskii–Petunin inequality
ZF	Zero forcing

Chapter 1
Introduction

This chapter introduces the main concepts behind the application of robust optimization theory to the allocation of resources in wireless networks. We begin by presenting important fundamental notions of wireless communications relevant to this book and proceed to show how the problem of robust allocation of resources in wireless environments can be methodically formulated. We also show how to simplify such problems and discuss different approaches and techniques to obtain their solutions.

1.1 Motivation

The growing proliferation of new and often high-speed services in wireless networks with inadequate and costly frequency spectra and other limited network resources, which require time-varying and at times unpredictable ancillary information, such as users' locations and movements, calls for innovative schemes to significantly improve the efficiency of data transmission in such networks. In doing so, one needs to consider new paradigms in designing the structure and topology of networks, new techniques in establishing communication links, new approaches to enhancing security, and new algorithms for allocating available resources in an efficient and fair manner.

A cellular network typically comprises multiple cells, where each cell is served by an access point (AP) or a base station (BS). Users do not expect their communications to be hampered by their movements within a cell or from one cell to another irrespective of their speed or other relevant circumstances. Cellular networks have evolved over time, with each generation surpassing its predecessor in terms of novel services, spectral efficiency, and data rates, for example. The fifth generation, expected to become operational by 2018–2019, is an exception in the

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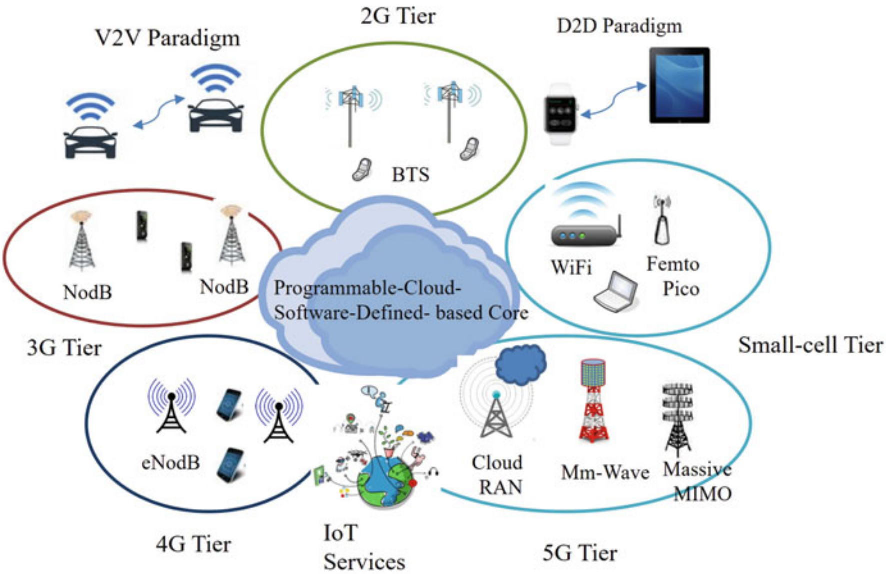

Fig. 1.1 Multi-tier and heterogeneous wireless paradigms and networks

sense that it represents a radically different concept, that is, it is not an evolution of fourth-generation cellular networks [1–5].

In cellular networks, a point-to-point link that carries information from a terminal to its AP is called an uplink (UL), and a point-to-point link from an AP to a terminal is called a downlink (DL). A broadcast link differs from a point-to point link in the sense that the former is used to simultaneously transmit information to all users. A relay node extends the reach of a link via either decode-and-forward (DF) or amplify-and-forward (AF) schemes. To overcome some of the existing limitations in available resources, and to provide more advanced services, other communication modalities, such as peer-to-peer transmissions that do not rely on APs or BSs, machine-type-communications (MTC), also called machine-to-machine (M2M) communications or Internet of things (IoT) or Internet of everything (IoE) for direct communications among electronic devices without human involvement, and device-to-device communications for cellular users that bypass APs or BSs after they are paired, have also been developed. Figure 1.1 is a pictorial depiction of multi-tier and heterogeneous wireless paradigms and networks.

In wireless networks, the frequency spectrum is an important, costly, and scarce resource. Other limited resources are transmit power, time, space, and codes. To increase efficiency via different multiple access techniques, such as time division multiple access (TDMA), frequency division multiple access (FDMA), orthogonal frequency division multiple access (OFDMA), space division multiple access (SDMA), code division multiple access (CDMA), and non-orthogonal multiple access (NOMA), have been developed.

It is of paramount importance to implement each generation of wireless networks in an efficient, optimal, and cost-effective manner using minimal resources while maintaining the required quality of service (QoS). In the past two decades, an enormous amount of research has focused on the development of optimal resource allocation schemes in wireless networks [6–9]. The common thread in all existing optimal resource allocation schemes in wireless networks is formulating the optimization problem in such a way as to improve the goodputs, for example, the total throughput, revenue, and fairness, and to reduce the badputs, for example, the transmit power, consumed energy, and cost, while maintaining the required QoS subject to certain constraints emanating from regulations, hardware and software limitations, and other pertinent restrictions [6].

In general, due to noise and interference in wireless channels, resource allocation problems are nondeterministic polynomial-time (NP)–hard, and the computational complexity of obtaining their solutions is high. To alleviate this, various techniques, including convex optimization [10], geometric programming (GP) [11], complementary GP (CGP) [12], difference of two convex functions (DC) approximation [13], arithmetic-geometric mean approximation [14], and successive convex approximation (SCA) [15], have been developed.

To maintain the required QoS, many existing resource allocation schemes require accurate and timely system information, which may not be available due to the time-varying nature of the environment or because of imperfect measurements. This problem is aggravated by changes in wave propagations, mobility of terminals, or variations in the number of active terminals, for example, which can result in significant deviations from optimality. Studying such impacts and developing effective schemes to ameliorate the unavailability of perfect measurements or exact system information in a timely manner are of practical and vital importance.

This book focuses on existing research that deals with such issues, and it identifies possible future directions of research in this area. We will describe how robust optimization theory can be utilized to develop resource allocation algorithms that would be robust against imperfect measurements or variations in system information. Although such schemes are beneficial in many aspects, they nevertheless involve additional computational complexities and possible undesirable performance degradations. It is yet another objective of this book to present practical ways and means of minimizing such unwanted side effects.

In general, by utilizing robust optimization theory, the nominal (i.e., not robust) optimization problem is mapped into its robust counterpart with a new set of parameters and constraints that depend on the nature of uncertainty in parameter values. There are two major and distinct approaches to robust optimal resource allocation in wireless networks [16–19]:

- Stochastic or Bayesian approach, where the statistics of errors are taken into account in obtaining a robust solution that is stochastically guaranteed to be robust against any occurrence of error
- Worst-case approach, where the error is assumed to be bounded in a specific region, called the uncertainty region, and a robust solution is obtained assuming the worst-case condition of error in that region

The appropriate approach for introducing robustness depends on the availability of information on the error in parameter values and the degree of imperfectness of measurements [17, 18, 20]. However, as stated earlier, the worst-case robust approach guarantees robustness in any given instance within the uncertainty region, while the stochastic approach provides robustness in a statistical manner over a period of time.

1.2 Formulating Resource Allocation Problems

A nominal (not robust) resource allocation problem can be formulated as [10, 21]

$$\max_{\mathbf{x}} f_0(\mathbf{x}, \mathbf{c}_0), \tag{1.1}$$

$$\text{subject to} \quad \begin{cases} f_y(\mathbf{x}, \mathbf{c}_y) = 0, & \forall y \in \mathscr{Y}, \\ f_z(\mathbf{x}, \mathbf{c}_z) \le 0, & \forall z \in \mathscr{Z}, \end{cases}$$

where

- $f_0(\mathbf{x}, \mathbf{c}_0)$ is the objective function, for example, throughput, energy efficiency, and fairness criteria;
- $\mathbf{x}$ contains the optimization variables and resources, including, for example, transmit power levels, subchannel assignments, user associations with different base stations, and antenna selection;
- $\mathbf{c}_w$ for $w = \{0, y, z\}$ contains system information (SI), including all channel state information, the number of active users, and user transmission constraints, for example. SI may be complete, meaning that the exact values are readily available or incomplete, that is, uncertain or inaccurate;
- $\mathscr{Y} = \{1, \ldots, Y\}$ is the set of variables pertaining to the equality constraints in the optimization problem whose yth element is $f_y(\mathbf{x}, \mathbf{c}_y) = 0$; and
- $\mathscr{Z} = \{1, \ldots, Z\}$ is the set of variables pertaining to the inequality constraints in the optimization problem whose zth element is $f_z(\mathbf{x}, \mathbf{c}_z) \le 0$.

Figure 1.2 is a pictorial representation of the pertinent elements in a typical resource allocation problem. The constraints in Eq. (1.1) can be categorized as follows:

- transmitter constraints, including, for example, the maximum transmit power and any other hardware or software limitations in the transmitter or in the receiver, such as frequency band or access technology;
- regulatory constraints, including, for example, the maximum allowable transmit power in any given frequency band, and the maximum allowable interference induced by each user;
- QoS constraints, including, for example, the minimum required signal-to-interference-plus-noise ratio (SINR) or throughput for each user or for all users, the maximum tolerable delay in wireless virtual networks, the maximum acceptable delay for each user, etc.

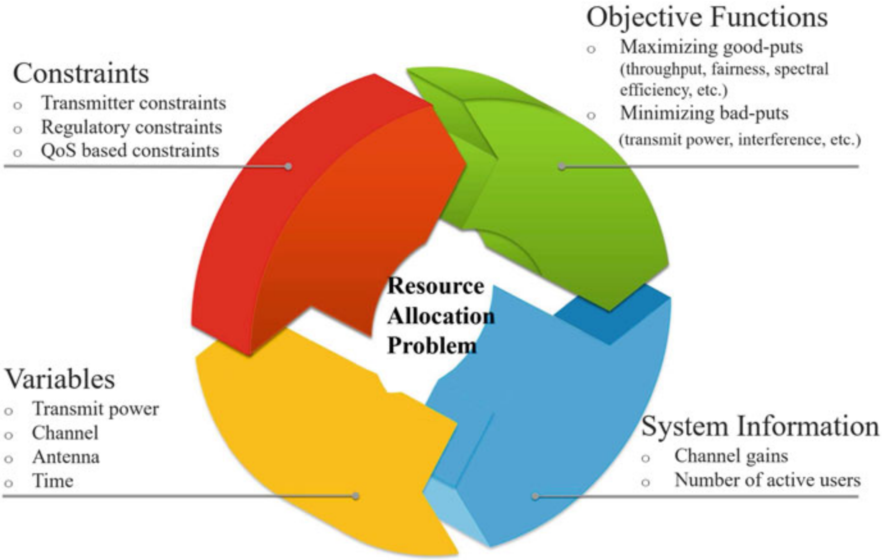

Fig. 1.2 Four intertwined tuples of resource allocation in wireless networks

Optimization problem (1.1) can represent any of the following scenarios:

- Network-centric scenario, where the aim is to achieve the network's objective, for example, maximizing the aggregate of all users' throughput. This scenario is also called the cooperative utility maximization problem [1, 22–25];
- User-centric scenario, where the objective is to maximize each user's utility. This scenario is also called noncooperative utility maximization [26–28].
- Network-aided scenario, where additional mechanisms, for example, pricing or intervention, are used for achieving objectives [29].

In the network-centric scenario, optimization is achieved via a central node, and the allocation of resources is globally optimal at the cost of increased computational complexity. In addition, it requires considerable message passing between each user and the central node to provide the central node with the required SI [6, 30].

In the user-centric scenario, each user calculates the optimal values of its transmission parameters by utilizing the locally available SI. In this way, calculations are distributed to users, and message passing to a central node is not required. Performance is measured at the equilibrium point where the optimization problems of all users converge to a fixed operating point. However, a major drawback is that the performance at equilibrium is suboptimal [11, 31, 32].

In the network-aided scenario, an effort is made to bridge the gap between the network-centric and user-centric scenarios. In doing so, the network intervenes to improve the performance of the user-centric scenario to the extent possible. This can be done by the application of pricing [31, 33–35]. There exist a number of cooperative game-theoretic approaches for introducing incentives to enhance performance while keeping the optimization problem as simple as possible and requiring minimal system information [7, 36–40].

In solving optimization problem (1.1), the following three points need to be carefully considered:

Theoretical Aspects and Computational Complexity: An important point in obtaining the globally optimal solution to (1.1) is its computational complexity. When (1.1) is convex, any local optima is also the global optima and is achievable via efficient numerical algorithms and existing software [41]. In practice, however, due to interference in wireless channels from multiple transmitters, (1.1) is not convex and the computational complexity of obtaining its globally optimal solution is high [22]. To alleviate this problem, different mathematical techniques and tools have been developed, such as GP [42, 43], variational inequalities (VIs) [44, 45], game theory [46, 47], SCA [15, 48], CGP [12, 49], relaxation algorithms (RAs) [25], SCA for low complexity (SCALE) [14, 50], DC approximation [51], nonnegative matrix theory [23–25], and the Lagrange dual function [22].

Distributed Algorithms: Distributed designs for resource allocation in multiuser networks that are scalable and low cost (in terms of computational complexity and the amount of required system message passing between transmitter–receiver pairs) have been extensively studied during the past decade, for example, [32, 34]. In addition, future wireless networks with a multilayer structure and the emerging concept of self-organizing networks (SONs) are the two major business drivers for distributed algorithms [32]. In distributed resource allocation schemes, each user implements a decision-making algorithm to choose the values of its decision parameters such as its transmit power and frequency band.

The following two approaches for distributed resource allocation have been used:

- *Decomposition Algorithms:* The original problem is decomposed into a number of solvable subproblems, each of which can be solved in a distributed manner while a higher-level agent coordinates optimal solutions to subproblems via a signaling scheme [52]. This approach has two variants: the primal decomposition method and the dual decomposition method. The former is based on decomposing the original primal optimization problem, whereas the latter is based on decomposing the corresponding Lagrangian dual problem. In the primal decomposition technique, also known as direct decomposition, a higher-level agent directly determines the amount of available resources, for example, the transmit power and the number of channels, for each subproblem. In contrast, in the dual decomposition technique, for each constraint in the optimization problem, a higher-level agent sets a corresponding price for utilizing the available resources by each subproblem;
- *Game Theory:* In this approach, the resource allocation problem is locally formulated for each user, where each player (transmitter–receiver pair) with local observations determines its transmit parameters via liaising with other players. The main issue is the network's performance at the convergence point of the game [9, 47, 53–55], where the concept of Nash equilibrium is widely used.

Table 1.1 Comparison of cooperative and noncooperative power control schemes

Reference(s)	Scenario	Approach	*Performance* computational complexity
[22]	Cooperative	Lagrange dual function	*Near optimal* Low
[23]	Cooperative	Nonnegative Matrix theory	*Globally optimal* High
[24, 25]	Cooperative	Relaxation Nonnegative matrix theory	*Locally optimal* Low
[50, 51]	Cooperative	Convex approximation	*Locally optimal* Low
[26, 27, 56–58]	Noncooperative	Strategic game theory	*Inefficient* Low
[33, 59]	Noncooperative	Strategic game theory	*Pareto optimal* Low

To choose a suitable method for solving a given resource allocation problem, the following attributes need to be considered: (1) the rate of convergence and its robustness against variations in parameter values, (2) the performance gap between the distributed and centralized schemes, (3) the amount of required message passing, and (4) the associated computational cost. In general, via a centralized scheme, a globally optimal solution can be obtained at the cost of significant message passing, whereas game-theoretic distributed algorithms need less message passing at the cost of deviating from the global optima. In the centralized scheme, it is assumed that all players cooperate and are obedient, whereas in distributed algorithms, players compete with each other (i.e., are noncooperating) and each player aims to maximize its own utility. In Table 1.1, we compare the salient features of some important existing works on resource allocation in wireless networks.

Availability of SI Existing schemes for optimal resource allocation require exact SI, for example, channel gains. However, due to user mobility, the existence of noise, interference, and other factors, as well as the time-varying and nonlinear nature of electromagnetic propagation, in practice it is not feasible to obtain accurate SI. As a result, the so-called optimal solution deviates from optimality, and some constraints may not be satisfied. This would negatively impact users' experienced QoS, as well as the network's performance and reliability. It is in this light that developing cost-effective schemes for resource allocation that would be robust against inherent uncertainties and inaccuracies in system information is of paramount importance. Our focus is to study robust resource allocation schemes for large-scale multiuser wireless networks with a view to identifying their pros and cons and demonstrating the impact of uncertainty in parameter values on the performance of such networks. In robust schemes, the performance is expected to be satisfactory to the extent possible, even in the presence of uncertainties and variations in parameter values.

1.3 Mathematical Background

We now present the mathematical preliminaries for robust resource allocation in wireless networks. Generally, the following factors are considered in any robust optimization problem:

- The uncertain parameter $\mathbf{c}_w$ (actual) for $w = \{0, y, z\}$, which includes all SI is modeled by

$$\mathbf{c}_w = \bar{\mathbf{c}}_w + \hat{\mathbf{c}}_w, \quad \forall w = \{0, y, z\}, \tag{1.2}$$

  where $\bar{\mathbf{c}}_w$ is the nominal (average, estimated, or mean) value and $\hat{\mathbf{c}}_w$ is the error;
- The nominal optimization problem, which is the optimization problem with no uncertainty in SI, that is, $\mathbf{c}_w = \bar{\mathbf{c}}_w$ for $w = \{0, y, z\}$, and (1.1) is called the nominal optimization problem;
- The robust counterpart problem, which is the modified version of the nominal optimization problem, in which uncertainty in SI is considered.

In essence, the nominal optimization problem is mapped into a robust optimization problem, and the way in which this mapping is formulated depends on the available information on the error and prevailing assumptions on the wireless network.

1.3.1 Stochastic Robust Optimization

In this approach, it is assumed that the probability density function of each uncertain parameter is Gaussian with known parameter values. This assumption is used in the stochastic formulation of the optimization problem. For example, the average-based formulation of resource allocation problem is[1]

$$\max_{\mathbf{x}} \mathbb{E}_{\mathbf{c}_0} \{\widetilde{f_0}(\mathbf{x}, \mathbf{c}_0)\}, \tag{1.3}$$

$$\text{subject to} \quad \begin{cases} \mathbf{E}_{\mathbf{c}_y} \{\widetilde{f_y}(\mathbf{x}, \mathbf{c}_y)\} = 0, & \forall y \in \mathscr{Y}, \\ \mathbf{E}_{\mathbf{c}_z} \{\widetilde{f_z}(\mathbf{x}, \mathbf{c}_z)\} \le 0, & \forall z \in \mathscr{Z}, \end{cases}$$

where $\mathbb{E}_{\mathbf{c}_0}(\mathbf{x})$ is the expected value of vector $\mathbf{x}$; $\widetilde{f_0}(\mathbf{x}, \mathbf{c}_0)$, $\widetilde{f_y}(\mathbf{x}, \mathbf{c}_y)$ and $\widetilde{f_z}(\mathbf{x}, \mathbf{c}_z)$ are the robust counterparts of $f_0(\mathbf{x}, \mathbf{c}_0)$, $f_y(\mathbf{x}, \mathbf{c}_y)$ and $f_z(\mathbf{x}, \mathbf{c}_z)$, respectively; and when $\hat{\mathbf{c}} = 0$, we have $\widetilde{f_y}(\mathbf{x}, \mathbf{c}_w) = f_y(\mathbf{x}, \bar{\mathbf{c}}_w) = f_y(\mathbf{x}, \mathbf{c}_w)$ for $w \in \{0, y, z\}$. Note that we use $\widetilde{f_y}(\mathbf{x}, \mathbf{c}_w)$ to highlight uncertainty in parameter values in the robust (counterpart) problems.

[1] The average-based formulation in (1.3) is not the only way to formulate the resource allocation problem; and the outage probability formulation, that is, $\Pr_{\mathbf{c}_y}\{f_y(\mathbf{x}, \mathbf{c}_y) = 0\} \ge 1 - \delta_y$, or the violation-based formulation, that is, $\Pr_{\mathbf{c}_y}\{f_y(\mathbf{x}, \mathbf{c}_y) \ne 0\} \le \delta_y$ can also be considered.

In this approach, the following factors should be considered:

- In practice, little may be known about the probability density function of error;
- The constraints of the resource allocation problem cannot be satisfied in all instances, which may not be acceptable in some wireless networks.

When error statistics are available and the instantaneous performance is not a major issue, the stochastic approach can be readily used [17, 60, 61]. In contrast, when the instantaneous performance, for example, the instantaneous QoS of each user, is of prime importance, allocating the resources based on (1.3) may not be acceptable. In such cases, worst-case robust optimization techniques, as described in what follows, are more appropriate [27, 62–67].

1.3.2 Worst-Case Robust Optimization

In worst-case robust optimization, it is assumed that the error at any instant is confined to the uncertainty region denoted by $\mathscr{R}_{\mathbf{c}_y}$ for $y = \{0, m, n\}$, centered at the nominal (estimated) $\bar{\mathbf{c}}_y$, and bounded by the worst-case realization of error.[2] The robust counterpart of (1.1) is

$$\max_{\mathbf{x}} \widetilde{f_0}(\mathbf{x}, \mathbf{c}_0), \quad \forall \mathbf{c}_0 \in \mathscr{R}_{\mathbf{c}_0}, \tag{1.4}$$

$$\text{subject to} \begin{cases} \widetilde{f_y}(\mathbf{x}, \mathbf{c}_y) = 0, & \forall \mathbf{c}_y \in \mathscr{R}_{\mathbf{c}_y} \quad \forall y \in \mathscr{Y}, \\ \widetilde{f_z}(\mathbf{x}, \mathbf{c}_z) \le 0, & \forall \mathbf{c}_z \in \mathscr{R}_{\mathbf{c}_z} \quad \forall z \in \mathscr{Z}. \end{cases}$$

The solution to (1.4) maximizes the objective function, is optimal under any instance of error in the uncertainty region, and satisfies all the constraints. Note that obtaining the optimal solution to (1.4) entails knowledge of the uncertainty region whose shape and size depend on the system model and the causes of uncertainty (Section 8.5.5. in [17]; [62]).

For any uncertain parameter $\mathbf{c}_w$ in (1.2), when the error has a known, for example Gaussian, probability density function (pdf) $\mathscr{D}(\hat{\mathbf{c}}_w)$, the value of the uncertain parameter for any realization of error is inside the uncertainty region with a probability π. This statement is formulated by

$$\int_{\|\hat{\mathbf{c}}_w\| \le \varepsilon_{\mathbf{c}_w}} \mathscr{D}(\hat{\mathbf{c}}_w) d\mathbf{c}_w \le \pi, \tag{1.5}$$

where $\|\mathbf{x}\|$ is the norm function and $\varepsilon_{\mathbf{c}_w}$ is the bound on the uncertainty region, which can be obtained from the probability of violation [17]. In general, the uncertainty set is assumed to be a nonempty, compact, and convex set. For different types

[2] In wireless networks, estimated values can be calculated by the receiver using pilot signals received from the corresponding transmitter.

of uncertainty, each with a different pdf, the respective uncertainty set can be obtained. It has been shown that for different pdfs, such as Gaussian or uniform, the uncertainty set can be obtained in this manner [17, 62].

When the uncertain parameters in the constraints and the uncertainty region can be formulated as specific and deterministic, the computational complexity of obtaining the optimal solution to (1.4) can be reduced. One approach to doing so is to use the concept of a protection function [68]. For instance, $\widetilde{f}_y(\mathbf{x}, \mathbf{c}_y)$ can be rewritten as

$$\widetilde{f}_y(\mathbf{x}, (\mathbf{c}_y - \bar{\mathbf{c}}_y + \bar{\mathbf{c}}_y)) = \widetilde{f}_y(\mathbf{x}, (\bar{\mathbf{c}}_y + \hat{\mathbf{c}}_y)). \tag{1.6}$$

When $f_y(\mathbf{x}, (\bar{\mathbf{c}}_y + \hat{\mathbf{c}}_y))$ can be decomposed into two parts, we write

$$\widetilde{f}_y(\mathbf{x}, (\bar{\mathbf{c}}_y + \hat{\mathbf{c}}_y)) = f_y(\mathbf{x}, \bar{\mathbf{c}}_y) \dagger \Delta(\mathbf{x}, \hat{\mathbf{c}}_y), \tag{1.7}$$

where $\dagger$ denotes a mathematical operation such as product or sum, and $\Delta(\mathbf{x}, \hat{\mathbf{c}}_y)$ is the protection function, which depends on the worst-case condition of error in the uncertainty region. In what follows, we will show how to obtain the protection function for any given uncertainty region.

1.3.2.1 General Norm

From (1.5), depending on the source of uncertainty, the uncertainty region can be modeled by the type or the order of a norm function. For example, the spherical uncertainty set that corresponds to uncertainties in Gaussian channels can be modeled by 2-norm (or the ellipsoid model [17]). Also, the cubic uncertainty set that corresponds to a quantization error can be modeled by $\|\hat{\mathbf{c}}_y\|_\infty$. The general norm covers all types of uncertainty in channel gains and system parameters in our system model [17].

1.3.2.2 Polyhedron Model

In the polyhedron model, the absolute value of each parameter is bounded [65], and the uncertainty region is

$$\Re_{\mathbf{c}_y} = \{\mathbf{c}_y | \mathbf{M}_y \cdot \mathbf{c}_y^{\mathrm{T}} \le \boldsymbol{\varepsilon}_{\mathbf{c}_y}\}, \tag{1.8}$$

where $\mathbf{M}_y$ is the weight matrix for $\mathbf{c}_y$, and $\boldsymbol{\varepsilon}_{\mathbf{c}_y}$ is an $R^{K\times 1}$ vector that represents the weighted maximum deviation of $\mathbf{c}_y$ from $\bar{\mathbf{c}}_y$. In Chapter 2, we will show that this type of uncertainty region can reduce the computational complexity of obtaining the optimal solution to (1.4).

Maxi-min Formulation for Worst-Case Robust Optimization:

- The optimal solution to (1.4) satisfies all the constraints under any condition of error within the uncertainty region. When the objective function is an increasing function of the uncertain parameters, (1.4) can be reformulated as

$$\max_{\mathbf{x}} \min_{\mathbf{c}_0 \in \mathscr{R}_{\mathbf{c}_0}} \widetilde{f_0}(\mathbf{x}, \mathbf{c}_0), \tag{1.9}$$

$$\text{subject to} \quad \begin{cases} f_y(\mathbf{x}, \mathbf{c}_y) = 0, & \forall y \in \mathscr{Y}, \\ f_z(\mathbf{x}, \mathbf{c}_z) \le 0, & \forall z \in \mathscr{Z}. \end{cases}$$

  Note that in (1.9), uncertainty is solely in the objective function, not in the constraints. This type of worst-case robust optimization formulation is called the maxi-min formulation.
- A solution to (1.9) guarantees an acceptable instantaneous performance for any error in the uncertainty region. A straightforward way to solve (1.9) is first to obtain the solution to the inner minimization problem and then solve the outer maximization problem. However, depending on the formulation of the uncertainty region, a closed-form solution to the inner optimization problem may not exist. In Chapters 2 and 4, we will show how to solve the inner optimization problem for some formulations of the uncertainty region.

1.3.3 Hybrid Approach: Bounded Uncertainty and Probabilistic Constraints

The stochastic approach guarantees performance in a statistical manner, and the worst-case approach guarantees performance for all instances of error within the uncertainty region. In situations that uncertainty is bounded and constraints are probabilistic, it is very desirable to devise a hybrid approach. Specifically, the following assumptions are considered in devising this hybrid approach:

- **A1**: The values of uncertain parameters are bounded and random.
- **A2**: Violation of each optimization constraint below a given threshold is statistically permissible, that is, $\Pr\{f_y(\mathbf{x}, \mathbf{c}_y) \neq 0\} \le \delta_y$, where δ_y is the acceptable probability of violating $f_y(\mathbf{x}, \mathbf{c}_y) = 0$ for any $y \in \mathscr{Y}$, and the pdf of error is $\mathscr{D}(\hat{\mathbf{c}}_w)$ for $w = \{0, y, z\}$.

Note that Assumption A1 stands in contrast to the corresponding assumption in the stochastic approach, and Assumption A2 stands in contrast to the corresponding assumption in the worst-case approach. In the sequel, we introduce two variants of this hybrid approach.

1.3.3.1 Chance-Constrained Approach

When Assumptions A1 and A2 hold, the robust counterpart of (1.1) is

$$\max_{\mathbf{x}} \widetilde{f_0}(\mathbf{x}, \mathbf{c}_0), \quad \forall \mathbf{c}_0 \in \mathscr{R}_{\mathbf{c}_0}, \tag{1.10}$$

$$\text{subject to} \quad \begin{cases} \Pr(\widetilde{f_y}(\mathbf{x}, \mathbf{c}_y) \neq 0) \leq \delta_y, & \forall \mathbf{c}_y \in \mathscr{R}_{\mathbf{c}_y}, \quad \forall y \in \mathscr{Y}, \\ \Pr(\widetilde{f_z}(\mathbf{x}, \mathbf{c}_z) \geq 0) \leq \delta_z, & \forall \mathbf{c}_z \in \mathscr{R}_{\mathbf{c}_z}, \quad \forall z \in \mathscr{Z}. \end{cases}$$

For an arbitrary $\mathscr{F}_{c_w}$, it is difficult to check whether or not the constraints of (1.10) are satisfied. In addition, for small values of δ_y and δ_z, the feasible set for (1.10) may be nonconvex, meaning that it may need excessive calculations. To overcome such difficulties, it has been shown [18] that when $f_y(\mathbf{x}, \mathbf{c}_y)$ or $f_z(\mathbf{x}, \mathbf{c}_z)$ is a linear function, the chance-constrained probabilities can be replaced by their convex approximations, leading to significantly fewer calculations.

In [18], it is shown that for independent and identically distributed (i.i.d.) values of uncertain parameters, when constraints are affine functions, they can be replaced by convex functions as their safe approximations. For example, when $f_z(\mathbf{x}, \mathbf{c}_z) = \mathbf{x} \cdot \mathbf{c}_z^{\mathrm{T}}$, where $\mathbf{x} \in \mathscr{C}^{1\times K}$ and $\mathbf{c}_z \in \mathscr{C}^{1\times K}$, we can write

$$\mathbf{x} \cdot \mathbf{c}_z^{\mathrm{T}} = \sum_{k=1}^{K} x^k \bar{c}_z^k + \sum_{k=1}^{K} \zeta_z^k x^k \hat{c}_z^k, \tag{1.11}$$

where $\zeta_z^k = \frac{c_z^k - \bar{c}_z^k}{\hat{c}_z^k}$ for each $z \in \mathscr{Z}$ is in the range $[-1, 1]$. For uncorrelated c_z^k, all values of ζ_z^k are independent of each other and belong to the pdf $\mathscr{D}(z)$ [69]. Hence, $\Pr\{f_z(\mathbf{x}, \mathbf{c}_z \geq 0\} \leq \delta_z$ can be replaced by the Bernstein approximations of the chance constraints [18], that is,

$$\sum_{k=1}^{K} x^k \bar{c}_z^k + \sum_{k=1}^{K} \eta_{\mathscr{D}(z)}^{+} x^k \hat{c}_z^k + \sqrt{2 \ln \delta_z^{-1}} \sqrt{\sum_{k=1}^{K} \varphi_{\mathscr{D}(z)}^2 (x^k \hat{g}_z^k)^2} \leq 0, \qquad \forall z \in \mathscr{Z}, \tag{1.12}$$

or

$$\sum_{k=1}^{K} x^k \bar{c}_z^k + \sum_{k=1}^{K} \eta_{\mathscr{D}(z)}^{+} x^k \hat{c}_z^k + \varphi_{\mathscr{D}(z)} \sqrt{2K \ln \delta_z^{-1}} \max_{\forall k \in \mathscr{K}} x^k \hat{c}_z^k \leq 0, \qquad \forall z \in \mathscr{Z}, \tag{1.13}$$

where $-1 \leq \eta_{\mathscr{D}(z)}^{+} \leq +1$ and $\varphi_{\mathscr{D}(z)} \geq 0$, which depend on $\mathscr{D}(z)$, are approximations of chance constraints. For some $\mathscr{D}(z)$, these parameters have fixed values, as shown in Table 1.2 [18].

Table 1.2 Values of $\eta^{+}_{\mathscr{D}(z)}$ and $\varphi_{\mathscr{D}(z)}$ for some $\mathscr{D}(z)$ [18]

$\mathscr{D}(z)$	$\eta^{+}_{\mathscr{D}(z)}$	$\varphi_{\mathscr{D}(z)}$
$\sup\{\mathscr{D}(z)\} \in [-1, 1]$	1	0
$\sup\{\mathscr{D}(z)\}$ is unimodal and $\sup\{\mathscr{D}(z)\} \in [-1, 1]$	1/2	$\frac{1}{\sqrt{12}}$
$\sup\{\mathscr{D}(z)\}$ is unimodal and symmetric	0	$\frac{1}{\sqrt{3}}$

Considering the preceding discussion, the protection functions for (1.12) and (1.13) are as follows:

$$\Delta(\mathbf{x}, \hat{\mathbf{c}}_z) = \sum_{k=1}^{K} \eta^{+}_{\mathscr{D}(z)} x^k \hat{c}^k_z + \sqrt{2 \ln \delta_z^{-1}} \sqrt{\sum_{k=1}^{K} \varphi^2_{\mathscr{D}(z)} (x^k \hat{g}^k_z)^2}, \qquad \forall z \in \mathscr{Z}, \tag{1.14}$$

and

$$\Delta(\mathbf{x}, \hat{\mathbf{c}}_z) = \sum_{k=1}^{K} \eta^{+}_{\mathscr{D}(z)} x^k \hat{c}^k_z + \varphi_{\mathscr{D}(z)} + \sqrt{2K \ln \delta_z^{-1}} \max_{\forall k \in \mathscr{K}} x^k \hat{c}^k_z, \qquad \forall z \in \mathscr{Z}. \tag{1.15}$$

1.3.3.2 D-Norm Approach

The D-norm was introduced in [68] for trading off between robustness and optimality by considering a protective function for constraints. To explain this approach, we again focus on the case of linear constraints for $f_z(\mathbf{x}, \mathbf{c}_z) = \mathbf{x} \cdot \mathbf{c}_z^{\mathrm{T}}$, where $\mathbf{x} \in \mathscr{C}^{1\times K}$ and $\mathbf{c}_z \in \mathscr{C}^{1\times K}$. Here, it is assumed that the exact values of SI fall in $\widetilde{c}^k_z = \{c^k_z \in [\bar{c}^k_z - \hat{c}^k_z, \bar{c}^k_z + \hat{c}^k_z]\}$ for all c^k_z. Moreover, the pdf of each uncertain parameter is symmetric but unknown, meaning that $\mathscr{F}_{\mathbf{c}_z}$ can be any symmetric density function such as Gaussian or uniform.[3] When the constraint involves a linear function, we have

$$\mathbf{x} \cdot \mathbf{c}_z^{\mathrm{T}} = \sum_{k=1}^{K} x^k \bar{c}^k_z + \Delta_z(\Gamma_z, \mathbf{x}),$$

where $\Delta_z(\Gamma_z, \mathbf{x})$ is the protection function for the linear constraint, and $0 \le \Gamma_z \le K$ is used to adjust the protection function [68]

$$\Delta_z(\Gamma_z, \mathbf{x}) = \max_{\mathscr{L}_z} \sum_{k \in \mathscr{L}_z} \hat{c}^k_z x^k, \tag{1.16}$$

[3] These assumptions are in line with other studies in the literature, which typically assume Gaussian or uniform densities for the error in wireless channel gains [17].

where $\mathscr{L}_z$ is any subset containing the values of c_z^k, of size $|\mathscr{L}_z| = \Gamma_z$. The number of c_z^k in $\mathscr{L}_z$ (i.e., the value of Γ_z) determines the magnitude of the protection function. For example, when $\Gamma_z = 0$, the protection function vanishes. In contrast, when $\Gamma_z = K$, all c_z^k are taken into account, and the value of $\Delta_z(\Gamma_z, \mathbf{x})$ is maximized. By adjusting Γ_z, one can control the cost of robustness in terms of the difference between the robust and nominal solutions.

Consequently, in the D-norm approach, when all constraints involve linear functions, the robust counterpart of the nominal optimization problem is

$$\max_{\mathbf{x}} \widetilde{f_0}(\mathbf{x}, \mathbf{c}_0), \quad \forall \mathbf{c}_0 \in \mathscr{R}_{\mathbf{c}_0}, \tag{1.17}$$

$$\text{subject to} \quad \sum_{k=1}^{K} x^k \bar{c}_z^k + \max_{\mathscr{L}_z} \sum_{k \in \mathscr{L}_z} \hat{c}_z^k x^k \leq 0, \qquad \forall z \in \mathscr{Z}.$$

Note that considering protection function (1.16) entails probabilistic compliance (or, equivalently, probabilistic violation) of the inequality constraint due to the fact that the protection function involves uncertain parameters.

To mathematically capture the preceding discussion, let us consider $\hat{c}_z^k = \varrho_z^k \bar{c}_z^k$, where $\varrho_z^k \in [-1, 1]$, and the pdf of $\hat{c}_z^k$ is unknown but symmetric. The probability of violating the inequality constraint in the D-norm approach is upper bounded [68] by

$$\Pr_z(\text{violating } f_z(\mathbf{c}_z, \mathbf{x}) \text{ in the D-norm approach}) \leq \left(1 - \Phi(\frac{\Gamma_z - 1}{\sqrt{K}})\right), \ \forall z \in \mathscr{Z}, \tag{1.18}$$

where Φ is the cumulative density function (cdf) of the standard normal (Gaussian) distribution with zero mean and variance θ. The upper bound on the violation probability in (1.18) is based on the independent and symmetrically distributed features of ϱ_z^k (Theorem 3 in [68]). By adjusting Γ_z, the probability of violating the interference constraint can be kept below the acceptable threshold. The relationship between the violation probability Γ_z and K for different values of δ_z via (1.18) is illustrated in Fig. 1.3.

When A1 and A2 hold, there is a predefined threshold for permissible violation of the constraints. To formulate the protection functions, different approaches can be considered [18, 69]. Among them, the chance-constrained and D-norm approaches are very appealing since they can be used to convert constraints into affine and convex forms and require no detailed information on the pdf of each uncertain parameter. In this way, the robust optimization problem can be solved efficiently. The assumptions are realistic in wireless networks [65, 70, 71], and the protection functions are not fixed as in the worst-case approach, that is, they can be formulated based on the prevailing conditions.

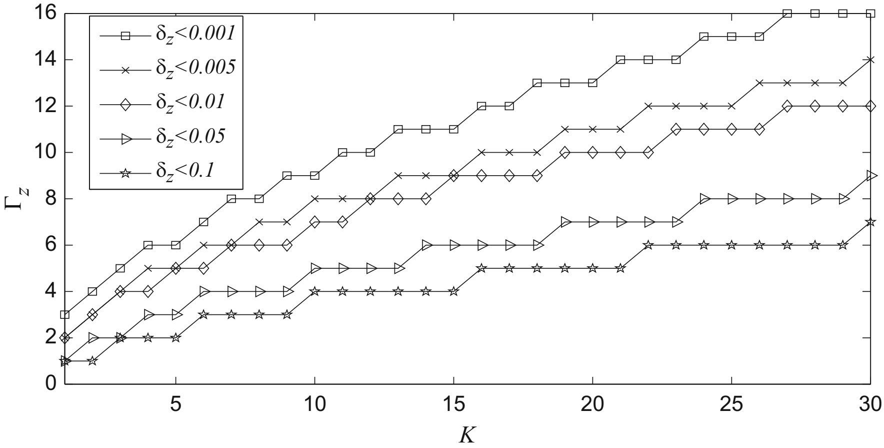

Fig. 1.3 Relationship between Γ_z and K for different values of δ_z

1.4 Generic System Model

In general, we may have different categories of users, each with its own specific priority, trustworthiness, and capabilities, as described in what follows:

- Legitimate[4] users, which are authorized to use the spectrum for their transmissions and have distinct priorities and capabilities, divided into two different types:
  - Primary users (PUs), which have the priority in using the frequency spectrum and expect a guaranteed QoS;
  - Secondary/cognitive users (SUs), which can opportunistically access the spectrum only when the QoS of PUs is preserved;
- External friendly nodes including
  - Friendly relays, which increase the throughput of legitimate users;
  - Friendly jammers, which prevent eavesdroppers from listening in on legitimate users' messages;
- Eavesdroppers that try to listen in on legitimate users' messages;
- Malicious jammers, which disrupt the communications of targeted legitimate users.

Consider a set of communication resources denoted by $\mathscr{K} = \{1, \ldots, K\}$ divided into K orthogonal dimensions, for example, frequency bands or channels, which are shared between users. The set of users is $\mathscr{N}$, whose size is $|\mathscr{N}| = N$. This set consists of all uniquely indexed transmitters and receivers.

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[4] A "user" is a terminal, node, a base station, or an access point.

The direct channel gain between transmitter n and its intended receiver m in channel k is h_{nm}^k. The interference channel gain from another transmitter n' to the same receiver m in channel k is $h_{n'm}^k$. The transmit power of transmitter n in channel k is p_n^k, and its SINR in channel k at receiver m is

$$\gamma_{nm}^k = \frac{p_n^k h_{nm}^k}{\sigma^2 + \sum_{n' \neq n,\, n' \in \mathscr{N}} p_{n'}^k h_{n'm}^k}, \quad \forall n \in \mathscr{N}, \tag{1.19}$$

where the denominator is the interference plus noise at receiver m, and σ^2 is the noise power assumed to be equal in all receivers in all channels. For simplicity and when there is no ambiguity, we may drop the receiver's index in our notations.

For user n, its transmit power p_n^k in channel k is the optimization variable of the resource allocation problem, and its strategy is the set of its transmit power levels in different channels, defined as

$$\mathscr{A}_n = \left\{ \mathbf{p}_n = (p_n^1, \ldots, p_n^K) | p_n^k \in [(p_n^k)^{\mathtt{min}}, (p_n^k)^{\mathtt{max}}], \text{ and } \sum_{k=1}^{K} p_n^k \leq (p_n)^{\mathtt{max}} \right\}, \tag{1.20}$$

where $(p_n^k)^{\mathtt{min}}$ and $(p_n^k)^{\mathtt{max}}$ are the minimum and maximum transmit power levels of user n in channel k, respectively, and $(p_n)^{\mathtt{max}}$ is the upper bound on the sum of transmit power levels of user n in all channels. In practice, $(p_n^k)^{\mathtt{min}}$ is much less than $(p_n^k)^{\mathtt{max}}$ and can even be negligible.

The goodput of user n, denoted by $v_n(\mathbf{p}_n, \mathbf{p}_{-n})$, is a function of its transmit power and other users' transmit power levels, where $\mathbf{p}_{-n}$ is the transmit power levels of other users. The transmit power levels for all users in all channels constitute the optimization variable $\mathbf{x}$ in (1.1). When the goodput (utility) of user n, denoted by v_n, is the sum of its throughputs in all subchannels, we have

$$v_n(\mathbf{p}_n, \mathbf{p}_{-n}) = \sum_{k \in \mathscr{K}} \log\left(1 + \gamma_n^k\right), \quad \forall n \in \mathscr{N}. \tag{1.21}$$

In this case, the following three assumptions hold [72]:

- $v_n(\mathbf{p}_n, \mathbf{p}_{-n})$ is a strictly concave and differentiable function, and its gradient is bounded;
- $\log\left(1 + \gamma_n^k\right)$ is a decreasing and convex function of other users' interference at the receiver of user n in channel k;
- The second-order mixed partial derivatives of $v_n(\mathbf{p}_n, \mathbf{p}_{-n})$ with respect to the additive impact of other terminals' interference plus noise, denoted by $\chi_n^k = \sigma^2 + \sum_{n' \neq n,\, n' \in \mathscr{N}} p_{n'}^k h_{n'}^k$, that is, $\frac{\partial^2 v_n^k}{\partial p_n^k \partial \chi_n^k}$ and $\frac{\partial^2 v_n^k}{\partial \chi_n^k \partial p_n^k}$, exist and are continuous.

When the parameter values are uncertain, the utility of user n is $u_n(\mathbf{p}_n, \mathbf{p}_{-n}, \mathbf{c}_n)$, where $\mathbf{c}_n$ denotes uncertain parameters whose values are bounded to $\varepsilon_{\mathbf{c}_n}$ within the uncertainty region $\mathscr{R}_{\mathbf{c}_n}$. When the uncertainty region shrinks to zero, we have

$$u_n(\mathbf{p}_n, \mathbf{p}_{-n}, \mathbf{c}_n)|_{\varepsilon_{\mathbf{c}_n}=0} = u_n(\mathbf{p}_n, \mathbf{p}_{-n}, \bar{\mathbf{c}}_n) = v_n(\mathbf{p}_n, \mathbf{p}_{-n}), \qquad \forall n \in \mathscr{N}.$$

We denote the optimal utility values of user n in the nominal and in the robust resource allocation problems by v_n^* and u_n^*, respectively, and the optimal social utility values for all users are

$$v^* = \sum_{n\in\mathscr{N}} v_n^* \quad \text{and} \quad u^* = \sum_{n\in\mathscr{N}} u_n^*.$$

Optimal solutions for the nominal and the robust power allocation problems for user n are denoted by $\mathbf{p}_n^*$ and $\widetilde{\mathbf{p}}_n^*$, respectively.

Users can cooperate or compete with each other. In the case of cooperating users, the resource allocation problem may be written from the network's perspective to maximize the sum of the total utility values of all users. In this case, the objective function of the resource allocation problem is

$$f_0 = \sum_{n\in\mathscr{N}} v_n(\mathbf{p}_n, \mathbf{p}_{-n}).$$

When each user's objective is to maximize its own utility, users compete with each other in utilizing the network's resources. In this case, the objective (utility) function of user n is $f_n = v_n(\mathbf{p}_n, \mathbf{p}_{-n})$. This case is modeled by a noncooperative strategic game, where each user n aims to maximize its own utility subject to its strategy space (transmit power) via $\max_{\mathbf{p}_n\in\mathscr{A}_n} v_n(\mathbf{p}_n)$. The strategic (power control) noncooperative game is denoted by $\mathscr{G} = \{\mathscr{N}, (v_n)_{n\in\mathscr{N}}, \mathscr{A}\}$.

When the parameter values are uncertain, the objective function is

$$\widetilde{f}_0 = \sum_{n\in\mathscr{N}} u_n(\mathbf{p}_n, \mathbf{p}_{-n}, \mathbf{c}_n),$$

where $u_n(\mathbf{p}_n, \mathbf{p}_{-n}, \mathbf{c}_n)$ is the utility function of user n, and the value of $\mathbf{c}_n$ is uncertain within the uncertainty region $\mathscr{R}_{\mathbf{c}_{nm}}$ whose bound is $\varepsilon_{\mathbf{c}_{nm}}$.

1.4.1 System Model for Wireless Networks with Homogeneous Users

In a wireless cellular network with homogeneous users, all terminals have the same priority and are equally capable of deriving the SI, but their minimum required QoS may be different, that is, each terminal may require a different SINR. We assume that the sum of transmit power levels of each terminal in all uplink channels is bounded [26, 73]. A typical cellular network with homogeneous users is depicted in Fig. 1.4, where all terminals are similar in type, and the SI, that is, $\mathbf{c}_w$ for $w = \{0, y, z\}$, includes uplink channel gains.

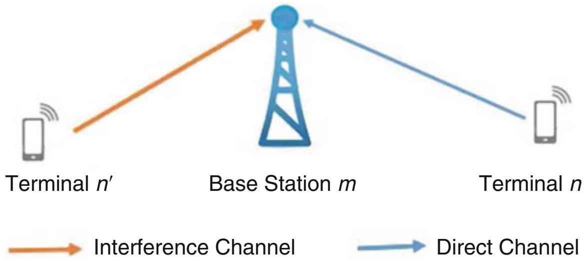

Fig. 1.4 Typical wireless network with homogeneous users

1.4.2 System Model for Wireless Networks with Heterogeneous Users

In a heterogeneous setup with multiple wireless networks, each network serves a different set of users, where each set is characterized by its own priority for utilizing the available resources, its own required QoS, and its capabilities in deriving system information. In this book, we focus on the following heterogenous scenarios.

1.4.2.1 Underlay Cognitive Radio Network

In an underlay cognitive radio network (CRN), two types of users exist:

- Primary users (PUs) or high-priority users who are licensed to use the frequency spectrum and expect to receive a guaranteed QoS. The set of PUs is $\mathcal{N}_\mathrm{q}$, and the total number of PUs is N_q;
- Secondary users (SUs) who have the capability to extract network side information, and their use of the frequency spectrum is subject to not violating the PUs' QoS. The set of SUs is $\mathcal{N}_\mathrm{s}$, and the total number of SUs is N_s.

Consider the case where there is one base station in the secondary network and the uplink channels are shared between the primary and secondary terminals (underlay CRN). The secondary base station's coverage area partially overlaps with the coverage area of the primary base stations. To guarantee the primary network's QoS in the underlay CRN, the interference of secondary terminals in each primary base station should be kept below a given threshold (interference threshold). The interference channel gain between the secondary terminal $n \in \mathcal{N}_\mathrm{s}$ and the primary base station $q \in \mathcal{N}_\mathrm{q}$ is g_{nq}^k. The interference constraint for the primary base station q in channel $k \in \mathcal{K}$ is

$$\sum_{n \in \mathcal{N}_\mathrm{s}} p_n^k g_{nq}^k \leq IT_q^k,$$

where IT_q^k is the interference threshold of primary base station q in channel k. Figure 1.5 illustrates a typical underlay CRN.

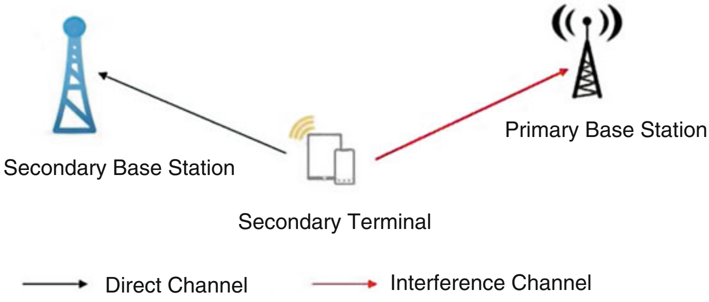

Fig. 1.5 Typical underlay CRN

In this setup, the throughputs of PUs or SUs can be considered the objective function of the optimization problem. The constraints are the bound on the terminals' transmit power levels and the interference threshold, and the SI is the channel gain between secondary terminal n and each base station.

1.4.2.2 Wireless Networks with Heterogeneous Users in Unlicensed Bands

In this case, all users have the same priority for using the unlicensed frequency spectrum, but they may have different capabilities to extract SI. Each user belongs to either of the following two groups:

- Leaders, which can extract side information, utility functions, and transmit power limitations of other users. The set of leaders is $\mathcal{N}_{\mathrm{L}}$, and the total number of leaders is N_{L};
- Followers, which can measure interference caused by other users on their respective receivers but cannot obtain side information pertaining to other users. The set of followers is $\mathcal{N}_{\mathrm{F}}$, and the total number of followers is N_{F}.

In this setup, the set of users is $\mathcal{N} = \mathcal{N}_{\mathrm{L}} \cup \mathcal{N}_{\mathrm{F}}$, and the side information obtained by user n is $\mathcal{I}_n$, which is empty for each follower (i.e., $\mathcal{I}_n = \emptyset$ if $n \in \mathcal{N}_{\mathrm{F}}$), or contains the side information pertaining to other users for each leader n, that is,

$$\mathcal{I}_n = \{\mathcal{A}_n, v_n, (\mathbf{H}_{nm})_{\forall m,n \in \mathcal{N}_{\mathrm{F}} \cup \mathcal{N}_{\mathrm{L}}}, \}, \quad \forall n \in \mathcal{N}_{\mathrm{L}},$$

where $\mathbf{H}_{nm} \triangleq \mathrm{diag}\{(h_{nm}^k)_{k=1}^K\}$ denotes channel gains between the transmitter of user n and the receiver of user m. The throughput of each group of users is obtained via (1.21).

Again, in this setup the objective function of leaders and followers can be their respective throughputs; and the constraints include the bound on the transmit power levels of leaders and followers. The system information includes both direct channel gains and interference channel gains.

1.4.3 Physical Layer Security in Wireless Channels

Preserving the security of communications over wireless channels against eavesdropping is an important issue and has been addressed in layers above the physical layer by way of message encryption [74], which involves the transmission and management of secret keys [75, 76]. Because of this, it is generally assumed that the computational resources of potential eavesdroppers is not sufficient to extract secret keys. The challenges in wireless distribution and management of secret keys, together with the fact that computing power is becoming increasingly accessible at much lower cost, have led to a growing interest in providing secrecy and confidentiality of communications in the physical layer.

The difference between the transmitter–receiver data rate (the desired rate) and the transmitter–eavesdropper data rate (eavesdropping rate) is defined as the secrecy rate, shown in Fig. 1.6. The secrecy rate is considered the utility of users, and its maximization is the objective of the resource allocation problem [75, 77–81]. The secrecy rate is negatively affected by interference at the intended receiver, that is, when the transmitter–eavesdropper's SINR is greater than the transmitter–receiver's SINR, the secrecy rate is zero [82]. When the transmitter has multiple antennas, different beam-forming techniques in the direction of intended receiver combined with transmitting noise in the direction of the eavesdropper are proposed in the literature to reduce the possibility of eavesdropping [83].

For a legitimate transmitter with a single-element antenna in the presence of multiple eavesdroppers, three cooperative approaches have been proposed in the literature in which the external nodes help the legitimate transmitter to increase its secrecy rate. These helpers can be multiple friendly relays or multiple friendly jammers targeting the eavesdropper.

The cooperative approaches to increasing the secrecy rate can be categorized as follows:

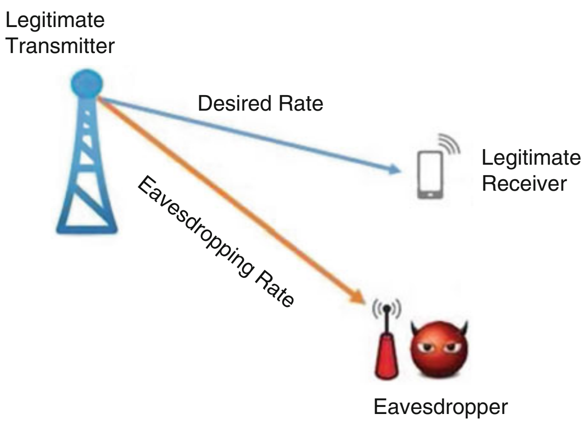

Fig. 1.6 Secrecy rate: $[\text{Desired rate} - \text{Eavesdropping rate}]^+$

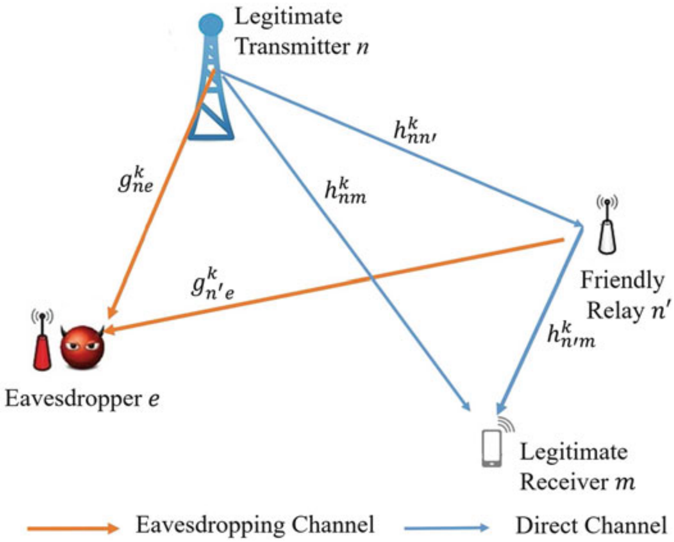

Fig. 1.7 Friendly relay approach

Friendly Relay Approach: One or more friendly relays help a legitimate transmitter to increase its secrecy rate by increasing its desired rate [75, 84–86]. In this approach, transmission is done in two consecutive hops. In the first hop, the legitimate transmitter transmits its data, and in the second hop, the friendly relay transmits the data received from the legitimate transmitter. To increase the secrecy rate, the intended receiver applies maximal ratio combining (MRC) on the signals received in the first and second hops. Friendly relays can adopt either an AF or DF strategy. Such a scheme is depicted in Fig. 1.7. The transmission rate between a legitimate transmitter n and a legitimate receiver m helped by a friendly relay n' is

$$R^{\mathrm{L}}_{nm} = \min\left\{\log\left(1+\frac{p^k_n h^k_{nn'}}{\sigma^2}\right), \log\left(1+\frac{p^k_{n'} h^k_{n'm}}{\sigma^2}\right)\right\}, \tag{1.22}$$

where $h^k_{nn'}$ and $h^k_{n'm}$ are the channel gains between transmitter n and friendly relay n' in channel k in the first hop, and the channel gain between friendly relay n' and receiver m in channel k in the second hop, respectively; and p^k_n and $p^k_{n'}$ are the transmit power levels of transmitter n and friendly relay n' in channel k, respectively. Note that for simplicity in notation, we have assumed equal noise power at all receivers, denoted by σ^2. When multiple eavesdroppers listen in on all transmissions in the two hops, the highest eavesdropping rate among multiple eavesdroppers and legitimate transmitter n is

$$R_n^{\mathrm{E}} = \max\left\{\log\left(1+\frac{p_n^k \max_e\{g_{ne}^k\}}{\sigma^2}\right), \log\left(1+\frac{p_{n'}^k \max_e\{g_{n'e}^k\}}{\sigma^2}\right)\right\}, \tag{1.23}$$

where g_{ne}^k is the channel gain between transmitter n and eavesdropper e in channel k in the first hop, and $g_{n'e}^k$ is the channel gain between friendly relay n' and eavesdropper e in channel k in the second hop. Consequently, the secrecy rate between transmitter n and receiver m is

$$R_{nm}^{\mathrm{Secrecy}} = \left[R_{nm}^{\mathrm{L}} - R_n^{\mathrm{E}}\right]^+. \tag{1.24}$$

The highest eavesdropping rate depends on the eavesdroppers' capabilities, for example, the ability to do MRC, and can be different from (1.23) [81].

Friendly Jammer Approach: One or more friendly jammers degrade eavesdroppers' listening channel by transmitting noise [87–89]. The transmission rate of legitimate transmitter n to legitimate receiver m helped by friendly jammer n' is

$$R_{nm}^{\mathrm{L}} = \log\left(1+\frac{p_n^k h_{nm}^k}{\sigma^2 + p_{n'}^k h_{n'm}^k}\right), \tag{1.25}$$

where $h_{n'm}^k$ is the channel gain between friendly jammer n' and legitimate receiver m on channel k, and $p_{n'}^k$ is the transmit power of friendly jammer n' on channel k. When multiple eavesdroppers listen in on the transmissions between transmitter n and receiver m, the highest eavesdropping rate among multiple eavesdroppers is

$$R_{nm}^{\mathrm{E}} = \max\left\{\log\left(1+\max_e\left\{\frac{p_n^k g_{ne}^k}{\sigma^2 + p_{n'}^k g_{n'e}^k}\right\}\right)\right\}, \tag{1.26}$$

where g_{ne}^k is the channel gain between transmitter n and eavesdropper e in channel k, and $g_{n'e}^k$ is the channel gain between friendly jammer n' and eavesdropper e in channel k. The secrecy rate can be derived via (1.24). This approach is depicted in Fig. 1.8.

Joint Friendly Relay and Friendly Jammer Approach Both friendly relays and friendly jammers help a legitimate transmitter to increase its secrecy rate [90]. The transmission rate between transmitter n and receiver m via DF relay n' and friendly jammer n'' is

$$R_{nm}^{\mathrm{L}} = \min\left\{\log\left(1+\frac{p_n^k h_{nn'}^k}{\sigma^2 + p_{n''}^k h_{n''n'}^k}\right), \log\left(1+\frac{p_{n'}^k h_{n'm}^k}{\sigma^2 + p_{n''}^k h_{n''m}^k}\right)\right\}. \tag{1.27}$$

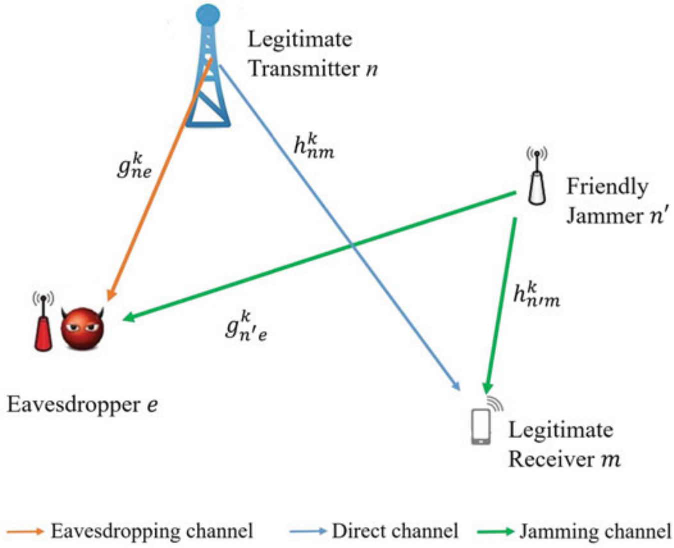

Fig. 1.8 Friendly jammer approach

When multiple eavesdroppers listen in on all transmissions in the two hops, the highest eavesdropping rate among multiple eavesdroppers is

$$R_n^{\mathrm{E}} = \max\left\{\log\left(1 + \max_e\left\{\frac{p_n^k g_{ne}^k}{\sigma^2 + p_{n''}^k g_{n''e}^k}\right\}\right), \log\left(1 + \max_e\left\{\frac{p_n^k g_{n'e}^k}{\sigma^2 + p_{n''}^k g_{n''e}^k}\right\}\right)\right\}, \tag{1.28}$$

where g_{ne}^k is the channel gain between transmitter n and eavesdropper e, $g_{n'e}^k$ is the channel gain between friendly relay n' and eavesdropper e, and $g_{n''e}^k$ is the channel gain between friendly jammer n'' and eavesdropper e. The secrecy rate is obtained via (1.24). This approach is shown in Fig. 1.9.

In Chapter 4, we will investigate the resource allocation problem for the aforementioned approaches with a view to maximizing the secrecy rate or maintaining the secrecy rate above a predefined threshold. For this type of problem, deriving the exact values of side information, for example, the channel gains of an eavesdropper, is not possible due to the fact that the eavesdroppers are passive (i.e., they do not transmit anything), and hence it is essential to consider uncertainty in the estimated values and propose robust schemes. We will do just this in Chapter 4.[5]

[5]The aforementioned approaches to increasing the secrecy rate between a legitimate transmitter and a legitimate receiver in the presence of an eavesdropper have been extended to the case of multiple antennas for all players, that is, eavesdroppers, legitimate users, friendly jammers, and relays, in [91–95]. In addition, the application of other techniques, such as full duplex transmissions for increasing the secrecy rate, is studied in [82, 96–100].

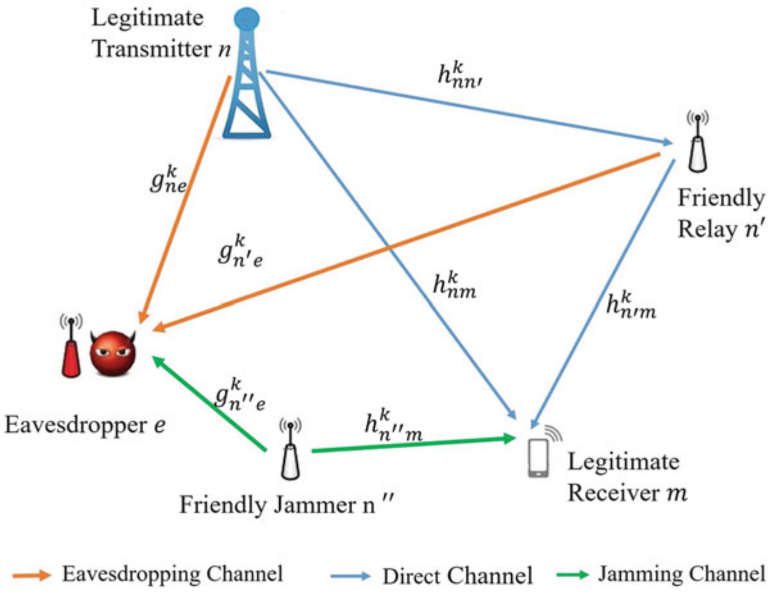

Fig. 1.9 Joint friendly relay and jammer approach

Table 1.3 Robust optimization problems in this book

Chapter	Objective function (f_0)	Optimization variable ($\mathbf{x}$)	Constraints (f_z) and (f_y)	System information $\mathbf{c}_w$ for $w = \{0, y, z\}$
Chapter 2	Throughput	$\mathbf{p}_n$	(1) Transmitter constraints	CSI, e.g., h^k_{nm}
			(2) Regulatory constraints	Noise and interference on secondary receivers
			(3) QoS constraints	
Chapter 3	Throughput	$\mathbf{p}_n$	(1) Transmitter constraints	CSI, e.g., h^k_{nm}
			(2) Maximizing followers' utility	
Chapter 4	Throughput	$\mathbf{p}_n$	(1) Transmitter constraints,	CSI, e.g., g^k_{ne}
	Secrecy rate		(2) Regulatory constraints	
	Power minimization		(3) QoS constraints, e.g., minimum secrecy rate	
	Energy efficiency			

Table 1.3 shows the coverage of various robust optimization problems and techniques in different chapters of this book.

1.5 Cost of Robustness

Applying robust resource allocation schemes in the presence of uncertainty in parameter values in wireless networks entails some costs. In what follows, we itemize the different costs of introducing robustness in wireless networks.

- **Computational Complexity:** Robust optimization in wireless networks entails consideration of additional constraints and variables, which require more calculations. Consideration of new variables that have couplings with the primal optimization variables, for example, the transmit power, and beam-forming parameters, means that conventional methods, such as relaxation techniques, cannot be used directly to solve nonconvex robust optimization problems, and additional calculations are required to derive robust optimal solutions. Whenever possible, we will make an effort to see how the computational complexity of robust schemes can be reduced. As we will show in Chapter 2, the ultimate objective is to reduce the computational cost to that of the nominal optimization problem. However, in some cases, the nominal optimization problem is inherently also nonconvex and NP-hard. In such cases, in Chapter 3, we will show how game theory can be used to reduce computational complexity. Moreover, in Chapter 4, we will discuss other techniques, such as semi-definite relaxation (SDR), nonlinear fractional programming (NLFP), DC approximation, and SCA to relax the robust optimization problem, leading to fewer calculations for obtaining the optimal solution.
- **Reduced Performance:** Robust optimization is inherently protective and conservative [58]. This point is obvious from the definition of the protection function given the constraints of the optimization problem, which shrinks the feasibility region of the optimization problem [10, 70], leading to a reduced goodput in wireless links. This is another cost of robustness, which we will discuss extensively in Chapters 2 and 3 with a view to elaborating on schemes to moderate this performance reduction by way of introducing tradeoffs between performance and robustness.
- **Distributed Algorithm:** Distributed algorithms for optimal (or near optimal) resource allocation in wireless networks are scalable, which is very desirable. However, in robust distributed schemes, there exist couplings between optimization variables, which means that decomposition algorithms or game-theoretic formulations cannot be directly applied. In devising robust distributed algorithms, the following issues require careful attention:
  - *Additional Message Passing in Decomposition-Based Distributed Algorithms*: In robust optimization, when protection functions are introduced or when the maxi-min approach is taken, there exist couplings between primal and robust variables. Hence, in such cases, there is a need for additional message passing to deal with such couplings [65]. This is another cost of utilizing decomposition techniques for robust optimization. One way to reduce this cost is infrequent message passing, which we will discuss in Chapter 2.

Table 1.4 Costs of robustness and related chapters in this book

Cost	Chapter 2	Chapter 3	Chapter 4
Computational complexity	√	√	√
Performance reduction	√	√	√
Additional message passing	√	–	–
Complexity of equilibrium analysis	–	√	–

– *Complexity of Equilibrium Analysis for Game-Theoretic Distributed Algorithms*: Conventional tools for convergence analysis, such as fixed point theory [101, 102], cannot be used for robust schemes because of the coupling between variables. Hence, other tools, such as sensitivity analysis and VIs [44, 103], should be used, which will incur additional costs for performance analysis. We will discuss this issue in Chapter 4.

Table 1.4 shows different costs of robustness and the corresponding chapters in this book that cover various techniques for reducing such costs.

1.6 Organization of This Book

In this book we present different robust resource allocation problems and seek their respective solutions. We also cover the issues pertaining to the implementation of robust schemes. The remainder of this book is organized as follows.

In Chapter 2, we present mathematical formulations of cooperative robust resource allocation problems that contain uncertain parameters, obtain their respective solutions, and show how the cost of robustness can be reduced. In doing so, we consider the case of transmit power control for single-channel cognitive radio networks, where channel gains and interference levels are uncertain. We also consider the case of transmit power control for multiple-channel cognitive radio networks, where channel gains are uncertain. Furthermore, we will show how to reduce the computational complexity of obtaining a robust solution. We will also study the performance gap between the solutions of nominal and robust optimization problems and discuss how to make tradeoffs between optimality and robustness for reducing this gap. Moreover, we present the distributed approach to solving the robust resource allocation problem via the dual decomposition technique. The need for additional message passing is the cost of using this approach, which, as we will show, can be alleviated by infrequent message passing while preserving the robustness of the solution.

Chapter 3 covers game-theoretic robust distributed resource allocation schemes for noncooperating users in wireless networks. Specifically, we formulate the power allocation problem for homogeneous users via strategic game theory and compare the equilibrium of nominal (nonrobust) and robust games. Since a closed-form solution to the power allocation problem of each user cannot be obtained, the conventional approach, that is, the best response approach, cannot be applied to investigate the uniqueness of the robust solution. We will show that robust-game equilibrium points can be analyzed using VIs and sensitivity analysis. In addition, we formulate the power allocation game for heterogeneous users (leaders and followers) via the robust Stackelberg game.

In Chapter 4, we will deal with nonconvex nominal resource allocation problems as well as nonconvex robust problems, where in both cases the solutions are nontractable. The general approach to solving such problems is to relax some constraint and convert nonconvex problems into convex or tractable formulations. We begin by presenting a taxonomy of existing approaches to such a conversion and demonstrate their applicability for the case of maximizing the secrecy rate.

In Chapter 5, we propose directions for future work and present a summary of previous chapters.

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Chapter 2
Robust Cooperative Resource Allocation

This chapter covers cooperative resource allocation in wireless networks. We discuss how this can be achieved by focusing on two specific cases. The first case is robust transmit power allocation for secondary users (SUs) in cognitive radio networks where a single uplink channel is shared between SUs and primary users (PUs) and channel gains between SUs and primary base stations are uncertain. We also consider robust transmit power allocation for device-to-device communication between SUs where PUs' multiple uplink channels are shared with SUs. In both cases, we assume worst-case uncertainty in channel gains, and show that under some conditions on the uncertainty region, the computational complexity of obtaining robust solutions can be the same as that of obtaining nominal nonrobust solutions. We also discuss algorithms to trade off between throughput reduction and robustness and present a scheme to reduce signaling in robust solutions. Our approach is applicable to other cases where the objective is the maximization of social utility, where users are cooperative.

2.1 Introduction

Robust solutions are desirable in the sense that optimal performance can be guaranteed even when exact parameter values are not available. However, robustness is achieved by considering additional constraints involving uncertain parameters, which complicates the problem. For practical reasons and so far, the preferred approach in the literature for introducing robustness has been to assume the worst-case bounded uncertainty in parameter values.

In general, there are two approaches to solving optimization problems: the centralized approach and the distributed approach. In some instances, centralized schemes are nonscalable and require measurements and parameter values pertaining to all users, that is, they need extensive message passing. In contrast, distributed

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schemes are scalable and, to the extent possible, utilize locally available measurements and information, that is, minimal message passing; but their performance in general is below that of a centralized scheme. To improve the performance of distributed schemes, decomposition algorithms have been developed, but they require additional message passing (as compared with conventional distributed algorithms) between distributed nodes to guarantee convergence. However, introducing robustness in decomposition algorithms increases the amount of additional message passing because of the coupling between optimization variables.

It is evident from the preceding statements that robust algorithms with less computational complexity and reduced signaling that can also trade off between performance and robustness are very desirable. Our objective in this chapter is to present such algorithms. We begin by formulating robust problems in the same manner that nominal (i.e., nonrobust) problems are formulated, but with additional constraints to include robustness requirements by using protection functions. This will enable us to simplify the robust problem for those cases where uncertainty is confined to a specific region, where, for example, the general definition of norm or the polyhedron model can be used. We also analyze the overhead of obtaining robust solutions and devise appropriate schemes to reduce such overhead to the extent possible.

We study the aforementioned issues by focusing on underlay cognitive radio networks (CRNs), where the secondary users (SUs) keep their interference on the primary users (PUs) below a given interference threshold (IT) [1–5]. In this way, we consider two scenarios in CRNs: (1) a single-channel scenario in which the SUs share one uplink channel with the PUs and (2) a multichannel scenario in which the SU pairs utilize device-to-device (D2D) communication. In both cases, the SUs cooperate with each other to maximize their total throughput, but the PUs are not obliged to provide their measurements and system information (including channel gains) to the SUs. Obtaining such information by the SUs is difficult and may entail uncertainty, which in turn may either violate the PUs' IT or result in inadequate SU signal-to-interference-plus-noise ratios (SINRs) [6], both because the SUs' allocated transmit power levels are inaccurate (higher in the former, lower in the latter), leading to undesirable fluctuations in the SUs' social utility value. Such concerns can be alleviated by applying robust schemes.

As indicated earlier, we formulate robust problems by considering additional constraints to include robustness requirements by utilizing protection functions. However, protection functions in many cases are nonlinear, which significantly increases the computational complexity of obtaining robust solutions. We will show that when uncertainty is confined to some specific uncertainty regions, nonlinear protection functions can be replaced either by their safe linear approximations or by a more tractable formulation. In this way, the complexity of solving the reformulated robust problem is significantly reduced, and pseudo water-filling formulations can be obtained in some cases.

We will also apply sensitivity analysis to show that throughput is reduced when uncertainty is increased [7], and we will demonstrate how to trade off between throughput and robustness via the D-norm approach and the chance constraint approach (introduced in Chapter 1, Section 1.3.3.1) [8, 9]. However, assuming less uncertainty makes the robust solution less resilient and increases the probability of violating the constraints, which may not be desirable for either SUs or PUs.

Moreover, we will show that in decomposition-based distributed robust algorithms for single-channel CRNs, it is possible to achieve convergence by infrequent message passing [8]. This is important because convergence of the conventional decomposition-based distributed schemes for both nominal and robust problems requires excessive message passing (which is not desirable).

We will conclude this chapter by providing a brief review of existing works on robust cooperative optimization and show that, although the development of schemes for reducing complexity, trading off between throughput and robustness, and reducing the amount of message passing in distributed robust algorithms is very desirable and beneficial, such schemes have not been extensively studied in the literature.

2.2 Single-Channel Cellular Cognitive Radio Networks

Consider a cellular CRN with one uplink channel, where the coverage area of a cognitive cell partially overlaps with the coverage areas of neighboring primary cells sharing the same frequency band. We assume that each user communicates with its own base station. The system model for this case is shown in Fig. 2.1. The set of PUs' base stations (PBSs) is $\mathcal{Q} = \{1, \ldots, Q\}$, and the set of SUs serviced by the secondary base station (SBS) is $\mathcal{S} = \{1, \ldots, S\}$. The channel gain between SU s and its SBS is h_s, and the interference channel gain between SU s and PBS q is g_{sq}. Cooperation between the SBS and PBSs is not assumed, but the SBS knows the predefined IT of PBSs. The SINR of SU s at its SBS is

$$\gamma_s = \frac{p_s h_s}{f_{\mathcal{Q}} + \sum_{m \in \mathcal{S},\, m \neq s,} p_m h_m + \sigma^2} \qquad \forall\, s \in \mathcal{S}, \tag{2.1}$$

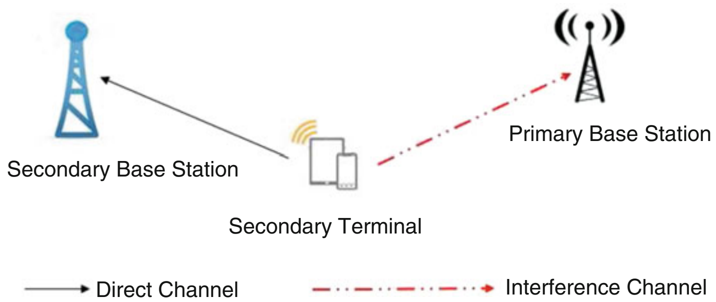

Fig. 2.1 Typical underlay CRN

where $p_s \in [p_s^{\mathtt{min}}, p_s^{\mathtt{max}}]$ is the transmit power of SU s, $f_{\mathscr{Q}}$ is the aggregate interference of PUs on the SBS, $\sum_{m \neq s,\, m \in \mathscr{S}} p_m h_m$ is the interference caused by other SUs on the SBS, and σ^2 is the noise power.

The resource allocation problem is

$$\max_{p_s^{\mathtt{min}} \leq p_s \leq p_s^{\mathtt{max}}} \quad \sum_{s \in \mathscr{S}} v_s(\gamma_s), \tag{2.2}$$

$$\text{subject to} \begin{cases} C_1 : \hat{\gamma}_s \leq \gamma_s & \forall\ s \in \mathscr{S}, \\ C_2 : \left(\sum_{s \in \mathscr{S}} g_{sq} p_s\right) \leq IT_q & \forall\ q \in \mathscr{Q}, \end{cases}$$

where C_1 represents the minimum required SINRs of SUs, C_2 is on the IT for PU q, and $v_s(\gamma_s)$ is the utility function of SU s. In general, (2.2) is nonconvex in p_s, but when two conditions are simultaneously satisfied—(a) $v_s(\gamma_s)$ is concave, strictly increasing, and twice continuously differentiable over $(0, \infty)$; and (b) $-\gamma_s v_s''(\gamma_s)/v_s'(\gamma_s) \geq 1$; problem (2.2) can be transformed into a convex optimization problem via geometric programming (GP) and logarithmic transformations [7, 10–12]. A utility function that simultaneously satisfies both conditions (a) and (b) is

$$v_s(\gamma_s) = w_s \ln \gamma_s, \tag{2.3}$$

where $w_s > 0$ is a per-user coefficient [10]. Note that by properly choosing w_s for each s, the utility function (2.3) can lead to proportionate fairness among SUs [10, 13, 14]. In [13, 15], it is shown that when the utility function (2.3) is applied, GP can be used to transform (2.2) into

$$\max_{p_s^{\mathtt{min}} \leq p_s \leq p_s^{\mathtt{max}}} \quad \sum_{s \in \mathscr{S}} v_s(h_s p_s \varrho_s^{-1}), \tag{2.4}$$

$$\text{subject to} \begin{cases} C_1 : \varrho_s \hat{\gamma}_s \leq h_s p_s, \\ C_2 : \left(\sum_{s \in \mathscr{S}} g_{sq} p_q \leq IT_q\right) & \forall q \in \mathscr{Q}, \\ C_3 : \left(f_{\mathscr{Q}} + \sum_{m \in \mathscr{S},\, m \neq s,} p_m h_m + \sigma^2\right) \leq \varrho_s, \end{cases}$$

where ϱ_s is an auxiliary variable corresponding to the interference plus noise for SU s. One can use the logarithmic transformations $p_s = e^{y_s}$ and $\varrho_s = e^{z_s}$ to convert (2.4) into a convex optimization problem [15, 16].

2.2.1 Robust Problem

In (2.4), the two parameters related to PBSs or PUs are $\mathbf{g}_q = [g_{1q}, \ldots, g_{Sq}]$ and $f_{\mathscr{Q}}$, the exact values of which are not easy to obtain by SUs because PBSs are not obliged to provide any information to SUs. To formulate the robust counterpart of (2.4), consider an uncertainty set for each uncertain parameter, containing the distances between the actual (uncertain) and the nominal (exact) values, as follows:

- The actual (uncertain) and the exact (nominal) channel gain vectors between the SUs and PBS q are $\mathbf{g}_q$ and $\overline{\mathbf{g}}_q$, whose elements for SU s are g_{sq} and $\overline{g}_{sq}$, respectively. The uncertainty set for channel gains between SUs and PBS q is

$$\mathscr{R}_{\mathbf{g}_q} = \{\mathbf{g}_q | \|\mathbf{g}_q - \overline{\mathbf{g}}_q\| \le \varepsilon_{\mathbf{g}_q}\}, \quad \forall q \in \mathscr{Q}, \tag{2.5}$$

  where $\varepsilon_{\mathbf{g}_q}$ is the upper bound on the uncertainty set for the channel gains, and $\|\mathbf{x}\|$ is the general norm.
- The actual (uncertain) and exact (nominal) interference levels caused by PUs on the SBS are $f_{\mathscr{Q}}$ and $\overline{f}_{\mathscr{Q}}$. The uncertainty set for interference levels is

$$\mathscr{R}_{f_{\mathscr{Q}}} = \{f_{\mathscr{Q}} | \|\overline{f}_{\mathscr{Q}} - f_{\mathscr{Q}}\| \le \varepsilon_{f_{\mathscr{Q}}}\}, \tag{2.6}$$

  where $\varepsilon_{f_{\mathscr{Q}}}$ is the upper bound on the uncertainty set of PUs' interference on the SBS.

Both g_{sq} and $f_{\mathscr{Q}}$ are present in the linear constraints C_2 and C_3 in (2.4). To deal with such uncertainties, we use the worst-case robust optimization for affine and convex constraints [17, 18], where the solution to (2.4) is robust if for any realization of $\mathbf{g}_s \in \mathscr{R}_{\mathbf{g}_q}$ and $f_{\mathscr{Q}} \in \mathscr{R}_{f_{\mathscr{Q}}}$ the optimal solution satisfies C_2 and C_3. Consequently, the robust counterpart of (2.4) is

$$\max_{p_s^{\min} \le p_s \le p_s^{\max}} \quad \sum_{s \in \mathscr{S}} \upsilon_s(h_s p_s \varrho_s^{-1}), \tag{2.7}$$

$$\text{subject to} \begin{cases} C_1 : \varrho_s \hat{\gamma}_s \le h_s p_s \\ C_2 : \left(\sum_{s \in \mathscr{S}} g_{sq} p_s\right) \le IT_q \qquad \forall\, q \in \mathscr{Q}, \\ C_3 : (f_{\mathscr{Q}} + \sum_{m \in \mathscr{S},\, m \ne s} p_m h_m + \sigma^2) \le \varrho_s \\ C_4 : \mathbf{g}_q \in \mathscr{R}_{\mathbf{g}_q}, \qquad f_{\mathscr{Q}} \in \mathscr{R}_{f_{\mathscr{Q}}}, \qquad \forall\, q \in \mathscr{Q}. \end{cases}$$

In the nominal problem (2.4), uncertainty is not considered, that is, the actual values are assumed to be exact, whereas in (2.7), the constraint C_4 assumes the worst-case uncertainty. In what follows, we state the conditions under which (2.7) is convex.

Proposition 2.1. *When $\mathscr{R}_{\mathbf{g}_q}$ and $\mathscr{R}_{f_{\mathscr{Q}}}$ are compact and convex sets,* (2.7) *is convex.*

Proof. This is true because C_2 and C_3 are satisfied when $\max_{\mathbf{g}_q \in \mathscr{R}_{\mathbf{g}_q}} \sum_{s \in \mathscr{S}} g_{sq} p_s \le IT_q$ and $\max_{f_{\mathscr{Q}} \in \mathscr{R}_{f_{\mathscr{Q}}}} (f_{\mathscr{Q}} + \sum_{m \in \mathscr{S},\, m \ne s} p_m h_m + \sigma^2) \le \varrho_s$. We rewrite these conditions as $(\sum_{s \in \mathscr{S}} \overline{g}_{sq} p_s + \max_{\mathbf{g}_q \in \mathscr{R}_{\mathbf{g}_q}} \sum_{s \in \mathscr{S}} (g_{sq} - \overline{g}_{sq}) p_s) \le IT_q$ and $(\overline{f}_{\mathscr{P}} + \max_{f_{\mathscr{Q}} \in \mathscr{R}_{f_{\mathscr{Q}}}} (f_{\mathscr{Q}} - \overline{f}_{\mathscr{P}}) + \sum_{m \in \mathscr{S},\, m \ne s} p_m h_m + \sigma^2) \le \varrho_s$, respectively. Since the `max` function over a convex set is a convex function (Section 3.2.4 in [19]), C_2 and C_3 [and, consequently, (2.7)] are convex.

The two terms $\Delta_q = \max_{\mathbf{g}_q \in \mathscr{R}_{\mathbf{g}_q}} \sum_{s\in\mathscr{S}} (g_{sq} - \overline{g}_{sq}) p_s$ and $\Delta_{f_{\mathscr{Q}}} = \max_{f_{\mathscr{Q}} \in \mathscr{R}_{f_{\mathscr{Q}}}} (f_{\mathscr{Q}} - \overline{f}_{\mathscr{Q}})$ are called the protection values against variations in channel gains and interference levels, respectively [20]. Using the preceding terms, the robust problem is written

$$\max_{p_s^{\min} \leq p_s \leq p_s^{\max}} \quad \sum_{s\in\mathscr{S}} v_s(h_s p_s \varrho_s^{-1}), \tag{2.8}$$

$$\text{subject to} \begin{cases} C_1 : q_s \hat{\gamma}_s \leq h_s p_s, \\ C_2 : \left(\sum_{s\in\mathscr{S}} \overline{g}_{sq} p_s + \Delta_q\right) \leq IT_q \qquad \forall q \in \mathscr{Q}, \\ C_3 : \left(\overline{f}_{\mathscr{Q}} + \sum_{\forall m\in\mathscr{S},\ m\neq s} p_m h_m + \Delta_{f_{\mathscr{Q}}} + \sigma^2\right) \leq \varrho_s. \end{cases}$$

Note that constraint C_4 in (2.7) is omitted in (2.8) by applying the protection values. However, since the protection values involve nonlinearity due to the use of `max`, the computational complexity of solving (2.7) is high. In what follows, we use the bounds on the uncertainty regions to avoid nonlinearity and reduce the computational complexity.

Proposition 2.2. *When the uncertainty set is stated in terms of the general norm, the protection values are*

$$\Delta_{f_{\mathscr{Q}}} = \varepsilon_{f_{\mathscr{Q}}} \|f_{\mathscr{Q}}\|^* \quad \textit{and} \quad \Delta_q = \varepsilon_{\mathbf{g}_q} \|\mathbf{p}\|^*,$$

where $\mathbf{p} = [p_1, \ldots, p_S]$, *and* $\|\mathbf{x}\|^*$ *is the dual norm of* $\|\mathbf{x}\|$ *(Definition 1 in [20]).*

Proof. See Appendix 1.

Proposition 2.1 establishes the conditions for the convexity of the robust power allocation problem. In Proposition 2.2, note that for a linear norm with order $a \geq 2$, that is, $\|\mathbf{x}\|_a = \sqrt[a]{\sum |x|^a}$, the dual norm is a linear norm with order $b = 1 + \frac{1}{a-1}$. In this way, the nonlinear function `max` in Δ_q and $\Delta_{f_{\mathscr{Q}}}$ in constraints C_2 and C_3 of (2.8) is avoided. Consequently, the robust power allocation problem is changed to the standard form of convex optimization with deterministic constraints, which can be solved very efficiently. For instance, with $a = 2$ (the ellipsoid norm), we have

$$\Delta_{f_{\mathscr{Q}}} = \varepsilon_{f_{\mathscr{Q}}} \|f_{\mathscr{Q}}\|_2 \quad \text{and} \quad \Delta_q = \varepsilon_{\mathbf{g}_q} \|\mathbf{p}\|_2,$$

and (2.8) becomes

$$\max_{p_s^{\min} \leq p_s \leq p_s^{\max}} \quad \sum_{s\in\mathscr{S}} v_s(h_s p_s \varrho_s^{-1}), \tag{2.9}$$

$$\text{subject to} \begin{cases} C_1 : q_s\hat{\gamma}_s \leq h_s p_s, \\ C_2 : (\sum_{s\in\mathcal{S}} \overline{g}_{sq} p_s + \varepsilon_{\mathbf{g}_q}\sqrt{\sum_{s\in\mathcal{S}} p_s^2}) \leq IT_q \qquad \forall q \in \mathcal{Q}, \\ C_3 : (\overline{f}_{\mathcal{P}} + \sum_{m\in\mathcal{S},\ m\neq s} p_m h_m + \Delta_{f_{\mathcal{Q}}} + \sigma^2) \leq \varrho_s. \end{cases}$$

Note that, as with (2.4), there are only two optimization variables, p_s and ϱ_s, in (2.9). Using the logarithmic transformations $p_s = e^{y_s}$ and $\varrho_s = e^{z_s}$, the convexity of problem (2.9) is established (Appendix 2). In the sequel, we will solve the robust problem (2.9) via its Lagrange function and show that the computational complexities of solving the nominal problem and its robust counterpart are the same.

2.2.1.1 Iterative Algorithm for Solving Nominal and Robust Problems

Considering the convexity of the nominal problem (2.2) and its robust counterpart (2.9), their respective optimal solutions $\mathbf{p}^* = [p_1^*, \ldots, p_S^*]$ and $\widetilde{\mathbf{p}}^* = [\tilde{p}_1^*, \ldots, \tilde{p}_S^*]$ can be obtained using analytical and numerical methods such as the interior point method and the Lagrange function [19] and the available software such as CVX [21], respectively. The computational complexity of solving the nominal problem is the same as that of its robust counterpart, that is, obtaining the robust solution does not entail additional calculations in this setup. As an example, in what follows, we use the Lagrange function to solve (2.9):

$$\begin{aligned} &L(\lambda_s, \mu_s, \nu_q, e^{z_s}, e^{y_s}) \\ &= -\sum_{s\in\mathcal{S}} v_s\left(h_s e^{y_s - z_s}\right) + \sum_{s\in\mathcal{S}} \lambda_s \left(\frac{e^{z_s}\hat{\gamma}_s}{h_s e^{y_s}} - 1\right) \\ &\quad + \sum_{q\in\mathcal{Q}} \nu_q \left(\frac{\overline{g}_{sq} e^{y_s} + \varepsilon_{\mathbf{g}_q}\sqrt{\sum_{s\in\mathcal{S}} e^{2y_s}}}{IT_q} - 1\right) \\ &\quad + \sum_{s\in\mathcal{S}} \mu_s \left(e^{-z_s}\left(\overline{f}_{\mathcal{P}} + \sum_{m\in\mathcal{S},\ m\neq s} e^{y_m} h_m + \Delta_{f_{\mathcal{Q}}} + \sigma^2\right) - 1\right), \end{aligned} \tag{2.10}$$

where λ_s, ν_q, and μ_s are the Lagrange multipliers for C_1, C_2, and C_3 in (2.9), respectively. This Lagrange function can be solved by applying a gradient-based algorithm to derive the primal and dual variables of the optimization problem. The gradient-based updates of variables and multipliers are

$$y_s(t+1) = \left[y_s(t) - \beta_{y_s} \times \partial L/\partial y_s\right]_{y_s^{\mathtt{min}}}^{y_s^{\mathtt{max}}}, \tag{2.11}$$

$$z_s(t+1) = \left[z_s(t) - \beta_{z_s} \times \partial L/\partial z_s\right]^+, \tag{2.12}$$

$$\lambda_s(t+1) = \left[\lambda_s(t) + \beta_{\lambda_s} \times \left(\frac{e^{z_s - y_s}\hat{\gamma}_s}{h_s} - 1\right)\right]^+, \tag{2.13}$$

$$\mu_s(t+1) = \left[\mu_s(t) + \beta_{\mu_s} \times \left(e^{-z_s} \left(\bar{f}_{\mathscr{Q}} + \sum_{m \in \mathscr{S},\ m \neq s} e^{y_m} h_m + \Delta_{f_{\mathscr{Q}}} + \sigma^2 \right) - 1 \right) \right]^+, \tag{2.14}$$

$$\nu_q(t+1) = \left[\nu_q(t) + \beta_{\nu_q} \times \left(\frac{\bar{g}_{sq} e^{y_s(t)} + \varepsilon_{\mathbf{g}_q} \sqrt{\sum_{s \in \mathscr{S}} e^{2y_s(t)}}}{IT_q} - 1 \right) \right]^+, \tag{2.15}$$

where β_{y_s}, β_{z_s}, β_{λ_s}, β_{μ_s}, and β_{ν_q} are small step sizes for the Lagrange multipliers, and $x^+ = \max\{0, x\}$. The partial derivatives of the Lagrange dual function with respect to y_s and z_s are

$$\begin{aligned} \partial L/\partial y_s = & -v_s'\left(h_s e^{y_s - z_s}\right) h_s e^{y_s - z_s} - \lambda_s \frac{e^{z_s} \hat{\gamma}_s}{h_s e^{y_s}} \\ & + \sum_{q \in \mathscr{Q}} \frac{\nu_q}{IT_q} \left(\bar{g}_{sq} e^{y_s} + \varepsilon_{\mathbf{g}_q} \frac{e^{2y_s}}{\sqrt{\sum_{s \in \mathscr{S}} e^{2y_s}}} \right) + e^{y_s} \sum_{m \in \mathscr{S},\ m \neq s} \mu_m h_m e^{-z_m} \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} \partial L/\partial z_s = & \ v_s'\left(h_s e^{y_s - z_s}\right) h_s e^{y_s - z_s} + \lambda_s \frac{e^{z_s} \hat{\gamma}_s}{h_s e^{y_s}} \\ & - \mu_s e^{-z_s} \left(\bar{f}_{\mathscr{P}} + \sum_{m \in \mathscr{S},\ m \neq s} e^{y_m} h_m + \Delta_{f_{\mathscr{Q}}} + \sigma^2 \right). \end{aligned} \tag{2.17}$$

Iterations in (2.11)–(2.17) lead to $\widetilde{\mathbf{p}}^*$. The same approach can be used to obtain $\mathbf{p}^*$ by setting the protection function to zero. Note that the computational complexity of solving the nominal problem is the same as that of solving its robust counterpart.

2.2.1.2 Reduced Throughput in Robust Solution

Tobtain the robust solution's throughput, we compare the feasibility region of (2.9) with that of the nominal problem. The uncertainties in C_2 and C_3 in (2.9) affect the SUs' transmit power levels in opposite directions. On the one hand, because of uncertainty in $\mathbf{g}_q$, the SUs' transmit power levels should be reduced to satisfy the PUs' IT. On the other hand, because of the uncertainty in $f_{\mathscr{Q}}$, the SUs' transmit power levels should be increased to improve the SUs' SINRs at the SBS, which increases interference. In either case, the transmit power allocation problem may become infeasible, meaning that a transmit power vector that simultaneously satisfies the constraints in (2.9) may not exist. To obtain the feasibility conditions, we use the notion of feasibility region, defined as follows:

Definition 2.1. The feasibility region, feasibility set, search space, or solution space of an optimization problem is the set of all possible points (sets of variables' values) that can satisfy all constraints in the optimization problem, including inequalities and equalities for integer and continuous variables. The optimal solution corresponds to a subset within the feasibility region whose performance is superior to those of other points in this region. The feasibility region for problem (1.1) in Chapter 1, denoted by **x**, is obtained by solving the following problem [19].

$$\text{Find} \quad \mathbf{x} \tag{2.18}$$

$$\text{subject to} \begin{cases} f_y(\mathbf{x}, \mathbf{c}_y) = 0, & \forall y \in \mathscr{Y}, \\ f_z(\mathbf{x}, \mathbf{c}_z) \leq 0, & \forall z \in \mathscr{Z}. \end{cases}$$

Note the following points concerning the feasibility region:

- Some constraints in resource allocation problems may not be satisfied for some parameter values, meaning that the problem may be infeasible (the feasibility set may be empty) [13, 22];
- When the feasibility region of the nominal problem contains the feasibility region of its robust counterpart, the performance of the nominal problem is higher than that of its robust counterpart [19, 23].

In what follows, we use the preceding points to compare the performance of the nominal problem and its robust counterpart in terms of the total throughput of a CRN. We begin by comparing the feasibility regions of (2.4) and (2.9). Consider $\upsilon = [\frac{\hat{\gamma}_1 n_0}{h_1}, \ldots, \frac{\hat{\gamma}_S n_0}{h_S}]^{\mathrm{T}}$ and $n_0 = \bar{f}_{\mathscr{Q}} + \sigma^2$; denote the $S \times Q$ matrix of channel gains between SUs and PBSs by **G**, whose elements are $[G]_{sq} = g_{sq}$, and the vector of ITs for PBSs by $\mathbf{IT} = [IT_1, \ldots, IT_Q]$. The upper bounds on the SUs' transmit power levels are $\mathbf{p}^{\max} = [p_1^{\max}, \ldots, p_S^{\max}]^{\mathrm{T}}$, and the normalized gain matrix of SUs is **F**, whose elements are

$$F_{sm} = \begin{cases} 0 & \text{if} \quad s = m, \\ \hat{\gamma}_s h_m / h_s & \text{if} \quad s \neq m. \end{cases}$$

In what follows, Proposition 2.3 states the feasibility conditions of (2.4) and (2.9), where $\rho(\mathbf{F})$ is the spectral radius of **F**:

Proposition 2.3. *Problem (2.4) is feasible [24] if and only if*

(1) $\rho(\mathbf{F}) < 1$,
(2) $\mathbf{b} \leq \mathbf{p}^{\max}$,
(3) $\mathbf{b}^{\mathrm{T}}\mathbf{G} \leq \mathbf{IT}$,

where $\mathbf{b} = (\mathbf{I} - \mathbf{F})^{-1}\upsilon$ *[8].*

The robust problem (2.9) is feasible [7, 22] if and only if

(4) $\rho(\mathbf{F}) < 1$,
(5) $\mathbf{b}_{\Delta} \leq \mathbf{p}^{\max}$,
(6) $\mathbf{b}_{\Delta}^{T}\overline{\mathbf{G}} + \boldsymbol{\varepsilon}\|\mathbf{b}_{\Delta}\|_2 \leq \mathbf{IT}$,

where $\mathbf{b}_{\Delta} = (\mathbf{I} - \mathbf{F})^{-1}(\upsilon + \boldsymbol{\Delta})$, $\boldsymbol{\Delta} = [\frac{\hat{\gamma}_1\varepsilon_{f_{\mathscr{Q}}}}{h_1}, \ldots, \frac{\hat{\gamma}_S\varepsilon_{f_{\mathscr{Q}}}}{h_S}]^T$, $\overline{\mathbf{G}}$ *is the exact channel gain matrix between SUs and PBSs, whose elements are* $[\overline{G}]_{sq} = \overline{g}_{sq}$ *and* $\boldsymbol{\varepsilon} = [\varepsilon_{\mathbf{g}_1}, \ldots, \varepsilon_{\mathbf{g}_Q}]$.

Proof. *See Appendix 3.*

Conditions (1)–(3) and (4)–(6) in Proposition 2.3 correspond to the feasibility regions of (2.4) and (2.9), respectively. Comparing conditions (2) and (4) reveals that the feasibility region of (2.9) depends on its protection function, and from condition (5) we see that increasing the value of $\varepsilon_{f_{\mathscr{Q}}}$ tightens condition (5) as compared to (2). Also, comparing conditions (3) and (6) reveals that increasing the values of $\varepsilon_{\mathbf{g}_q}$ and $\varepsilon_{f_{\mathscr{Q}}}$ shrinks the feasibility region of Eq. (2.9) as compared to that of Eq. (2.4). Consequently, the SUs' total throughput for the robust solution is less than that for the nominal solution [8].

To illustrate the relationship between the uncertainty and feasibility regions, let us consider a simple example in which a CRN consists of one PBS and one SU, and $\bar{f}_{\mathscr{P}} = 2\sigma^2$. The impact of $\varepsilon_{\mathbf{g}}$ on the feasibility set of (2.9) is shown in Fig. 2.2. In this setup, since we only have one SU and one PBS, for simplicity the indices are omitted from $\hat{\gamma}_s$, p_s, IT_q, $\overline{g}_{sq}$, and other variables, as well as in Fig. 2.2. Now, without considering C_2 and when $\varepsilon_{f_{\mathscr{Q}}} = 0$, the SU reaches its $\hat{\gamma}$ for $0.5 \times p^{\max}$. As can be seen in Fig. 2.2, by increasing $IT/\overline{g}$ and $\varepsilon_{\mathbf{g}}$, constraint C_2 shrinks the feasibility set of (2.9) until the transmit power that satisfies C_2 and $\boldsymbol{\varepsilon}$ falls bellow $0.5 \times p^{\max}$, meaning that for those values of $IT/\overline{g}$ and $\varepsilon_{\mathbf{g}}$ that correspond to the transmit power levels below $0.5 \times p^{\max}$ there is no transmit power for the SU to satisfy both C_1 and C_2 in (2.9), that is, the robust power allocation problem is infeasible when the SU's transmit power is less than $0.5 \times p^{\max}$.

Next, we study the feasibility region for the SU's transmit power when two parameters are uncertain. Let us start by assuming $\varepsilon_{f_{\mathscr{Q}}} > 0$ when C_2 is not considered. The required transmit power for reaching $\hat{\gamma}$ is

$$p = 0.5 \times \left(1 + \frac{1}{3}\varepsilon_{f_{\mathscr{Q}}}\right)p^{\max}. \tag{2.19}$$

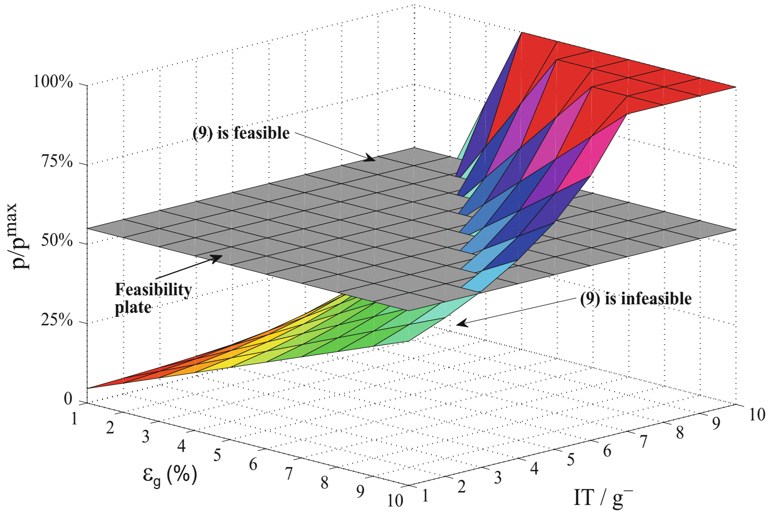

Fig. 2.2 Feasibility set of (2.9) versus $\varepsilon_{\mathbf{g}}$ and $IT/\overline{g}$

When both C_1 and C_2 are considered, $\varepsilon_{f_{\mathcal{Q}}} = 90\%$, $IT/\overline{g} = 9$, and $\varepsilon_{\mathbf{g}} = 4\%$, from (2.19), the SU should transmit at $0.65 \times p^{\max}$ to reach $\hat{\gamma}$. However, from Fig. 2.2, when $IT/\overline{g} = 9$ and $\varepsilon_{\mathbf{g}} = 4\%$, the SU's transmit power should be less than $0.6 \times p^{\max}$ to satisfy the IT. Due to this contradiction, the problem is infeasible. From the preceding considerations, when both $\varepsilon_{f_{\mathcal{Q}}}$ and $\varepsilon_{\mathbf{g}}$ have nonzero values, they affect the SU's transmit power in opposite directions, which may lead to the infeasibility of the robust problem, whereas the nominal problem is feasible.

Note also that increasing $\varepsilon_{\mathbf{g}}$ for any given value of $IT/\overline{g}$ causes (2.9) to be infeasible, and the value of $\varepsilon_{\mathbf{g}}$ that makes the robust problem infeasible depends on the value of $IT/\overline{g}$. In a tight IT, for example, when $IT/\overline{g} < 2$, the value of $\varepsilon_{\mathbf{g}}$ is smaller compared to that for $IT/\overline{g} \geq 2$.

Throughput reduction in the worst-case approach is reported in [8] for a CRN with two partially overlapping cells, one for the PUs and one for the SUs, as shown in Fig. 2.3a, b, where the index q is dropped for simplicity and $\varepsilon_{\mathbf{g}}$ and ε_f are expressed in percentages as $\varepsilon_{\mathbf{g}} = \frac{\|\mathbf{g}-\overline{\mathbf{g}}\|_2}{\|\overline{\mathbf{g}}\|_2}$ and $\varepsilon_{f_{\mathcal{Q}}} = \varepsilon_f = \frac{\|f_{\mathcal{Q}}-\overline{f}_{\mathcal{Q}}\|_2}{\|\overline{f}_{\mathcal{Q}}\|_2}$. The simulation scenario for these two figures are as follows. The PBS is located at the center of a circular cell with a radius of 2 km, and the SBS is located 0.5 km from the PBS and has a 1 km radius. There are three active SUs at $d = [150, 200, 350]$ m from the SBS and $D = [550, 300, 400]$ m from the PBS. The transmit power for each SU is within $[0.001, 1]$ W, $\hat{\gamma}_s = 40$ dBm, $w_s = 1$ for all SUs, $\sigma^2 = -110$ dBm, and $\overline{f}_{\mathcal{Q}} = 2 \times \sigma^2$. The channel gains are $h_s = \frac{k}{d_s^4}$ and $g_{sq} = \frac{k}{D_{sq}^4}$ with $k = 0.09$. Note that expanding the uncertainty region (i.e., increasing $\varepsilon_{\mathbf{g}}$ and ε_f) reduces the robust throughput.

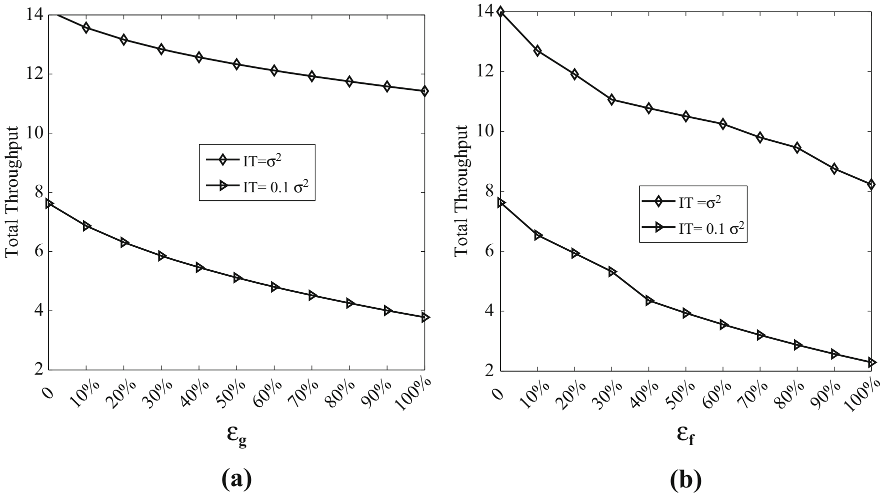

Fig. 2.3 Total throughput of SUs versus (**a**) $\varepsilon_{\mathbf{g}}$ and (**b**) $\varepsilon_{f_{\mathcal{Q}}} = \varepsilon_f$

As stated in Proposition 2.3 and shown in Fig. 2.2, satisfying the PUs' IT while considering uncertainties in $\mathbf{g}_q$ shrinks the feasibility set of (2.8) compared to that of (2.4) and forces the SUs to reduce their transmit power levels, which in turn reduces the SUs' social utility when Proposition 2.3 holds. This is not desirable from the SUs' point of view and calls for a trade-off between increasing the SUs' total throughput and preserving the IT for all instances of channel uncertainties. In practice, the uncertainty does not always correspond to its worst case, and moderation of the worst-case approach is important and desirable to increase the SUs' throughput [17, 25]. This can be achieved via a probabilistic approach in which the uncertainty set is chosen in such a way that the probability of violating the IT is kept below a predefined level, and the SUs' total throughput is kept close to the optimal value of the nonrobust case.

Introducing robustness reduces the SUs' total throughput, measured by

$$d_{\Delta} = \|v^* - v^*_{\Delta}\|_2, \tag{2.20}$$

where v^* and v^*_{Δ} are the SUs' optimal total throughput values for (2.4) and (2.8), respectively. In the following lemma, we will show how d_{Δ} can be obtained.

Lemma 2.1. *Let* $\boldsymbol{\nu}^* = \{\nu_1^*, \ldots, \nu_Q^*\}$ *and* $\boldsymbol{\mu}^* = \{\mu_1^*, \ldots, \mu_S^*\}$ *be the optimal values of Lagrange multipliers for* C_2 *and* C_3 *in (2.4), respectively. For all values of* Δ_q *and* $\Delta_{f_{\mathscr{Q}}}$ *in Proposition 2.2 we have [7, 26]*

$$d_\Delta \approx \sum_{q \in \mathscr{Q}} \nu_q^* \Delta_q + \sum_{s \in \mathscr{S}} \mu_s^* \Delta_{f_{\mathscr{Q}}}. \tag{2.21}$$

Proof. See Appendix 4.

By adjusting Δ_q, one can control d_Δ. To do so, we apply the D-norm approach [25], where the uncertainty in each channel gain is $|\hat{g}_{sq}| \leq \varepsilon_{sq}$ with a symmetric (e.g., uniform) distribution, that is,

$$g_{sq} \in [\bar{g}_{sq} - \hat{g}_{sq}, \bar{g}_{sq} + \hat{g}_{sq}], \quad \forall\, s \in \mathscr{S}, \quad \forall\, q \in \mathscr{Q}. \tag{2.22}$$

For a nonnegative integer $\Gamma_q \leq |\mathscr{S}|$, the protection value [25] is

$$\Delta_q = \max_{e_q \in \mathscr{S},\, |e_q| = \Gamma_q,\, |\hat{g}_{sq}| \leq \varepsilon_{sq}} \sum_{s \in e_q} \hat{g}_{sq} p_s, \qquad \forall\, q \in \mathscr{Q}, \tag{2.23}$$

where e_q is the subset of all SUs that affect the protection value in the D-norm approach. The size of e_q is Γ_q, and from (2.23) the value of Γ_q determines the total number of uncertain parameters in the protection value. Note that the values of g_{sq} for all s and q are uncertain, but only those values that pertain to the set e_q are considered in the protection function of the D-norm approach. By adjusting Γ_q, a trade-off is made between robustness and throughput. When $\Gamma_q = 0$, no protection is considered, and increasing Γ_q means more protection, that is, increasing the number of uncertain parameters results in a higher protection value. Now, the robust counterpart of (2.4) is

$$\max_{p_s^{\min} \leq p_s \leq p_s^{\max}} \sum_{s \in \mathscr{S}} \upsilon_s(h_s p_s \varrho_s^{-1}), \tag{2.24}$$

$$\text{subject to} \begin{cases} C_1 : \varrho_s \hat{\gamma}_s \leq h_s p_s, \\ C_2 : \left(\sum_{s \in \mathscr{S}} \bar{g}_{sq} p_s + \Delta_q\right) \leq IT_q, & \forall\, q \in \mathscr{Q}, \\ C_3 : \left(\bar{f}_{\mathscr{Q}} + \sum_{m \in \mathscr{S},\, m \neq s} p_m h_m + \Delta_{f_{\mathscr{Q}}} + \sigma^2\right) \leq \varrho_s, & \forall\, s \in \mathscr{S}, \end{cases}$$

where $\Delta_q = \max_{e_q \in \mathscr{S},\, |e_q| = \Gamma_q|,\, \hat{g}_{sq}| \leq \varepsilon_{sq}} \sum_{s \in e_q} \hat{g}_{sq} p_s$. Since the protection value in the D-norm approach is a special form of the norm function [20], the convexity of (2.24) is preserved. Now the question is how to expand the feasibility region, which would increase the SUs' total throughput in the D-norm approach. To answer this question, we state the feasibility conditions of (2.24) in the following proposition.

Proposition 2.4. *The robust problem (2.24) is feasible for* $\varepsilon_q = \varepsilon_{sq}, \forall\, s \in \mathscr{S}$ *when*

(1) $\rho(\mathbf{F}) < 1$,
(2) $\mathbf{b}_\Delta \leq \mathbf{p}^{\mathtt{max}}$,
(3) $\mathbf{b}_\Delta^T \overline{\mathbf{G}} + \boldsymbol{\varepsilon} \times \max(\|\mathbf{b}_\Delta\|_\infty, \|\mathbf{b}_\Delta\|_1/\Gamma_q) \leq \mathbf{IT}$,

where $\mathbf{b}_\Delta = (\mathbf{I} - \mathbf{F})^{-1}(\upsilon + \boldsymbol{\Delta})$, $\boldsymbol{\Delta} = [\frac{\hat{\gamma}_1 \delta}{\overline{h}_1}, \ldots, \frac{\hat{\gamma}_S \delta}{\overline{h}_S}]^T$, $\boldsymbol{\varepsilon} = [\varepsilon_1, \ldots, \varepsilon_Q]$, *and* $\|\mathbf{b}_\Delta\|_1 = \sum_{s=1}^{S} \|(b_\Delta)_s\|$, *in which* $(b_\Delta)_s$ *is the sth element of* $\mathbf{b}_\Delta$.

Proof. See Appendix 5.

From Proposition 2.4 we note that Γ_q adjusts the constraints that affect the feasibility region. In what follows, we focus on the conditions that expand the feasibility set in the D-norm approach, leading to a higher total throughput for the SUs.

Lemma 2.2. *When the uncertainty boundaries in the ellipsoid uncertainty model and the D-norm uncertainty model are the same, the optimal total throughput of* (2.7) *for the D-norm uncertainty model is higher than or equal to the optimal total throughput for the ellipsoid uncertainty model if*

$$\Gamma_q \leq \sqrt{|\mathscr{S}|}, \quad \forall q \in \mathscr{Q}, \tag{2.25}$$

where $|\mathscr{S}|$ *is the size of* $\mathscr{S}$.

Proof. See Appendix 6.

Reducing the protection value means that the PUs' interference constraint may not be satisfied for all instances of uncertainties. From C_2 in (2.24), the probability of violating the IT for a given protection value in the D-norm approach can be derived via (1.18) in Chapter 1.

By fixing the probability of violating C_2 to δ_q, the lower bound of Γ_q can be obtained from (1.18) in Chapter 1, as shown in Fig. 1.3 of Chapter 1. Note that reducing δ_q increases the value of Γ_q, and vice versa. From (1.18) in Chapter 1, when the number of SUs increases, the SBS needs a higher value of Γ_q to keep the violation probability acceptable. In addition, unlike (2.7), which satisfies the PBSs' interference constraint for all instances of uncertainties, by using the D-norm in (2.24), the interference constraint is probabilistically satisfied. The SBS can calculate the value of Γ_q via (1.18) in Chapter 1 for any value of the violation probability for PBSs and use it to obtain the protection value. In practice, the value of δ_q can be set by the regulatory authority to trade off between the throughput of SUs and the violation probability of PUs.

2.2.1.3 Distributed Robust Solutions and Extra Message Passing

Using the Lagrange function (2.10) for solving (2.9) has three features that are very useful for developing distributed algorithms:

- Each user's utility function only depends on that user's primal variables;
- Dual variables can be divided into local dual variables for each SU (i.e., μ_s and λ_s) and global dual variables of all SUs (i.e., ν_q);
- Protection functions are simple and linear, and each user can calculate its own protection value for the corresponding constraint by using a broadcasted scaler.

Based on the preceding points, we now present a distributed algorithm for solving (2.9) using the gradient-based updates of primal and dual variables (2.11)–(2.17). We make the following assumptions:

- Each SU knows its normalized SINR at the SBS and receives its $\varepsilon_{\mathbf{g}_q}$ and $\varepsilon_{f_{\mathscr{Q}}}$;
- Each SU needs the two global scalar variables $b_1(t) = \sum_{s\in\mathscr{S}} \mu_s h_s e^{-z_s(t)}$ and $b_2(t) = \sqrt{\sum_{s\in\mathscr{S}} e^{2y_s(t)}}$ to update its primal and dual local variables.

The global dual variable ν_q is updated at the SBS because we need to know all values of the SUs' transmit power levels, channel gains g_{sq}, and IT_q. The distributed Algorithm 1 for power control based on the Lagrange dual decomposition is as follows:

Distributed Algorithm 1
Initialization: At $t = 0$, initialize all primal values with small positive random values and Lagrange multipliers with positive random values.
At SBS:
For $t = 1, 2, \ldots$, the values of $b_1(t)$, $b_2(t)$, and $\nu_q(t)$ are updated using the SUs' parameter values and broadcasted to all SUs.
At each SU s:
For $t = 1, 2, \ldots$:

1. Each SU s receives $b_1(t)$, $b_2(t)$ and $\nu_q(t)$ from the SBS.
2. Each SU s locally calculates its $z_s(t)$ and $y_s(t)$ using (2.11)–(2.12) and updates its Lagrange multipliers using (2.13)–(2.14).
3. Each SU s transmits $\mu_s(t)h_s e^{-z_s(t)}$ and $y_s(t)$ to the SBS.

Since (2.9) is a convex problem, the distributed Algorithm 1 converges to the optimal solution when the conditions in Proposition 2.3 hold and the step size is sufficiently small (Theorem 3.5 in [16]) [10, 16, 23, 27]. The distribution of calculations among SUs is a key advantage of distributed algorithms compared to centralized schemes. However, the distributed Algorithm 1 requires message passing in each iteration between the SUs and the SBS, as well as one additional variable passing for (2.9), that is, b_2, compared to (2.4). If message passing can be done at a slower rate without affecting the convergence and optimality of the allocated transmit power, the algorithm's signaling overhead is reduced.

2.2.1.4 Infrequent Message Passing

To reduce signaling in the robust distributed algorithm, we now present an algorithm with infrequent message passing. Let D denote the time difference between two successive message passings for $b_1(t)$, $b_2(t)$, and $\nu_q(t)$. The algorithm with infrequent message passing is as follows:

Distributed Algorithm 2
Initialization: At $t = 0$, initialize the primal values with small positive random values and the Lagrange multipliers with positive random positive values.
At SBS:

1. For $\xi = 0, 1, \ldots,$ at $t = \xi D + \tau$ for $0 \leq \tau < D$, the value of $\nu_q(t)$ in (2.15) is updated using $y_s(\xi D)$ and $\mu_s(\xi D) h_s e^{-z_s(\xi D)}$ received from all users.
2. At $t = D, 2D, \ldots$:
   - The values of $b_1(t)$ and $b_2(t)$ are updated using $\mu_s(\xi D) e^{-z_s(nD)}$ received from all users;
   - The values of $y_s(\xi D)$ and $\nu_q(t)$ for all s and q are broadcasted to SUs.

At each SU:

1. For $\xi = 0, 1, \ldots$ at $t = \xi D + \tau$ for $0 \leq \tau < D$, each SU s locally calculates its $z_s(t)$, $y_s(t)$, and Lagrange multipliers using $b_1(\xi D)$, $b_2(\xi D)$, and $\nu_q(\xi D)$;
2. At $t = D, 2D, \ldots$, each SU s transmits $\mu_s(t) h_s e^{-z_s(t)}$ and $y_s(t)$ to its SBS.

Obviously, a higher value for D leads to less message passing but may cause nonconvergence of the distributed algorithm. Hence, we need to find the maximum value of D for which the distributed Algorithm 2 converges to the optimal solution of (2.9).

Lemma 2.3. *Let Λ denote the total number of dual and primal variables of the Lagrange dual function (2.10), and let $\boldsymbol{\Theta}$ denote the vector of all step sizes of the gradient algorithm in (2.11)–(2.15). The distributed Algorithm 2 converges to the optimal value of (2.9) when*

$$D \leq \left(\frac{1}{\|\boldsymbol{\Theta}\|_\infty} - 1 \right) \times \frac{1}{\Lambda - 1}, \tag{2.26}$$

where $\|\boldsymbol{\Theta}\|_\infty$ is the maximum absolute value of the elements of $\boldsymbol{\Theta}$.

Proof. See Appendix 7.

The distributed Algorithms 1 and 2 can be slightly modified to solve (2.24) via the D-norm approach in the following manner. In the modified Algorithm 1, instead of b_2, the vector $\mathbf{b}_2 = [\widetilde{b}_1, \ldots, \widetilde{b}_Q]$, in which $\widetilde{b}_q$ is the protection value Δ_q in (2.23), is calculated by the SBS and broadcasted to all SUs. Since (2.24) is convex, the modified Algorithm 1 converges to the optimal solution when the conditions in Proposition 2.4 hold and the step size is sufficiently small. Algorithm 2 can also be slightly modified to reduce the signaling by way of infrequent message passing in the D-norm approach. The modified Algorithm 2 also converges to the optimal solution when the conditions in Proposition 2.4 hold and the upper bound on D in Lemma 2.3 is satisfied.

Convergence of ν and the transmit power in the distributed Algorithms 1 and 2 are shown in Figs. 2.4(a) and 2.4(b), respectively, where both $\varepsilon_{f_{\mathscr{Q}}}$ and $\varepsilon_{\mathbf{g}}$ are 10 % and $IT = \sigma^2$. Other simulation parameters are the same as those for Figs. 2.3(a) and 2.3(b). Figure 2.4(a) shows that Algorithm 1 converges relatively fast, whereas Fig. 2.4(b) shows that Algorithm 2 cannot converge to its optimal value with a large step size, for example, 0.5, and a large delay, for example, $D = 20$. By reducing the step size to 0.01 and delay to $D = 5$, Algorithm 2 converges but may take an order of magnitude more iterations as compared to Algorithm 1, which is in line with Lemma 2.3.

2.3 Multi-channel Cognitive Radio Networks

Now we consider a multi-channel CRN with D2D communications between the secondary transmitters and receivers in the presence of PBSs, as shown in Fig. 2.5. Each D2D secondary transmitter and its receiver are considered as one SU pair. The set of all D2D SU pairs is $\mathscr{S} = \{1, 2, \ldots, S\}$, the set of PBSs is $\mathscr{Q} = \{1, 2, \ldots, Q\}$, and the set of uplink channels shared between SU pairs and PBSs is $\mathscr{K} = \{1, 2, \ldots, K\}$. For simplicity, we limit our consideration to D2D communications only and neglect cellular communications between SUs and the SBS.

We assume that the SBS solves the network utility maximization (NUM) problem for SU pairs subject to the IT of PBSs and the power constraint of SUs. The NUM's objective function is the sum of utility values of all SU pairs. The utility of SU s is its total throughput over all K channels, defined as

$$v_s(\gamma_s^k) = \sum_{k=1}^{K} \log(1 + \gamma_s^k), \quad \forall\, s \in \mathscr{S}, \tag{2.27}$$

where γ_s^k is the SINR of SU s defined as

$$\gamma_s^k = \frac{p_s^k h_{ss}^k}{\sigma^2 + f_{\mathscr{Q}}^k + \sum_{m=1, m\neq s}^{S} p_m^k h_{ms}^k}, \quad \forall\, s \in \mathscr{S}, \tag{2.28}$$

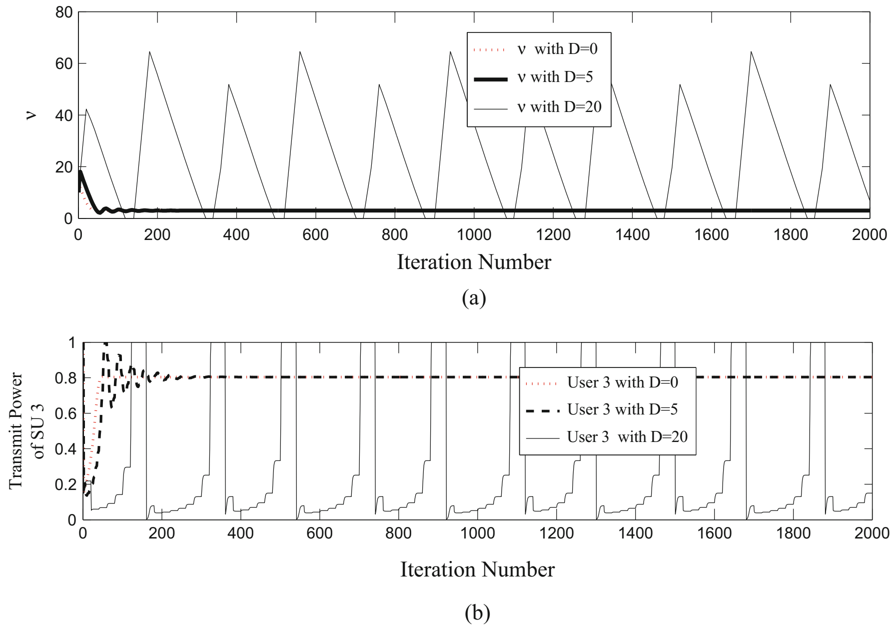

Fig. 2.4 Convergence of Algorithms 1 and 2 for the transmit power of SU 3 (**b**) and ν (**a**). In Algorithm 1, $D = 0$ and all step sizes are 0.5. In Algorithm 2, when $D = 5$, all step sizes are 0.1; and when $D = 20$, all step sizes are 0.5

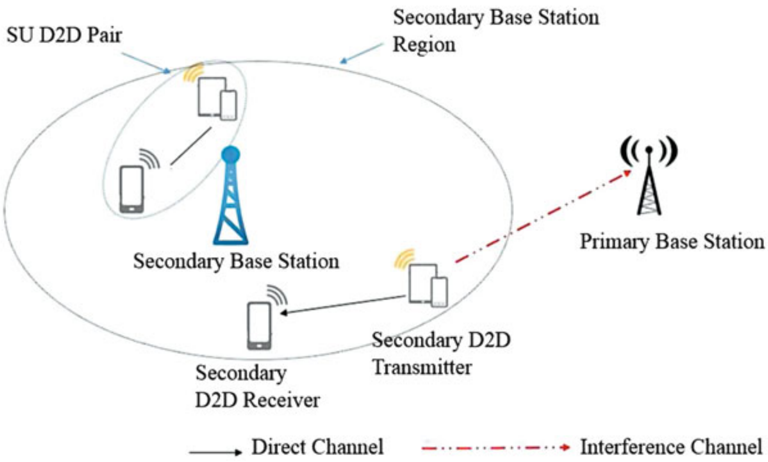

Fig. 2.5 A CRN with D2D SU pairs

where p_s^k is the transmit power of SU s in channel k, h_{ms}^k is the channel gain between transmitter SU m to receiver SU s in channel k, and $\sigma^2 + f_{\mathscr{Q}}^k$ is additive white Gaussian noise (AWGN) power plus interference caused by PUs to SU s in channel k. Now, the NUM is

$$\max_{[\mathbf{p}_1,\ldots,\mathbf{p}_S]} \sum_{s=1}^{S} v_s(\gamma_s^k), \tag{2.29}$$

$$\text{subject to} \begin{cases} C_1 : \sum_{k=1}^{K} p_s^k \leq p_s^{\max}, & \forall\ s \in \mathscr{S}, \\ C_2 : \mathbf{p}_s \ldots \mathbf{g}_{sq}^{\mathrm{T}} \leq IT_{sq}, & \forall\ s \in \mathscr{S}, \quad \forall\ q \in \mathscr{Q}, \end{cases}$$

where $\mathbf{p}_s = [p_s^1, \ldots p_s^K]$ and $p_s^{\max}$ are the transmit power vector of SU s over all channels and the maximum transmit power of SU s, respectively. Also, $\mathbf{g}_{sq} = [g_{sq}^1, \ldots, g_{sq}^K]$ is the channel gain vector between SU s and PU q over all channels, and IT_{sq} is the IT of SU s on PU q. Even without considering C_2, (2.29) is nonconvex and NP-hard [28]. In order to solve this class of problems, one may use their Lagrange dual function [29]. The duality gap between the optimal solution and its upper bound obtained by the Lagrange dual function approaches zero as the number of channels tends to infinity [29]. Hence, the solution obtained by the Lagrange dual function can be considered as close to the globally optimal solution for large values of K.

When the constraint on the IT of PBSs is relaxed, the Lagrange dual function of (2.29) is

$$D(\mathbf{p}, \boldsymbol{\omega}) = \max_{[\mathbf{p}_1,\ldots,\mathbf{p}_S]} \min_{\omega_{sq}} \left\{ \sum_{s=1}^{S} \sum_{k=1}^{K} \log(1+\gamma_s^k) - \sum_{s=1}^{S} \sum_{q=1}^{Q} \omega_{sq} \mathbf{p}_s \cdot \mathbf{g}_{sq}^{\mathrm{T}} + \sum_{s=1}^{S} \sum_{q=1}^{Q} \omega_{sq} IT_{sq} \right\}, \tag{2.30}$$

$$\text{subject to} \quad C_1 : \sum_{k=1}^{K} p_s^k \leq p_s^{\max}, \quad \forall s \in \mathscr{S},$$

where $\omega_{sq} \geq 0$ is the Lagrange multiplier for constraint C_2 for SU s and PBS q. To solve the preceding problem, one can use the following iterative formula:

$$\omega_{sq}^{r+1} = \left[\omega_{sq}^r + \kappa(\mathbf{p}_s \cdot \mathbf{g}_{sq}^{\mathrm{T}} - IT_{sq})\right]^+, \quad \forall\ s \in \mathscr{S}, \quad \forall\ k \in \mathscr{K}, \tag{2.31}$$

where κ is a small step size in the iterative formula, and r is the iteration number. By solving the following subproblem, the optimal $\mathbf{p}_s$ can be iteratively obtained [29]:

$$\max_{[\mathbf{p}_1,\ldots,\mathbf{p}_S]} \left(\sum_{s=1}^{S} \sum_{k=1}^{K} \log(1+\gamma_s^k) - \sum_{s=1}^{S} \sum_{q=1}^{Q} \omega_{sq}^r \mathbf{p}_s \cdot \mathbf{g}_{sq}^{\mathrm{T}} \right), \tag{2.32}$$

$$\text{subject to} \quad C_1 : \sum_{k=1}^{K} p_s^k \leq p_s^{\max}, \quad \forall\ s \in \mathscr{S}.$$

2.3.1 Robust Problems

To introduce robust interference control in (2.29), we treat $\mathbf{g}_{sq}$ as an uncertain parameter that can be modeled by $\mathbf{g}_{sq} = \overline{\mathbf{g}}_{sq} + \hat{\mathbf{g}}_{sq}$, where $\overline{\mathbf{g}}_{sq}$ and $\hat{\mathbf{g}}_{sq}$ are the exact value and the error in the parameter value, respectively. For each pair of SU s and PBS q, the uncertain gain is $\mathbf{g}_{sq} = (\overline{\mathbf{g}}_{sq} + \hat{\mathbf{g}}_{sq}) \in \mathscr{R}_{\mathbf{g}_{sq}}$, where $\mathscr{R}_{\mathbf{g}_{sq}}$ is the uncertainty region for $\mathbf{g}_{sq}$. Now, the robust counterpart of (2.29) is

$$\max_{[\mathbf{p}_1,\ldots,\mathbf{p}_S]} \sum_{s=1}^{S} v_s(\gamma_s^k), \tag{2.33}$$

$$\text{subject to} \begin{cases} C_1 : \sum_{k=1}^{K} p_s^k \le p_s^{\max}, & \forall\, s \in \mathscr{S}, \\ C_2 : \mathbf{p}_s \cdot \mathbf{g}_{sq}^{\mathrm{T}} \le IT_{sq}, & \forall\, s \in \mathscr{S}, \quad \forall\, q \in \mathscr{Q}, \\ C_3 : \mathbf{g}_{sq} \in \mathscr{R}_{\mathbf{g}_{sq}}, & \forall\, s \in \mathscr{S}, \quad \forall\, q \in \mathscr{Q}. \end{cases}$$

In what follows, we will show how to solve (2.33) using the polyhedron model and the general norm introduced in Chapter 1, Section 1.3.2.1. For the polyhedron model, the uncertainty region for the pair SU s and PU q is

$$\mathscr{R}_{\mathbf{g}_{sq}} = \{\mathbf{g}_{sq} | \mathbf{M}_{sq} \cdot \mathbf{g}_{sq}^{\mathrm{T}} \le \boldsymbol{\varepsilon}_{\mathbf{g}_{sq}}\},$$

where $\mathbf{M}_{sq}$ is the $R^{K\times K}$ weight matrix for $\mathbf{g}_{sq}$, and $\boldsymbol{\varepsilon}_{\mathbf{g}_{sq}}$ is the $R^{K\times 1}$ vector. Next, we transform C_2 in (2.33) into a linear constraint [18]. When C_2 holds for a given fixed value of $[\mathbf{p}_1, \ldots, \mathbf{p}_S] = [\tilde{\mathbf{p}}_1, \ldots, \tilde{\mathbf{p}}_S]$, the value of IT_{sq} must be higher than the optimal objective value of the following optimization problem for each SU [9, 18]:

$$\max_{\mathbf{g}_{sq}} \tilde{\mathbf{p}}_s \cdot \mathbf{g}_{sq}^{\mathrm{T}}, \tag{2.34}$$

$$\text{subject to} \quad \mathbf{M}_{sq} \cdot \mathbf{g}_{sq}^{\mathrm{T}} \le \boldsymbol{\varepsilon}_{\mathbf{g}_{sq}} \quad \forall\, s \in \mathscr{S}, \quad \forall\, q \in \mathscr{Q}.$$

Since (2.34) is feasible, its optimal objective value, denoted by Δ_{sq}^*, can be obtained via its Lagrange dual function [18, 25],

$$\min_{\mathbf{y}_{sq} \ge \mathbf{0}} \quad \boldsymbol{\varepsilon}_{\mathbf{g}_{sq}} \cdot \mathbf{y}_{sq}^{\mathrm{T}},$$

$$\text{subject to} \quad \tilde{\mathbf{p}}_s^{\mathrm{T}} \le \mathbf{M}_{sq} \cdot \mathbf{y}_{sq}^{\mathrm{T}}, \quad \forall\, s \in \mathscr{S}, \quad \forall\, q \in \mathscr{Q}, \tag{2.35}$$

where $\mathbf{y}_{sq} = [y_{sq}^1, \ldots, y_{sq}^K]$ is the qth Lagrange multiplier for (2.34), and $\mathbf{0}$ is the zero vector with the same size as $\mathbf{y}_{sq}$. A feasible solution to (2.35), denoted by $\mathbf{y}_{sq}^*$, satisfies $\boldsymbol{\varepsilon}_{\mathbf{g}_{sq}} \cdot \mathbf{y}_{sq}^{*\mathrm{T}} \le IT_{qs}$ and also satisfies $\Delta_{sq}^* \le \mathbf{c}_{sq} \cdot \mathbf{y}_{sq}^{*\mathrm{T}} \le IT_{sq}$ for all uncertainty in $\mathbf{g}_{sq}$. In this case, we can replace C_2 in (2.33) with the following constraints:

$$\boldsymbol{\varepsilon}_{\mathbf{g}_{sq}} \cdot \mathbf{y}_{sq}^{\mathrm{T}} \le IT_{sq}, \qquad \mathbf{p}_s^{\mathrm{T}} \le \mathbf{M}_{sq} \cdot \mathbf{y}_{sq}^{\mathrm{T}}, \qquad \mathbf{y}_{sq} \ge \mathbf{0}, \quad \forall\, q \in \mathscr{Q}, \quad \forall\, s \in \mathscr{S}. \tag{2.36}$$

In this way, one new constraint and one new optimization variable are added for each SU and PBS in each channel for solving (2.33), as compared to (2.29). An iterative approach that uses the nominal problem's dual function (2.30) can also be applied to (2.33), except that (2.31) is replaced by

$$\widetilde{\omega}_{sq}^{r+1} = \left[\omega_{sq}^{r} + \kappa_1(\boldsymbol{\varepsilon}_{\mathbf{g}_{sq}} \cdot \mathbf{y}_{sq}^{\mathrm{T}} - IT_q)\right]^{+}, \qquad \forall\, s \in \mathscr{S}, \quad \forall\, k \in \mathscr{K}, \tag{2.37}$$

and

$$\widehat{\omega}_{sq}^{r+1} = \left[\widehat{\omega}_{sq}^{r} + \kappa_2\left(\mathbf{p}_s^{\mathrm{T}} - (\mathbf{M}_{sq} \cdot \mathbf{y}_{sq}^{\mathrm{T}})\right)\right]^{+}, \quad \forall\, s \in \mathscr{S}, \quad \forall\, k \in \mathscr{K}, \tag{2.38}$$

where κ_1 and κ_2 are small step sizes, r is the iteration number, and $\widetilde{\omega}_{sq} \leq 0$ and $\widehat{\omega}_{sq} \leq 0$ are the Lagrange multipliers for the linear constraints in (2.36).

When all primal and dual variables are iteratively updated to solve (2.29) and (2.33), for each new constraint or each new variable added to (2.29) to introduce robustness, there is one additional iterative formula. Additional calculations for each new iterative formula are not significant due to the fact that it is a linear function. Hence, we only consider the number of new iterative formulas to obtain the order of additional calculations. The order of additional calculations for solving (2.33) as compared to (2.29) is $O(2\times S\times K\times Q)$. Hence, the order of additional calculations for solving (2.33) as compared to (2.29) is linearly increased by increasing the number of SUs for the polyhedron model, meaning that the problem is tractable.

For the general norm, consider the uncertainty region as

$$\mathscr{R}_{\mathbf{g}_{sq}} = \{\mathbf{g}_{sq} \parallel \mathbf{M}_{sq} \cdot (\mathbf{g}_{sq} - \bar{\mathbf{g}}_{sq})^{\mathrm{T}} \parallel \leq \boldsymbol{\varepsilon}_{\mathbf{g}_{sq}}\},$$

where $\mathbf{M}_{sq}$ is an invertible $R^{K\times K}$ weight matrix of channel gains between the transmitter of SU s and the receiver of PBS q, and $\boldsymbol{\varepsilon}_{\mathbf{g}_{sq}}$ is the bound on the uncertainty region. Now, using the protection function, (2.33) is equivalent to

$$\max_{[\mathbf{p}_1,\ldots,\mathbf{p}_S]} \sum_{s=1}^{S} v_s(\gamma_s^k), \tag{2.39}$$

$$\text{subject to} \begin{cases} C_1 : \sum_{k=1}^{K} p_s^k \leq p_s^{\max}, & \forall\, s \in \mathscr{S}, \\ C_2 : \left(\mathbf{p}_s \cdot \bar{\mathbf{g}}_{sq}^{\mathrm{T}} + \Delta_{sq}(\mathbf{p}_s)\right) \leq IT_{sq}, & \forall\, q \in \mathscr{Q}, \end{cases}$$

where $\Delta_{sq}(\mathbf{p}_s) = \max_{\mathbf{g}_{sq}\in\mathscr{R}_{\mathbf{g}_{sq}}} \mathbf{p}_s \cdot (\mathbf{g}_{sq} - \bar{\mathbf{g}}_{sq})$ is the protection function for the pair SU s and PU q. Additional calculations in (2.33) or (2.39) compared to (2.29) are due to uncertainty in C_2 and C_3 for each SU. When C_2 and C_3 in (2.33) can be replaced by deterministic functions, or their corresponding protection function in (2.39) can be replaced by a set of linear constraints, the problem becomes tractable [9].

For the linear norm with order a, we have $\Delta_{sq}(\mathbf{p}_s) = \varepsilon_{\mathbf{g}_{sq}} \left(\sum_{k=1}^{K} (\mathbf{m}_{sq}^k \cdot \mathbf{p}_s^{\mathrm{T}})^b \right)^{\frac{1}{b}}$, where $\mathbf{m}_{sq}^k$ is the k^{th} row of $\mathbf{M}_{sq}^{-1}$, and $b = 1 + 1/(1-a)$. In this case, since there is no additional optimization variable or constraint, the optimal solution to (2.39) can be obtained with no significant additional calculations as compared to (2.29).

Feasibility of Nominal Problem and Its Robust Counterpart: Note that (2.29) and (2.33) [or (2.39)] are always feasible even for a large number of SUs in a CRN. Let the set of transmit power levels of SU s in (2.29) and (2.33) be

$$\mathscr{A}_s = \left\{ \mathbf{p}_s | \sum_{k=1}^{K} p_s^k \leq p_s^{\max}, \qquad \mathbf{p}_s \cdot \mathbf{g}_{sq}^{\mathrm{T}} \leq IT_{sq},\ \forall\, q \in \mathscr{Q} \right\}$$

and

$$\widetilde{\mathscr{A}}_s = \left\{ \mathbf{p}_s | \sum_{k=1}^{K} p_s^k \leq p_s^{\max}, \qquad \left(\mathbf{p}_s \cdot \overline{\mathbf{g}}_{sq}^{\mathrm{T}} + \Delta_{sq}(\mathbf{p}_s) \right) \leq IT_{sq} \right\},$$

respectively, where $\Delta_{sq}(\mathbf{p}_s) = \max_{\mathbf{g}_{sq} \in \mathscr{R}_{\mathbf{g}_{sq}}} \mathbf{p}_s \cdot (\mathbf{g}_{sq} - \overline{\mathbf{g}}_{sq})$ for all $q \in \mathscr{Q}$. The sets $\mathscr{A}_s$ and $\widetilde{\mathscr{A}}_s$ are nonempty due to the fact that at least one power vector $\mathbf{p}_s = \mathbf{0}$ exists that satisfies the constraints on $\mathscr{A}_s$ and $\widetilde{\mathscr{A}}_s$. In addition, the constraints on each SU only depend on that SU's transmit power and are decoupled from other SUs' constraints. Consequently, the feasibility set of (2.29) (i.e., $\mathscr{A} = \bigcup_{s=1}^{S} \mathscr{A}_s$) and the feasibility set of (2.33) (i.e., $\widetilde{\mathscr{A}} = \bigcup_{s=1}^{S} \widetilde{\mathscr{A}}_s$) are nonempty, meaning that (2.29) and (2.33) [or its equivalent (2.39)] are feasible [19]. Note also that the transmit power of SU s can be zero (i.e., no transmission), depending on the size of the uncertainty region, the values of IT_{sq}, and the channel gains $\mathbf{g}_{sq}$. Uncertainty in $\mathbf{g}_{sq}$ causes the feasibility set of (2.33) to be smaller than that of (2.29). When quality of service (QoS) is considered in the constraints of the optimization problem, for example, C_1 in (2.2), the feasibility region for the nominal problem (and for its robust counterpart) depends on the channel gains between users, system information, and each user's transmit power limitation [22], as already indicated in Proposition 3. In such a case, the problem may be infeasible, as in Fig. 2.2, where the minimum required SINR for each SU may not be attained. On the other hand, when QoS is not considered in the constraints, the feasibility region can be obtained in a straightforward manner.

Now, the question is whether the robust solution can be obtained in a manner similar to how the nominal solution was obtained. To answer this question, we focus on a single SU scenario and, for notational simplicity, drop the index s in variables. The nominal problem (2.29) is transformed into

$$\max_{\mathbf{p}} \sum_{k=1}^{K} \log\left(1 + \frac{p^k h^k}{\sigma^2 + f_{\mathscr{Q}^k}}\right), \tag{2.40}$$

$$\text{subject to} \begin{cases} C_1 : \sum_{k=1}^{K} p^k \leq p^{\max}, \quad p^k \geq 0, \quad \forall\, k \in \mathscr{K}, \\ C_2 : \mathbf{p} \cdot \mathbf{g}_q^{\mathrm{T}} \leq IT_q, \quad \forall\, q \in \mathscr{Q}. \end{cases}$$

Since (2.40) is convex, its solution obtained via the Lagrange dual function [30] is

$$p^k = \left[\frac{1}{\lambda + \sum_{q=1}^{Q} \nu_q g_q^k} - \frac{\sigma^2 + f_{\mathscr{Q}}^k}{h^k}\right]^+, \tag{2.41}$$

where λ and ν_q are nonnegative Lagrange multipliers associated with C_1 and C_2 in (2.40), and $x^+ = \mathtt{max}\{0, x\}$. The optimal transmit power and Lagrange multipliers should satisfy the following Karush–Kuhn–Tucker (KKT) conditions [19]:

$$\lambda\left(\sum_{k=1}^{K} p^k - p^{\max}\right) = 0, \quad \text{and} \quad \nu_q(\mathbf{p} \cdot \mathbf{g}_q^{\mathrm{T}} - IT_q) = 0, \quad \forall\, q \in \mathscr{Q}. \tag{2.42}$$

We now present a pseudo water-filling algorithm for the robust problem. Consider the uncertainty region for the polyhedron model as

$$\mathscr{R}_{\mathbf{g}_q} = \{\mathbf{g}_q | \mathbf{M}_q \cdot \mathbf{g}_q^{\mathrm{T}} \leq \boldsymbol{\varepsilon}_{\mathbf{g}_q}\}, \tag{2.43}$$

where $\mathbf{M}_q$ is the weight matrix for $\mathbf{g}_q$, and $\boldsymbol{\varepsilon}_{\mathbf{g}_q} = [\varepsilon^1_{\mathbf{g}_q}, \ldots, \varepsilon^K_{\mathbf{g}_q}]$ is an $R^{K\times 1}$ vector that represents the weighted maximum deviation of $\mathbf{g}_q$ from $\bar{\mathbf{g}}_q$. In this case, the robust counterpart of (2.40) is

$$\max_{\mathbf{p}} \sum_{k=1}^{K} \log\left(1 + \frac{p^k h^k}{\sigma^2 + f_{\mathscr{Q}}^k}\right), \tag{2.44}$$

$$\text{subject to} \begin{cases} C_1 : \sum_{k=1}^{K} p^k \leq p^{\max}, \quad p^k \geq 0, \quad \forall\, k \in \mathscr{K}, \\ C_2 : \max_{(\mathbf{g}_q | \mathbf{M}_q \cdot \mathbf{g}_q^{\mathrm{T}} \leq \boldsymbol{\varepsilon}_{\mathbf{g}_q})} \left(\mathbf{p} \cdot (\mathbf{g}_q^{\mathrm{T}} - \bar{\mathbf{g}}_q^{\mathrm{T}})\right) \leq IT_q, \quad \forall\, q \in \mathscr{Q}. \end{cases}$$

From (2.36), C_2 can be replaced by

$$\boldsymbol{\varepsilon}_{\mathbf{g}_q} \cdot \mathbf{y}_q^{\mathrm{T}} \leq IT_q, \qquad \mathbf{p}^{\mathrm{T}} \leq \mathbf{M}_q \cdot \mathbf{y}_q^{\mathrm{T}}, \qquad \mathbf{y}_q \geq \mathbf{0}, \quad \forall\, q \in \mathscr{Q}.$$

Now, when the uncertainty region is (2.43), problem (2.44) is equivalent to

$$\max_{\mathbf{p}} \sum_{k=1}^{K} \log\left(1 + \frac{p^k h^k}{\sigma^2 + f_{\mathcal{Q}}^k}\right), \tag{2.45}$$

$$\text{subject to} \begin{cases} C_1 : \sum_{k=1}^{K} p^k \le p^{\max}, & \\ C_2 : \sum_{k=1}^{K} \varepsilon_{\mathbf{g}_q}^k y_q^k \le IT_q, & \forall\, q \in \mathcal{Q}, \\ C_3 : p^k \le \sum_{i=1}^{K} [M_q^{\mathrm{T}}]^{ki} y_q^i, \quad \mathbf{y}_q \ge \mathbf{0}, & \forall\, q \in \mathcal{Q}, \end{cases}$$

where $[M_q^{\mathrm{T}}]^{ki}$ is the element of $\mathbf{M}_q^{\mathrm{T}}$ in its k^{th} row and i^{th} column. Since (2.45) is a convex optimization problem with linear constraints, it can be solved with fewer calculations than (2.44) with the same uncertainty region (2.43).

Proposition 2.5. *When $\sum_{i=1}^{K} \mu_q^i [M_q^T]^{ki} = \hat{\nu}_q \varepsilon_{\mathbf{g}_q}^k$ for all q and k, the optimal solution to* (2.45) *is*

$$\tilde{p}^{*k} = \left[\frac{1}{\lambda + \sum_{q=1}^{Q} \mu_q^k} - \frac{\sigma^2 + f_{\mathcal{Q}}^k}{h^k}\right]^+, \tag{2.46}$$

where λ, $\hat{\nu}_q$, and μ_q^k are nonnegative Lagrange multipliers for C_1, C_2, and C_3 in (2.45), respectively.

Proof. See Appendix 8.

From Proposition 2.5, the robust problem can be transformed into an optimization problem with linear constraints whose solution is obtained by a pseudo water-filling formula. In an orthogonal frequency division multiple access (OFDMA) system, when the frequency and time interleaver with sufficient interleaving depth is applied, fading can be considered uncorrelated across channels [31]. In this case, we can assume that uncertainty in $\mathbf{g}_q$ can be modeled by independent and identically distributed (i.i.d.) random variables [31]. Consequently, $\mathbf{M}_q$ becomes diagonal, and the power allocation problem (2.45) becomes

$$\max_{\mathbf{p}} \sum_{k=1}^{K} \log\left(1 + \frac{p^k h^k}{\sigma^2 + f_{\mathcal{Q}}^k}\right), \tag{2.47}$$

$$\text{subject to} \begin{cases} C_1 : \sum_{k=1}^{K} p^k \le p^{\max}, & \\ C_2 : \sum_{k=1}^{K} \varepsilon_{\mathbf{g}_q}^k y_q^k \le IT_q, & \forall\, q \in \mathcal{Q}, \\ C_3 : p^k \le [M_q^{\mathrm{T}}]^{kk} y_q^k, \quad \mathbf{y}_q \ge \mathbf{0}, & \forall\, q \in \mathcal{Q}. \end{cases}$$

At the optimal point, from the KKT conditions we have $[M_q^{\mathrm{T}}]^{kk} y_q^k = p^k$, and the allocated transmit power (2.46) is simplified to the following pseudo water-filling formula:

$$\tilde{p}^{*k} = \left[\frac{1}{\lambda + \sum_{q=1}^{Q} \frac{\hat{\nu}_q \varepsilon_{\mathbf{g}_q}^k}{[M_q^{\mathrm{T}}]^{kk}}} - \frac{\sigma^2 + f_{\mathscr{Q}}^k}{h^k} \right]^+, \tag{2.48}$$

where λ and $\hat{\nu}_q$ are the Lagrange multipliers for C_1 and C_2 in (2.47), respectively. The robust transmit power is obtained by (2.48) in a straightforward manner.

The robust counterpart of (2.40) can be written using the protection function. When $\mathscr{R}_{\mathbf{g}_q}$ is a compact set, the robust counterpart of (2.40) is

$$\max_{\mathbf{p}} \sum_{k=1}^{K} \log \left(1 + \frac{p^k h^k}{\sigma^2 + f_{\mathscr{Q}}^k} \right), \tag{2.49}$$

$$\text{subject to} \begin{cases} C_1 : \sum_{k=1}^{K} p^k \leq p^{\max}, & p^k \geq 0, \quad \forall k \in \mathscr{K}, \\ C_2 : \left(\mathbf{p} \cdot \bar{\mathbf{g}}_q^{\mathrm{T}} + \Delta_q(\mathbf{p}) \right) \leq IT_q, & \forall q \in \mathscr{Q}, \end{cases}$$

where $\Delta_q(\mathbf{p}) = \max_{\mathbf{g}_q \in \mathscr{R}_{\mathbf{g}_q}} \mathbf{p} \cdot (\mathbf{g}_q - \bar{\mathbf{g}}_q)^{\mathrm{T}}$ is the protection function in C_2, whose value (the protection value) depends on the uncertain parameters [18, 20]. For the general norm, e.g., for the ellipsoid norm, the protection function can be replaced by bounds on the uncertainty region and transmit power, which are more tractable. In this way, uncertainty is removed from the optimization variables, resulting in less complexity.

In light of the preceding discussion, we now study the throughput reduction in the worst-case robust optimization approach. In doing so, we consider $d_\Delta = v^* - v_\Delta^*$, where v^* and v_Δ^* are the optimal throughput values for (2.40) and (2.49), respectively. Note that the difference between (2.40) and (2.49) is in their protection function in C_2, which we use to derive the relationship between d_Δ and the protection function for sensitivity analysis. Similar to Lemma 2.1, one can show that [9]

$$d_\Delta \approx \sum_{q=1}^{Q} \nu_q^* \Delta_q(\mathbf{p}). \tag{2.50}$$

From (2.50), the value of d_Δ depends on the size of the uncertainty region and on the interference constraint in (2.40). The approximation (2.50) gives us a quantitative measure of the impact of C_2 and $\Delta_q(\mathbf{p})$ on the throughput in (2.49). Let $\mathbf{p}^*$ be the optimal transmit power in (2.40). When $\mathbf{p}^* \cdot \mathbf{g}_q^{\mathrm{T}} < IT_q$ for all q, constraint C_2 is not satisfied and the protection function does not affect the optimal solution to (2.49), and $d_\Delta \approx 0$. On the other hand, when $\mathbf{p}^* \cdot \mathbf{g}_q^{\mathrm{T}} = IT_q$, the value of ν_q^* indicates the impact of the protection function on the optimal solution to (2.49).

To demonstrate the relationship between λ, ν, and d_Δ, we simulate a CRN with one PBS and one SU pair for D2D communication. In this case, for notational simplicity, we drop indices s and q in the formulas. The number of channels used by the SU is 64, $\sigma^2 + f_{\mathcal{Q}}^k$ is -70 dB, and $p^{\max}$ is 10 dBm. The gain in each channel k is $h^k = \frac{\beta}{D^\alpha}$ and $g^k = \frac{\beta}{d^\alpha}$, where $\alpha = 4$, β is a frequency-dependent coefficient, and D and d are distances from the transmitting SU to its receiving SU and from the transmitting SU to the receiving PBS, respectively. To consider both constraints C_1 and C_2, we take $D = d = 500$ m. This is because when $D \ll d$, the value of $\mathbf{g}_q$ is very small, and C_2 in (2.40) can be neglected. Also, when $d \ll D$, the value of $\mathbf{g}_q$ is very high and C_1 in (2.40) can be ignored. In other words, when the values of D and d are very disparate, only one of the two constraints in (2.40) is applicable, and the problem is much simpler. When the values of D and d are close to one another, both C_1 and C_2 in (2.40) should be satisfied, and we can study the impacts of other parameters, for example, $p^{\max}$ and IT, on the allocated robust transmit power to the SU. In simulations, the IT for the PBS's receiver is proportional to its receiving noise level, and the SU's throughput is measured in bits per symbol per Hertz.

We now study the dependence of d_Δ on IT and $p^{\max}$ in (2.40). To do so, and as an example, we assume that the normalized error in channel gain is $\frac{\mathbf{g}-\bar{\mathbf{g}}}{\bar{\mathbf{g}}} = 0.5$, that is, the error in the channel gain is not more than 50 % of its exact value. In Fig. 2.6, the average values of λ^* [i.e., the optimal Lagrange multiplier for C_1 in (2.40)], the average values of $\boldsymbol{\nu}^*$, and the average values of d_Δ are shown. Note the following three distinct regions for different values of $IT/p^{\max}$ in Fig. 2.6:

1. Region I: The value of IT is much higher than $p^{\max}$, that is, $\frac{p^{\max}}{IT} \ll 1$, which means that in (2.40), C_2 can be ignored, but C_1 should be satisfied. In other words, $\lambda^* > 0$ and $\boldsymbol{\nu}^* = 0$.
2. Region II: The interference constraint is moderate, that is, $\frac{p^{\max}}{IT} \approx 1$, which means that in (2.40), both C_1 and C_2 must be considered.

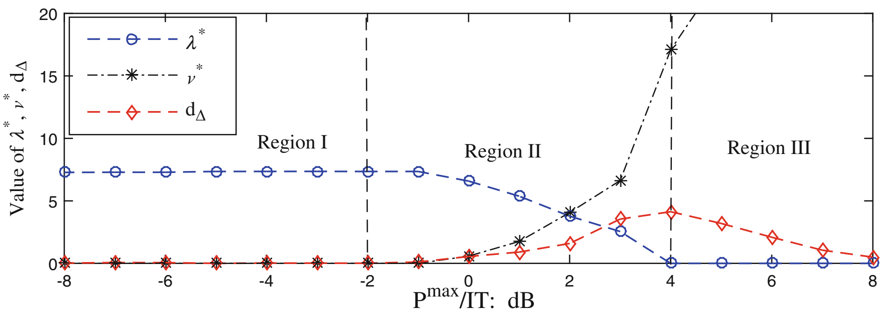

Fig. 2.6 Values of λ^* and ν^* in (2.40) and d_Δ versus $\frac{p^{\max}}{IT}$

3. Region III: The value of IT is much less than $p^{\max}$, that is, $\frac{p^{\max}}{IT} \gg 1$, which means that in (2.40), C_2 should be satisfied but C_1 can be ignored. In other words, $\lambda^* = 0$ and $\boldsymbol{\nu}^* > 0$.

In Region I, where C_2 is not relevant, we have $\nu^* = 0$, meaning that $d_\Delta = 0$ and $\upsilon^* = \upsilon^*_\Delta$. However, in Regions II and III, we have $\nu^* \neq 0$, and consequently $d_\Delta \neq 0$. Therefore, $\upsilon^*_\Delta < \upsilon^*$. Implementing the foregoing considerations in an underlay CRN has the following consequences. In Region I, uncertainty in $\mathbf{g}$ and the corresponding trade-off do not affect the SU's throughput. But in Regions II and III, a trade-off algorithm is desirable to increase the SU's throughput. The impact of robustness on the SU's throughput in Regions II and III depends on the values of $\mathbf{h}$ and $\mathbf{g}$, as well as on the values of the trade-off parameters. Note that for a fixed $p^{\max}$ in Region III, increasing the ratio $\frac{p^{\max}}{IT}$ is equivalent to reducing IT, which reduces the SU's allocated transmit power. For very small values of the allocated transmit power, the difference d_Δ in the achieved throughput in the nominal and robust approaches diminishes.

2.3.1.1 Social Utility of Robust Solutions Versus Uncertainty Levels

Now the question concerns the impact of uncertainty levels on the robust solution's social utility. In Fig. 2.7, the polyhedron approach is simulated for D2D communications in a multiple-SU single-PBS scenario with multiple OFDMA channels

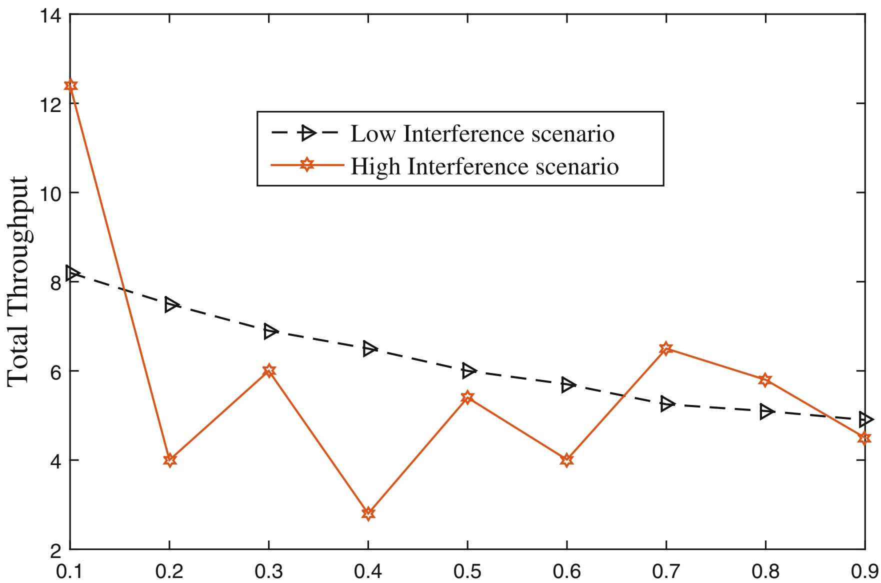

Fig. 2.7 Throughput values for polyhedron model for low- and high-interference cases versus ξ

when interference between SUs is low and when it is high. Simulation parameters are $K = 64$, $Q = 1$, $S = 2$, $D_{ss} \in [450, 550]$ m (distance between the transmitting and receiving SU s), and $d_{sq} \in [450, 500]$ m (distance between the transmitting SU s and receiving PBS q). In this setup, $D_{ss} \approx d_{sq}$, and hence both C_1 and C_2 in (2.29) are applicable. The distance between the transmitting SU s and the receiving SU m, denoted by D_{sm}, is $D_{sm} \in [800, 950]$ m for the low-interference case and $D_{sm} \in [300, 250]$ m for the high-interference case. We assume $\hat{g}^k_{sq} = \xi \bar{g}^k_{sq}$ for all $s \in \mathscr{S}$ and $q \in \mathscr{Q}$ and $k \in \mathscr{K}$.

Note that in the low-interference case, increasing ξ results in monotonically reducing the SUs' total throughput (social utility). This is not the case for the high-interference case, where introducing robustness may reduce each SU's transmit power, and, consequently, the interference of each SU on the other SU may be reduced. Hence, it is possible that the throughput of each SU may increase with increasing ξ due to the reduced interference between SUs. In other words, in a high-interference scenario, it may be possible to increase the total throughput by increasing uncertainty when the SUs reduce their transmit power levels to satisfy the constraints.

2.3.2 *Trade-Off Algorithms*

In what follows, we discuss the trade-off algorithms for the D-norm and chance-constrained approaches. We begin by considering a CRN with one SU and then extend our formulations for multiple SUs.

2.3.2.1 Trade-Off in D-Norm Approach

The protection value for the D-Norm approach is

$$\Delta_q = \max_{\mathscr{L}_q} \sum_{k \in \mathscr{L}_q} \hat{g}^k_q p^k, \tag{2.51}$$

where $\mathscr{L}_q$ is any subset of channels of size Γ_q. The number of channels in $\mathscr{L}_q$ determines the protection value, meaning that a smaller Γ_q results in a smaller d_Δ. For $\Gamma_q = 0$, the protection value vanishes, and $d_\Delta = 0$. In contrast, for $\Gamma_q = K$, all channels are taken into account, and d_Δ is maximized. By adjusting Γ_q, one can control the cost of robustness from the SU's point of view.

For the D-norm approach, as with the polyhedron uncertainty region, a linear protection function can be obtained [9], resulting in fewer calculations. From Proposition 2.1 in [25], for a given $\mathbf{p} = \tilde{\mathbf{p}}$, the protection function (2.51) for the constraint on the interference on PBS q is

$$\max_{0\leq l_q^k\leq 1}\sum_{k=1}^{K}\hat{g}_q^k l_q^k \tilde{p}^k \tag{2.52}$$

$$\text{subject to}\sum_{k=1}^{K} l_q^k \leq \Gamma_q,$$

where l_q^k is an auxiliary variable that indicates whether the corresponding error $\hat{g}_q^k$ is considered in the protection function of the D-norm approach, and $\tilde{p}^k$ is the kth element of $\tilde{\mathbf{p}}$. Note that l_q^k is by nature a binary variable, but to simplify the problem we take it as a continuous variable, which will eventually be converted into a binary variable. Let $\tilde{\Delta}_q^*$ be the optimal protection value of (2.52), which can be obtained from its dual function [25]

$$\min_{z_q,\boldsymbol{\chi}_q}\left(\Gamma_q z_q+\sum_{k=1}^{K}\chi_q\right), \tag{2.53}$$

$$\text{subject to}\begin{cases}\hat{g}_q^k x_q^k \leq (z_q+\chi_q), \\ \tilde{p}^k \leq x_q^k & \forall\ k\in\mathcal{K},\end{cases}$$

where z_q, $\boldsymbol{\chi}_q=[\chi_q^1,\ldots,\chi_q^K]$, and x_q^k are positive variables. Since (2.52) is feasible and bounded for all $0\leq\Gamma_q\leq K$, its dual, that is, (2.53), is also feasible and bounded, and by strong duality, their optimal protection values coincide. Feasible solutions to (2.53), denoted by z_q^* and $\boldsymbol{\chi}_q^*=[\chi_q^{*1},\ldots,\chi_q^{*K}]$, satisfy $(\Gamma_q z_q^*+\sum_{k=1}^{K}\chi_q^{*k})\leq IT_q-\mathbf{p}.\overline{\mathbf{g}}_q^{\mathrm{T}}$ and also satisfy $(\tilde{\Delta}_q^*\leq\Gamma_q z_q^*+\sum_{k=1}^{K}\chi_q^{*k})\leq IT_q-\mathbf{p}.\overline{\mathbf{g}}_q^{\mathrm{T}}$. Therefore, C_2 in (2.49) can be replaced by the following constraints:

$$\left(\mathbf{p}.\overline{\mathbf{g}}_q^{\mathrm{T}}+\Gamma_q z_q+\sum_{k=1}^{K}q_q^k\right)\leq IT_q,\quad \hat{g}_q^k x_q^k\leq\left(z_q+q_q^k\right),\quad p^k\leq x_q^k,\qquad\forall\ k\in\mathcal{K}.$$

Consequently, the robust optimization problem is

$$\max_{\mathbf{p}}\sum_{k=1}^{K}\log\left(1+\frac{p^k h^k}{\sigma^2+f_{\mathcal{Q}}^k}\right), \tag{2.54}$$

$$\text{subject to}\begin{cases}C_1:\sum_{k=1}^{K}p^k\leq p^{\max},\\ C_2:\left(\sum_{k=1}^{K}p^k\overline{g}_q^k+z_q\Gamma_q+\sum_{k=1}^{K}\chi_q\right)\leq IT_q,\\ C_3:\hat{g}_q^k x_q^k\leq(z_q+\chi_q),\\ C_4:p^k\leq x_q^k,\\ C_5:0\leq x_q^k,\quad 0\leq z_q,\quad 0\leq\chi_q^k,\quad\forall\ k\in\mathcal{K},\quad\forall\ q\in\mathcal{Q}.\end{cases}$$

Since all the constraints in (2.54) are affine functions, a pseudo water-filling formulation can be developed for obtaining p^k using their corresponding Lagrange dual functions. When $\hat{g}_q^k \tilde{\mu}_q^k = w_q^k$, $\tilde{\nu}_q \Gamma_q = \sum_{k=1}^{K} \tilde{\mu}_q^k$, and $\tilde{\mu}_q^k = \tilde{\nu}_q$, as shown in [9, 19], the optimal solution to (2.54) is

$$\tilde{p}^{*k} = \left[\frac{1}{\lambda + \sum_{q=1}^{Q} (\tilde{\nu}_q \bar{g}_q^k + w_q^k)} - \frac{\sigma^2 + f_{\mathscr{Q}}^k}{h^k} \right]^+ ,$$

where λ, $\tilde{\nu}_q$, $\tilde{\mu}_q^k$, and w_q^k are the nonnegative Lagrange multipliers for C_1, C_2, C_3, and C_4 in (2.54), respectively.

2.3.2.2 Trade-off in Chance-Constrained Approach

When A_1 and A_2 in Section 1.3.3.1, Chapter 1 hold, the robust counterpart of (2.40) in the chance-constrained approach is

$$\max_{\mathbf{p}} \sum_{k=1}^{K} \log \left(1 + \frac{p^k h^k}{\sigma^2 + f_{\mathscr{Q}}^k} \right), \tag{2.55}$$

$$\text{subject to} \begin{cases} C_1 : \sum_{k=1}^{K} p^k \le p^{\max}, \quad p^k \ge 0, & \forall\, k \in \mathscr{K}, \\ C_2 : \Pr\left(\mathbf{p} \cdot \mathbf{g}_q^{\mathrm{T}} \ge IT_q\right) \le \delta_q, & \forall\, q \in \mathscr{Q}, \\ C_3 : \mathbf{g}_q \in \mathscr{R}_{\mathbf{g}_q}, & \forall\, q \in \mathscr{Q}. \end{cases}$$

Again, adjusting δ_q in (2.55) results in a trade-off between robustness and optimality. By reducing δ_q, the system becomes more robust against uncertainty, and by increasing δ_q, the SU's throughput is increased. In the chance-constrained approach [17], C_2 in (2.55) can be replaced by its convex safe approximation. To do so, we rewrite the SU's interference on PBS q as

$$\mathbf{p} \cdot \mathbf{g}_q^{\mathrm{T}} = \sum_{k=1}^{K} p^k \bar{g}_q^k + \sum_{k=1}^{K} \zeta_q^k p^k \hat{g}_q^k,$$

where $\zeta_q^k = \frac{g_q^k - \bar{g}_q^k}{\hat{g}_q^k}$ for each $q \in \mathscr{Q}$ is known to be in the range $[-1, 1]$. For uncorrelated fading channels [31], all the values of ζ_q^k are independent of each other and their probability distribution function (pdf)is $\mathscr{D}(q)$ (Section 1.3.3.1 in Chapter 1). Hence, C_2 in (2.55) can be replaced by the Bernstein approximations of chance constraints [17], that is,

$$\left(\sum_{k=1}^{K} p^k \bar{g}_q^k + \sum_{k=1}^{K} \eta_{\mathscr{D}(q)}^{+} p^k \hat{g}_q^k + \sqrt{2 \ln \delta_q^{-1}} \left(\sum_{k=1}^{K} \varphi_{\mathscr{D}(q)}^2 (p^k \hat{g}_q^k)^2 \right)^{\frac{1}{2}} \right) \le IT_q, \quad \forall\, q \in \mathscr{Q}, \tag{2.56}$$

or

$$\left(\sum_{k=1}^{K} p^k \overline{g}_q^k + \sum_{k=1}^{K} \eta_{\mathscr{D}(q)}^{+} p^k \hat{g}_q^k + \varphi_{\mathscr{D}(q)} \sqrt{2K \ln \delta_q^{-1}} \max_{\forall\, k \in \mathscr{K}} p^k \hat{g}_q^k \right) \leq IT_q, \quad \forall\, q \in \mathscr{Q}, \tag{2.57}$$

where $-1 \leq \eta_{\mathscr{D}(q)}^{+} \leq +1$ and $\sigma_{\mathscr{D}(q)}^2 \geq 0$, which depend on $\mathscr{D}(q)$, are used for safe approximations of the chance constraints. For a specific $\mathscr{D}(q)$, these parameters have fixed values [17], as shown in Table 1.2 in Chapter 1. Note that when $\mathbf{p}$ satisfies (2.56) and (2.57), it certainly satisfies C_2 and C_3 in (2.55). Hence, from (2.56) we have

$$\max_{\mathbf{p}} \sum_{k=1}^{K} \log\left(1 + \frac{p^k h^k}{\sigma^2 + f_{\mathscr{Q}}^k}\right), \tag{2.58}$$

$$\text{subject to} \begin{cases} C_1 : \sum_{k=1}^{K} p^k \leq p^{\max}, \\ C_2 : \left(\sum_{k=1}^{K} p^k \overline{g}_q^k + \sum_{k=1}^{K} \eta_{\mathscr{D}(q)}^{+} p^k \hat{g}_q^k + \sqrt{2 \ln \delta_q^{-1}} \sqrt{\sum_{k=1}^{K} \varphi_{\mathscr{D}(q)}^2 \left(p^k \hat{g}_q^k\right)^2}\right) \leq IT_q. \end{cases}$$

Note that (2.58) can be rewritten as a conic quadratic programming problem, which can be solved with few calculations.

We use (2.57) to rewrite (2.55) as

$$\max_{\mathbf{p}} \sum_{k=1}^{K} \log\left(1 + \frac{p^k h^k}{\sigma^2 + f_{\mathscr{Q}}^k}\right), \tag{2.59}$$

$$\text{subject to} \begin{cases} C_1 : \sum_{k=1}^{K} p^k \leq p^{\max}, \\ C_2 : \left(\sum_{k=1}^{K} p^k \overline{g}_s^k + \sum_{k=1}^{K} \eta_{\mathscr{D}(q)}^{+} p^k \hat{g}_q^k + \varphi_{\mathscr{D}(q)} \sqrt{2K \ln \delta_q^{-1}} \max_{\forall k \in \mathscr{K}} p^k \hat{g}_q^k\right) \leq IT_q, \end{cases}$$

which is equivalent to

$$\max_{\mathbf{p}} \sum_{k=1}^{K} \log\left(1 + \frac{p^k h^k}{\sigma^2 + f_{\mathscr{Q}}^k}\right), \tag{2.60}$$

$$\text{subject to} \begin{cases} C_1 : \sum_{k=1}^{K} p^k \leq p^{\max}, \\ C_2 : \left(\sum_{k=1}^{K} p^k \overline{g}_q^k + \sum_{k=1}^{K} \eta_{\mathscr{D}(q)}^{+} p^k \hat{g}_q^k + \varphi_{\mathscr{D}(q)} \sqrt{2K \ln \delta_q^{-1} u_q}\right) \leq IT_q, \quad \forall\, q \in \mathscr{Q}, \\ C_3 : p^k \hat{g}_q^k \leq u_q, \quad \forall\, q \in \mathscr{Q}, \quad \forall\, k \in \mathscr{K}, \end{cases}$$

where $u_q > 0$ is an auxiliary variable representing the maximum error in the estimated interference on PU q in all channels. Utilizing the Lagrange dual function, the optimal solution to (2.60) is

$$\tilde{p}^{*k} = \left[\frac{1}{\lambda + \sum_{q=1}^{Q} \left(\overline{\nu}_q (\overline{g}_q^k + \eta_{\mathscr{D}(q)}^{+} \hat{g}_q^k) + \overline{\mu}_q^k \hat{g}_q^k \right)} - \frac{\sigma^2 + f_{\mathscr{Q}}^k}{h^k} \right]^+, \tag{2.61}$$

where λ, $\overline{\nu}_q$, and $\overline{\mu}_q^k$ are the nonnegative Lagrange multipliers for C_1, C_2, and C_3, respectively, which satisfy

$$\lambda \left(\sum_{k=1}^{K} \tilde{p}^{*k} - p^{\max} \right) = 0,$$

$$\overline{\nu}_q \varphi_{\mathscr{D}(q)} \sqrt{2K \ln \delta_q^{-1}} = \sum_{k=1}^{K} \overline{\mu}_q^k,$$

$$\overline{\mu}_q^k (\tilde{p}^{*k} \hat{g}_q^k - u_q) = 0,$$

$$\overline{\nu}_q \left(\sum_{k=1}^{K} \tilde{p}^{*k} \overline{g}_q^k + \sum_{k=1}^{K} \eta_{\mathscr{D}(q)}^{+} \tilde{p}^{*k} \hat{g}_q^k + \varphi_{\mathscr{D}(q)} \sqrt{2K \ln \delta_q^{-1}} u_q - IT_q \right) = 0.$$

In (2.59), the protection function is

$$\Delta_q = \sum_{k=1}^{K} \eta_{\mathscr{D}(q)}^{+} p^k \hat{g}_q^k + \varphi_{\mathscr{D}(q)} \sqrt{2K \ln \delta_q^{-1}} \max_{\forall\, k \in \mathscr{K}} p^k \hat{g}_q^k. \tag{2.62}$$

Sensitivity Analysis of d_Δ: Protection functions in the D-norm and chance-constrained approaches depend on Γ_q and δ_q, respectively. In what follows, we study the sensitivity of d_Δ on Γ_q in the D-norm approach and on δ_q in the chance-constrained approach by differentiating d_Δ with respect to Γ_q and δ_q.

- D-Norm Approach: Since $\Delta_q = z_q \Gamma_q + \sum_{q=1}^{Q} q_q^k$, from (2.50) we have

$$d_\Delta \approx \sum_{q=1}^{Q} \nu_q^* \left(z_q \Gamma_q + \sum_{k=1}^{K} q_q^k \right), \tag{2.63}$$

  which means that d_Δ directly relates to Γ_q. Hence, for a given value of IT_q and in a given channel uncertainty region, increasing the value of Γ_q proportionally reduces the SU's throughput, and vice versa. This is shown in Fig. 2.8 for a single SU and a single PBS, where $\hat{g}^k = \xi \overline{g}^k$ and $IT = -3$ dB. Note that increasing Γ_q and expanding the uncertainty region ξ

(continued)

proportionally reduces the SU's throughput, as expected from (2.50). The simulation setup for Fig. 2.8 is the same as for Fig. 2.6.

- Chance-Constrained Approach: In this case, the protection function is

$$\Delta_q = \sum_{k=1}^{K} \eta^{+}_{\mathscr{D}(q)} p^k \hat{g}^k_q + \sqrt{2 \ln \delta_q^{-1}} \sqrt{\sum_{k=1}^{K} \varphi^2_{\mathscr{D}(q)} (p^k \hat{g}^k_q)^2}, \tag{2.64}$$

and we have

$$d_\Delta \approx \sum_{q=1}^{Q} \nu_q^* \left(\sum_{k=1}^{K} \eta^{+}_{\mathscr{D}(q)} p^k \hat{g}^k_q \right) + \sqrt{2 \ln \delta_q^{-1}} \sqrt{\sum_{k=1}^{K} \varphi^2_{\mathscr{D}(q)} (p^k \hat{g}^k_q)^2}. \tag{2.65}$$

Based on the preceding expressions, the sensitivity of d_Δ to δ_q is

$$S_{\delta_q}(d_\Delta) = \frac{\partial d_\Delta}{\partial \delta_q} = \frac{1}{\delta_q \sqrt{2 \ln \delta_q^{-1}}}. \tag{2.66}$$

Figure 2.9 shows that for a given IT_q and a channel uncertainty region, when $\delta_q < 0.2$, d_Δ is very sensitive to δ_q. This is not the case for higher values of δ_q. The same is true of (2.59). The sensitivity of d_Δ to δ_q is shown in Fig. 2.10, where for $\delta \geq 0.2$, the SUs' throughput does not increase very much.

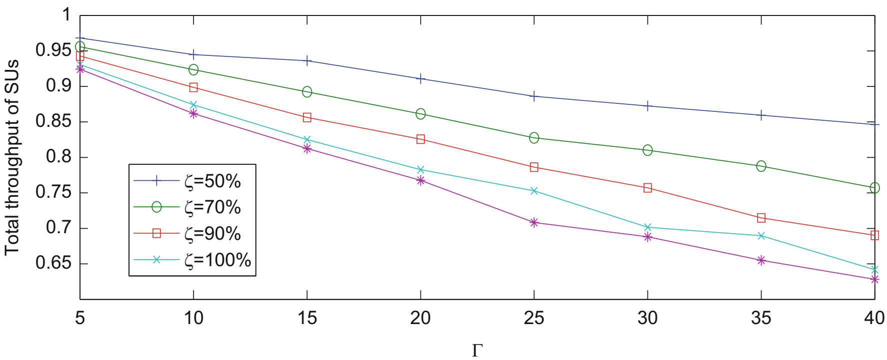

Fig. 2.8 Throughput in D-norm approach versus ξ and $\Gamma = \Gamma_q$ for all $q \in \mathscr{Q}$ and for $IT = \frac{1}{2} p^{\max}$ dB

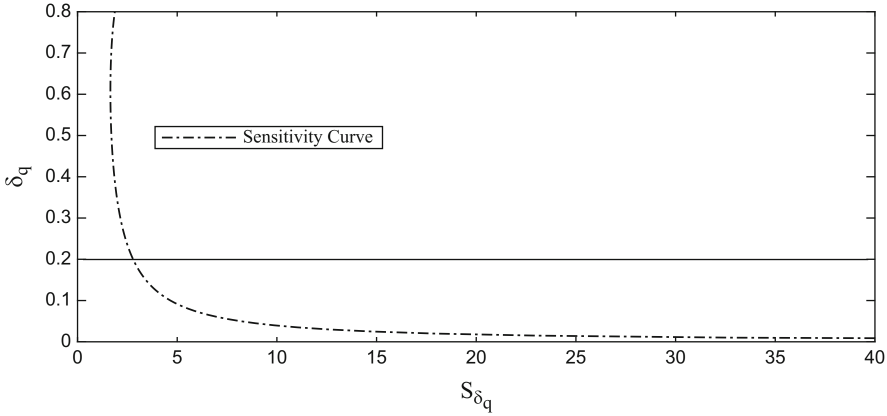

Fig. 2.9 δ_q versus sensitivity of d_Δ in (2.65)

Choosing a Trade-Off Approach: Now we consider the criteria for choosing a trade-off approach for a given scenario. We begin by noting that the computational cost in both choices, namely, in the trade-off algorithm for the D-norm approach and in the trade-off algorithms for the two safe approximations in the chance-constrained approach, is moderate and not a determining factor. However, the *protection value* plays a major role in the choice of trade-off algorithm. An algorithm with less protection has a smaller d_Δ and a higher throughput. Less protection also means a higher probability of violating the constraints. One can keep the violation probability below a given threshold in the D-norm approach, as shown by (1.18) in Chapter 1, and in both safe approximations in the chance-constrained approach. In practice, however, the violation probability may be even less than the given threshold, as reported in [9]. Hence, when high throughput is important, a trade-off algorithm with less protection should be chosen, and when it is important to guarantee the violation probability, a trade-off algorithm with higher protection should be selected. In [9], it is shown that in a single-SU single-PBS CRN, the protection value for (2.56) is higher than those for (2.51) and (2.57), which means (2.56) is preferred by the PBS but is disadvantageous for the SU. The opposite is true of (2.57), where the SU enjoys its highest throughput at the expense of minimal protection for the PBS. While the throughput in the D-norm approach is close to that of (2.57), the probability of violating the IT in the former is considerably lower than that in the latter because the former has higher protection. Hence, the trade-off algorithm in the D-norm approach can be a better choice from the perspectives of both the SU and the PBS. In a similar manner, one could argue which trade-off algorithm is a better choice for other cases.

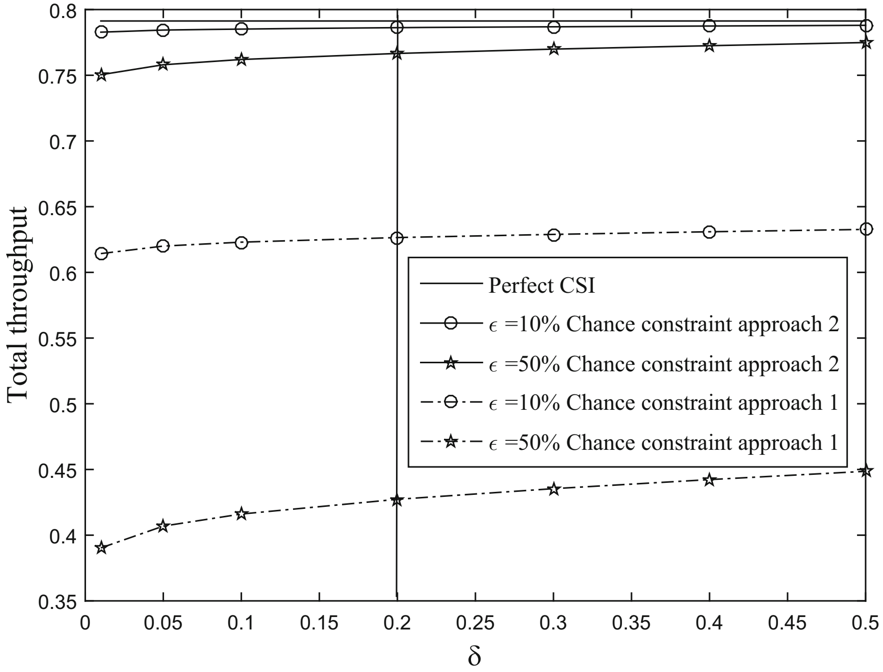

Fig. 2.10 Throughput in chance-constrained approach versus $\delta = \delta_q$ and $\varepsilon = \varepsilon_{\mathbf{g}_q}$ for all $q \in \mathcal{Q}$ for $IT = \frac{1}{2}p^{\max}$

Note that the linear constraint for the D-norm approach in (2.54) and for the chance-constrained approach in (2.60) for the single-SU scenario can also be applied for multiple SUs in (2.39). For instance, in the D-norm approach, (2.39) is transformed into

$$\max_{[\mathbf{p}_1,\ldots,\mathbf{p}_S]} \sum_{s=1}^{S}\sum_{k=1}^{K} \log\left(1 + \frac{p_s^k h_{ss}^k}{\sigma^2 + f_{\mathcal{Q}}^k}\right), \tag{2.67}$$

$$\text{subject to} \begin{cases} C_1: \sum_{k=1}^{K} p_s^k \le p_s^{\max}, \\ C_2: \sum_{k=1}^{K}\left(p_s^k \overline{g}_{sq}^k + z_{sq}\Gamma_{sq} + \sum_{k=1}^{K}\chi_{sq}\right) \le IT_{sq}, \\ C_3: \hat{g}_{sq}^k x_{sq}^k \le (z_{sq} + \chi_{sq}), \\ C_4: p_s^k \le x_{sq}^k, \\ C_5: 0 \le x_{sq}^k, \quad 0 \le z_{sq}, \quad 0 \le \chi_{sq}^k, \quad \forall\, k \in \mathcal{K}, \ \forall\, s \in \mathcal{S}, \ \forall\, q \in \mathcal{Q}, \end{cases}$$

where Γ_{sq} is the D-norm parameter for protecting PBS q from SU s, and χ_{sq} and x_{sq}^k are nonnegative parameters for converting the D-norm's protection function into linear constraints.

Similarly, in the chance-constrained approach, (2.39) is transformed into

$$\max_{[\mathbf{p}_1,\dots,\mathbf{p}_S]} \sum_{s=1}^{S}\sum_{k=1}^{K} \log\left(1+\frac{p_s^k h_{ss}^k}{\sigma^2+f_{\mathscr{Q}}^k}\right), \tag{2.68}$$

$$\text{subject to} \begin{cases} C_1: \sum_{k=1}^{K} p_s^k \le p_s^{\max}, \\ C_2: \left(\sum_{k=1}^{K} p_s^k \bar{g}_{sq}^k + \sum_{k=1}^{K}\left(\eta_{\mathscr{D}(sq)}^{+} p_s^k \hat{g}_{sq}^k\right) + \varphi_{\mathscr{D}(sq)}\sqrt{2K\ln \delta_{sq}^{-1}} u_{sq}\right) \le IT_{sq},\ \forall\, q \in \mathscr{Q}, \\ C_3: p_s^k \hat{g}_{sq}^k \le u_{sq}, \qquad \forall\, s \in \mathscr{S},\quad \forall\, q \in \mathscr{Q},\quad \forall\, k \in \mathscr{K}, \end{cases}$$

where $u_{sq} > 0$ is an auxiliary variable representing the maximum error in the estimated interference by SU s on PU q in all channels, δ_{sq} is the violation probability for the IT of SU s on PBS q, and $\mathscr{D}(sq)$ is the pdf for the uncertainty in channel gain $\mathbf{g}_{sq}$ between SU s and PBS q (defined in Table 1.2, Chapter 1).

The preceding discussion shows that, by using linear constraints instead of protection functions, the order of additional calculations for introducing robustness in the D-norm approach and in the chance-constrained approach is $O(Q\times(4\times Q\times K+Q))$ and $O(S\times(Q+K\times Q))$, respectively, compared to (2.29) [9], meaning that the computational complexity of obtaining the nominal and robust solutions are similar. To obtain the order of computational complexity associated with solving (2.60) or (2.67), all primal and dual variables are iteratively updated, much like the iterative algorithm for solving (2.30). Therefore, for each new constraint or each new variable in (2.60) or (2.67), there is one additional iterative formula compared to (2.29).

So far, we have shown that replacing nonlinear protection functions in the constraints of robust problems with linear functions can reduce the computational complexity associated with solving robust problems. We have also shown that introducing robustness entails throughput reduction, and sensitivity analysis was used to demonstrate that it is possible to trade off between throughput and robustness.

2.4 Overview of Other Works on Robust Cooperative Resource Allocation

In the preceding sections, we presented robust cooperative schemes for optimal transmit power allocation in wireless networks and explained how to trade off between total throughput and robustness. In what follows, we provide an overview of other works on this topic, highlighted in Table 2.1, and show that while we covered trade-off algorithms for complexity, throughput, and signaling, in general they have not been extensively addressed in the literature.

In [32], the worst-case ergodic resource allocation in the uplink of an underlay CRN is studied, where the objective is to maximize an SU's total throughput subject to PUs' interference threshold. In this work, the uncertainty in channel

Table 2.1 Overview of other existing works on robust cooperative resource allocation

References	Complexity	Throughput	Signaling	Trade-off algorithms	
				D-norm	Chance-constrained
[32]	√	√	–	–	√
[33]	√	√	–	–	√
[34]	√	√	–	–	–
[35]	√	√	√	–	√
[36]	√	√	–	–	√
[37]	√	√	–	–	–
[38]	√	√	–	–	–

gains between SU transmitters and SU receivers and in channel gains between SU transmitters and PU receivers are considered. To reduce the computational complexity, in [32] a middle way between the worst-case and Bayesian approaches is applied for robustness–throughput trade-off via a chance constraint, and the impact of the uncertainty region's bound on SUs' total throughput is investigated via simulations only (no analytical study).

In [33, 34], physical layer security in the presence of eavesdroppers is studied, where the impact of expanding the uncertainty region on the throughput is investigated, and reducing a problem's computational complexity is discussed. In [33], the robustness–throughput trade-off via a chance constraint is considered, but no such trade-off algorithm is presented in [34]. In [35], to maximize power efficiency while maintaining the required signal-to-interference ratio (SIR) for each user in interference-limited wireless networks with uncertain parameters, the problem is formulated via the chance-constrained approach. It is also shown that the proposed approach maintains the convexity and tractability of the problem by solving it in a distributed manner by the Lagrange dual decomposition method. An infrequent message passing algorithm is also proposed to reduce the signaling in the proposed robust scheme.

In [36], worst-case robust optimization is applied in relay-assisted two-hop D2D communications in LTE-A cellular networks, where the objective is to maximize the network's throughput in shared channel environments. The constraints are the bounded transmit power, the interference on relays and receiving users, and user QoS requirements. Uncertainty in channel gains, stated by the linear norm, is assumed to be within the uncertainty region. It is shown that the robust problem is convex and a gradient-based dual decomposition algorithm can be applied to allocate transmit power in a distributed manner. However, [36] is silent on additional message passing in the robust distributed algorithm, but the chance-constrained approach is applied to ameliorate the throughput reduction in the robust scheme by trading off between throughput and robustness.

In [37], a robust resource allocation scheme is proposed for a cooperative CRN in which a set of relays assist SUs, and strict SUs' ITs on PUs are set. Assuming that channel gains are uncertain, the robust resource allocation problem is formulated to maximize the SUs' sum rate while satisfying the interference constraints on PUs via both the probabilistic and worst-case approaches. It is shown that the original robust optimization problem can be approximated and transformed into a convex deterministic form, and a closed-form solution for the SUs' transmit power levels is derived.

In [38], optimal robust resource allocation in the downlink of renewable-powered HetNets is studied, where channel gains are uncertain due to noise. In this work, robust energy management and transmit beam-forming are proposed to minimize worst-case energy transactions subject to a worst-case guarantee for each user's QoS. The authors use semi-definite programming (SDP) relaxation to transform the problem into a convex approximation, which is solved by applying the gradient method in a distributed manner. In this way, computational complexity is reduced, but the paper is silent on additional message passing in the robust solution. In addition, using simulations, it is shown that expanding the uncertainty region increases the robust transmit power levels.

As can be seen in Table 2.1, while the impact of uncertainty in parameter values on the total throughput or on transmit power levels has been extensively studied, the performance gap between the robust and nominal solutions similar to (2.21) in Lemma 2.1 in this chapter has not been considered in the literature. Note that deriving a mathematical expression for the aforementioned performance gap enables one to see how each constraint in the robust problem affects the performance of the robust solution. Moreover, as shown in Table 2.1, although the D-norm approach is easy to implement and has some interesting aspects, as mentioned in Section 3.2.2, it has not been a very popular topic. Nevertheless, utilizing (1.18) in Chapter 1 for the D-norm approach to make a connection between the violation probability and the protection value seems to be an interesting topic.

2.5 Concluding Remarks

In this chapter, we showed how network-centric robust resource allocation problems in wireless networks with cooperative users can be solved via the dual decomposition approach; in addition, we discussed how the costs of robustness can be mitigated. In what follows, we summarize our conclusions.

- **Computational Complexity**: We demonstrated how to simply robust problems and reduce their computational complexity to the same order of the nominal problem by utilizing protection functions; and showed that for the general norm and for the polyhedron uncertainty region, the robust counterpart problem is convex. We also obtained the optimal solution for some specific cases by way of utilizing a pseudo water-filling formula, whose computational complexity is not significant.

- **Throughput Reduction**: We derived mathematical formulations for a given amount of throughput reduction in robust solutions when the nominal problem is convex. We utilized sensitivity analysis to demonstrate that when the uncertainty region is shrank, the throughput gap between the nominal and robust solutions can be reduced. Also, we showed the application of the D-norm and chance-constrained approaches to trading off between robustness and throughput reduction in different scenarios and compared them by way of sensitivity analysis on the size of the protection function for each approach. We also explained the criteria for choosing a trade-off approach for each specific scenario.
- **Extra Message Passing**: In solving cooperative distributed resource allocation problems, the drawback of applying dual decomposition techniques is the need for extra message passing. To reduce this additional signaling, we proposed a distributed scheme with infrequent message passing and derived the conditions for its convergence to the optimal solution of the original distributed algorithm.

Appendices

Appendix 1: Proof of Proposition 2.2

Recall that the dual norm of $\|\mathbf{x}\|$ is $\|\mathbf{x}\|^* \triangleq \max_{\|\mathbf{s}\|\leq 1} \mathbf{s}^{\mathrm{T}}\mathbf{x}$. For the uncertainty set in $\mathscr{R}_{\mathbf{g}_{sq}}$, we use $\mathbf{y}_q = \frac{\bar{\mathbf{g}}_q - \mathbf{g}_q}{\varepsilon_{\mathbf{g}_q}}$ to rewrite the uncertainty set as

$$\mathscr{R}_{\mathbf{g}_q} = \{\mathbf{y}_q | \|\bar{\mathbf{y}}_q - \mathbf{y}_q\| \leq 1\} \qquad \forall\, q \in \mathscr{Q}. \tag{2.69}$$

We also rewrite the protection value as

$$\max_{\mathbf{g}_q \in \mathscr{R}_{\mathbf{g}_q}} \sum_{s\in\mathscr{S}} \big((g_{sq} - \bar{g}_{sq})p_s\big) = \varepsilon_{\mathbf{g}_q} \max_{\mathbf{y}_q \in \mathscr{R}_{\mathbf{g}_q}} \sum_{s\in\mathscr{S}} \big((y_{sq} - \bar{y}_{sq})p_s\big), \tag{2.70}$$

which corresponds to the dual norm. The same is true of the uncertainty set pertaining to PU interference on the SBS [20].

Appendix 2: Convexity of (9)

By logarithmic transformation of the optimization variable in (2.9), the upper and lower bounds on the transmit power of each SU, as well as C_1 and C_3, are changed to

$$p_s^{\mathtt{min}} e^{-y_s} \leq 1,$$
$$(p_s^{\mathtt{max}})^{-1} e^{y_s} \leq 1,$$
$$\hat{\gamma}_s h_s^{-1} e^{z_s - y_s} \leq 1,$$

$$e^{-z_s}(f_{\mathscr{Q}} + \sum_{m \neq s} e^{y_m} h_m + \sigma^2) \leq 1.$$

The left-hand side of the preceding constraints are the affine nonnegative sum of exponential variables and constitute convex functions ([10] and Section 3.2 in [19]). Also, the interference constraint C_2 in (2.9) is changed to

$$\left(\sum_{s \in \mathscr{S}} \overline{g}_{sq} e^{y_s} + \varepsilon_{\mathbf{g}_q} \sqrt{\sum_{s \in \mathscr{S}} e^{2y_s}} \right) \leq IT_q, \quad \forall\, q \in \mathscr{P}.$$

The left-hand side of the preceding expression is the linear norm of the vector of exponential variables with order 2, which is convex (Section 3.2.4 in [19]). Note that (2.4) is a concave function under logarithmic transformation [10, 12]. Hence, (2.9) is convex.

Appendix 3: Proof of Proposition 2.3

Problem (2.9) is feasible when there is at least one transmit power vector that simultaneously satisfies all constraints. The feasibility of problem (2.7) can be written (Section 2.1.3 in [16]) as

$$\min_{1 \leq \varsigma,\ p_s^{\mathtt{min}} \leq p_s \leq p_s^{\mathtt{max}}} \varsigma, \tag{2.71}$$

$$\text{subject to} \begin{cases} C_1 : \chi_{1n}(z_s, y_s) = e^{z_s - y_s} \frac{\hat{\gamma}_s}{h_s} \leq \varsigma, \\ C_2 : \chi_{2n}(z_s, y_s) = \frac{1}{IT_q} (\sum_{s \in \mathscr{S}} g_{sq} e^{y_s}) \leq \varsigma \qquad \forall\, q \in \mathscr{Q}, \\ C_3 : \chi_{3n}(z_s, y_s) = e^{-z_s}(f_{\mathscr{Q}} + \sum_{m \neq s,\ m \in \mathscr{S}} e^{y_m} h_m + \sigma^2) \leq \varsigma, \\ C_4 : \mathbf{g}_q \in \mathscr{R}_{\mathbf{g}_q}, \quad f_{\mathscr{Q}} \in \mathscr{R}_{f_{\mathscr{Q}}}, \end{cases}$$

where ς is an auxiliary variable representing the upper bound on all constraints. By solving (2.71) and obtaining $(\varsigma^*, \mathbf{p}^* = [p_1^*, \ldots, p_S^*])$, when $\varsigma^* = 1$, the set of posynomial constraints in C_1, C_2, and C_3 are feasible, and the corresponding transmit power vector of all SUs, that is, $\mathbf{p}^* = [p_1^*, \ldots, p_S^*]$ is the feasibility set of (2.7). Otherwise, the set of posynomial constraints and, consequently, (2.7) are infeasible [16]. Based on the preceding expressions, to obtain the feasibility conditions of (2.7), we set $\varsigma = 1$ and obtain $\mathbf{p}^* = [p_1^*, \ldots, p_S^*]$. In doing so, we divide problem (2.71) into two subproblems, one for C_1, C_3, and $f_{\mathscr{Q}} \in \mathscr{R}_{f_{\mathscr{Q}}}$, and the other for C_2 and $\mathbf{g}_q \in \mathscr{R}_{\mathbf{g}_q}$.

Sub-problem 1. This problem is

$$\min_{1\leq\varsigma,\ p_s^{\min}\leq p_s\leq p_s^{\max}} \varsigma, \tag{2.72}$$

$$\text{subject to} \begin{cases} C_1: e^{z_s - y_s}\frac{\hat{\gamma}_s}{h_s} \leq \varsigma, \\ C_3: \left(e^{-z_s}(f_{\mathscr{Q}} + \sum_{m\in\mathscr{S},\ m\neq s} e^{y_m} h_m + \sigma^2)\right) \leq \varsigma, \\ C_4: f_{\mathscr{Q}} \in \mathscr{R}_{f_{\mathscr{Q}}}. \end{cases}$$

If we use $\varDelta_F$ instead of the uncertainty set in (2.7) and consider $\varsigma = 1$, this problem becomes a standard transmit power allocation problem [22] with a protection value. The solution to (2.72) satisfies

$$\upsilon + \boldsymbol{\varDelta} \leq (\mathbf{I} - \mathbf{F})\mathbf{p}, \tag{2.73}$$

where $\mathbf{p}$ is the transmit power vector of all SUs. This inequality is satisfied [22] when

$$\rho(\mathbf{F}) < 1, \tag{2.74}$$

where ρ is the spectral radius of $\mathbf{F}$, minimized for $\mathbf{b}_{\varDelta} = (\upsilon + \boldsymbol{\varDelta})(\mathbf{I} - \mathbf{F})^{-1}$. Also, the transmit power vector obtained in (2.73) should be below its upper bound for each user, that is,

$$\left((\nu + \boldsymbol{\varDelta})(\mathbf{I} - \mathbf{F})^{-1}\right) \leq \mathbf{p}^{\max}. \tag{2.75}$$

Sub-problem 2. For C_2, we obtain the feasibility condition by solving

$$\min_{1\leq\varsigma,\ p_s^{\min}\leq p_s\leq p_s^{\max}} \varsigma, \tag{2.76}$$

$$\text{subject to} \begin{cases} C_2: \frac{1}{IT_q}(\sum_{s\in\mathscr{S}} g_{sq} e^{y_s}) \leq \varsigma \qquad \forall q \in \mathscr{Q}, \\ C_4: \mathbf{g}_q \in \mathscr{R}_{\mathbf{g}_q}, \end{cases}$$

which is equivalent to solving the following linear programming problem:

$$\left(\bar{\mathbf{g}}_q \mathbf{p} + \varepsilon_{\mathbf{g}_q} \max_{\|\mathbf{r}\|\leq 1} \mathbf{r}\mathbf{p}\right) \leq IT_q, \tag{2.77}$$

where $\mathbf{r} = (\mathbf{g}_q - \bar{\mathbf{g}}_q)/\varepsilon_{\mathbf{g}_q}$. Note that the second term in (2.77) is the dual norm [19]. For (2.9), the dual norm of a linear norm with order 2 is a linear norm with order 2. Hence,

$$\left(\bar{\mathbf{g}}_q \mathbf{p} + \varepsilon_{\mathbf{g}_q} \|\mathbf{p}\|_2\right) \leq IT_q. \tag{2.78}$$

When (2.74), (2.75), and (2.78) hold for $\mathbf{b}_{\Delta}$, (2.9) is feasible. Thus, the feasibility conditions for (2.9) are items **(4)–(6)** in Proposition 2.3.

Appendix 4: Proof of Lemma 2.1

Note that (2.8) is a perturbed version of (2.4) with protection values in C_2 and C_3. To obtain the relationship between d_Δ and the protection value, we perform a local sensitivity analysis of the optimization problem by perturbing its constraints [7, 19, 26]. Let

$$v^*(\mathbf{a},\mathbf{b}) = \inf \Bigg\{ - \max_{p_s^{\min} \le p_s \le p_s^{\max}} \sum_{s\in\mathcal{S}} v_s(h_s p_s \varrho_s^{-1})| \ , \\ \varrho_s \hat{\gamma}_s \le h_s p_s, \left(\sum_{s\in\mathcal{S}} g_{sq} p_s + \Delta_q \right) \le IT_q, \\ \left(f_{\mathcal{Q}} + \sum_{m\in\mathcal{S},\ m\neq s} p_s h_s + \Delta_{f_{\mathcal{Q}}} + \sigma^2 \right) \le q_s \Bigg\},$$

where $\mathbf{a}$ is a vector whose qth element is Δ_q, and $\mathbf{b}$ is a vector whose elements are all equal to $\Delta_{f_{\mathcal{Q}}}$. When Δ_q and $\Delta_{f_{\mathcal{Q}}}$ are small, $v^*(\mathbf{a},\mathbf{b})$ is differentiable with respect to the perturbation vectors $\mathbf{a}$ and $\mathbf{b}$ [26]. Using a Taylor series, we write

$$v^*(\mathbf{a},\mathbf{b}) = v^*(\mathbf{0},\mathbf{0}) \\ + \sum_{q\in\mathcal{P}} a_q \frac{\partial v^*(\mathbf{0},\mathbf{b})}{\partial a_q} + \sum_{s\in\mathcal{S}} b_s \frac{\partial v^*(\mathbf{a},\mathbf{0})}{\partial b_s} + o, \tag{2.79}$$

where $v^*(\mathbf{0},\mathbf{0})$ is the social utility for (2.4), and $\mathbf{0}$ is the all zero vector. Note that $v^*(\mathbf{a},\mathbf{b})$ and $v^*(\mathbf{0},\mathbf{0})$ are equal to v^*_Δ and v^*, respectively. Since (2.4) is convex, from the sensitivity analysis in [26] we have $\frac{\partial v^*(\mathbf{0},\mathbf{b})}{\partial a_q} \approx -\nu_q^*$ and $\frac{\partial v^*(\mathbf{a},\mathbf{0})}{\partial b_s} \approx -\mu_s^*$ for all $q \in \mathcal{Q}$ and $s \in \mathcal{S}$. Hence,

$$v^*_\Delta - v^* \approx -\sum_{q\in\mathcal{Q}} \mu_q^* \Delta_q - \sum_{s\in\mathcal{S}} \mu_s^* \Delta_{f_{\mathcal{Q}}}. \tag{2.80}$$

Since ν_q^* and μ_s^* are nonnegative optimum Lagrange multipliers, the SUs' social utility is reduced compared to when the exact channel gains are known.

Appendix 5: Proof of Proposition 2.4

Problem (2.24) is feasible (Section 2.1.3 in [16]) when the following optimization problem is feasible:

$$\min_{1\le\varsigma,\ p_s^{\min}\le p_s\le p_s^{\max}} \varsigma, \tag{2.81}$$

$$\text{subject to} \begin{cases} C_1 : e^{y_s - z_s} \frac{\hat{\gamma}_s}{h_s} \le \varsigma, \\ C_2 : \left(\frac{e^{-z_s}}{IT_q} (\sum_{s \in \mathscr{S}} \bar{g}_{sq} e^{y_s} + \Delta_q) \right) \le \varsigma \quad \forall\, q \in \mathscr{Q}, \\ C_3 : \left(e^{-z_s} (\bar{f}_{\mathscr{P}} + \sum_{m \in \mathscr{S},\, m \neq s} e_m^{y_m} h_m + \Delta_{f_{\mathscr{Q}}} + \sigma^2) \right) \le \varsigma, \end{cases}$$

where $\Delta_q = \max_{e_q \in \mathscr{S},\, |\hat{g}_{sq}| \le \varepsilon_{sq},\, |e_q| = \Gamma_q} \sum_{s \in e_q} \hat{g}_{sq} e^{y_s}$. As in Appendix 3, the feasibility of this problem can be established by dividing it into two subproblems. The first subproblem is similar to subproblem 1 in Appendix 3. Also, subproblem 2 in Appendix 3 can be utilized for the D-norm approach, where (2.77) is changed to

$$\bar{\mathbf{g}}_q \mathbf{p} + \varepsilon_q \|\mathbf{p}\|^*_{\Gamma_q} \le IT_q, \tag{2.82}$$

where $\|\mathbf{p}\|_{\Gamma_q} = \max_{e_q \in \mathscr{S},\, |e_q| = \Gamma_q} \sum_{k \in e_q} p^k$, and $\|\mathbf{p}\|^*_{\Gamma_q}$ is the dual norm of $\|\mathbf{p}\|_{\Gamma_q}$, defined [20] by

$$\|\mathbf{p}\|^*_{\Gamma_q} = \max \left(\|\mathbf{p}\|_\infty, \frac{\|\mathbf{p}\|_1}{\Gamma_q} \right). \tag{2.83}$$

From the preceding expressions, when (2.74), (2.75), and (2.82) hold for $\mathbf{b}_\Delta$, (2.24) is feasible. Thus, the feasibility conditions for (2.24) are items **(1)–(3)** in Proposition 2.4.

Appendix 6: Proof of Lemma 2.2

Part 1: Since $\left(\min\{1, \frac{\Gamma_q}{\sqrt{|\mathscr{S}|}}\} \|\mathbf{x}\|_2 \right) \le \max \left(\|\mathbf{x}\|_\infty, \frac{\|\mathbf{x}\|_1}{\Gamma_q} \right)$, the feasibility set of (2.24) is greater than that of (2.9), provided that (2.25) holds. Therefore, there is a transmit power vector for (2.24), denoted by $\mathbf{p}^{\text{D-norm}}$, that satisfies all constraints in (2.9), and

$$\mathbf{p}^{\text{D-norm}} \ge \mathbf{p}^{\text{Ellipsoid}}, \tag{2.84}$$

where $\mathbf{p}^{\text{Ellipsoid}}$ is the allocated transmit power vector for (2.9) in the ellipsoid uncertainty set, meaning that the ellipsoid's feasibility set for (2.9), that is, $\mathscr{F}_{\text{Ellipsoid}}$, is a subset of the D-norm feasibility set for (2.23), that is, $\mathscr{F}_{\text{D-norm}}$.
Part 2: Let the optimal social utility of (2.24) subject to $\mathscr{F}_{\text{D-norm}}$ be $v_{\text{D-norm}}$ and the optimal social utility of (2.9) subject to $\mathscr{F}_{\text{Ellipsoid}}$ be $v_{\text{Ellipsoid}}$. We assume

$$\begin{aligned} &\exists \quad \mathbf{p} \in \mathscr{F}_{\text{Ellipsoid}}, \\ &v_{\text{Ellipsoid}}(\mathbf{p}) > v_{\text{D-norm}}(\mathbf{p}'), \qquad \forall\, \mathbf{p}' \in \mathscr{F}_{\text{D-norm}}. \end{aligned} \tag{2.85}$$

Since $\mathscr{F}_{\text{Ellipsoid}} \subseteq \mathscr{F}_{\text{D-norm}}$, we have $\mathbf{p} \in \mathscr{F}_{\text{D-norm}}$, and from the assumption in (2.85) and convexity of the robust transmit power allocation for the general norm, the value of $v_{\text{Ellipsoid}}(\mathbf{p})$ is also the optimal solution to (2.24) subject to $\mathscr{F}_{\text{D-norm}}$, that is, $v_{\text{Ellipsoid}}(\mathbf{p}) = v_{\text{D-norm}}(\mathbf{p})$. However, from the assumption in (2.85) we have $v_{\text{Ellipsoid}}(\mathbf{p}) > v_{\text{Ellipsoid}}(\mathbf{p})$. This contradiction implies that our assumption was wrong, and thus $v_{\text{D-norm}} \geq v_{\text{Ellipsoid}}$. The preceding point generally holds for any form of convex optimization problems. For example, the social utility of (2.9) is always equal to or less than that of (2.4) due to the fact that the feasible set of (2.9) is smaller than that of (2.4).

Appendix 7: Proof of Lemma 2.3

Let $\boldsymbol{\omega}_i$ be an iterative vector with n_i steps that can be obtained from (2.11)–(2.15) as

$$\boldsymbol{\omega}_i = \begin{cases} [y_s, z_s, \lambda_s, \mu_s] & \text{if} \quad i \in \mathscr{S}, \\ \nu_q & \text{if} \quad i \in \mathscr{P}. \end{cases}$$

Algorithm 1 can be decomposed into $\underline{\Pi} = \sum_{i \in \mathscr{S}+\mathscr{Q}} \Pi_i$ block components $\boldsymbol{\omega}_1,\ldots,\boldsymbol{\omega}_{S+Q}$. Note that in Algorithm 1, $\Pi_i = 4$ for $i \in \mathscr{S}$ and $\Pi_i = 1$ for $i \in \mathscr{Q}$. In addition, for the iterative primal and dual variables obtained from (2.11)–(2.15), the following points hold:

- The Lagrange dual function $-L(\lambda_s, \mu_s, \nu_q, e^{z_s}, e^{y_s})$ is positive.
- The Lagrange dual function is Lipschitz continuous. Since p_s is upper bounded for all users, all Lagrange multipliers are upper bounded as well, and the Lagrange dual function is continuously differentiable.
- From (2.11)–(2.15), Algorithm 1 is a gradient projection algorithm [10]. Hence, Lemma 5.1 in Section 7.5 of [23] holds for Algorithm 1, and Assumptions 5.1 and 5.2 in Section 7 of [23] also hold. In addition, Algorithm 2 is a partially asynchronous implementation of Algorithm 1 with maximum delay D (Section 7.1 in [23]). Hence, there is a step size β_0 such that when all step sizes are smaller than β_0, Algorithm 2 converges to the optimal point of Algorithm 1. The value of β_0 depends on D and N (Propositions 5.1 and 5.3, Section 7 in [23]), that is,

$$\beta_0 \leq \frac{1}{1 + D + N \times D}. \tag{2.86}$$

From the preceding expression, (2.26) can be obtained with some rearrangements. Note that the preceding conditions are also valid for the D-norm approach. Hence, the same constraint (2.26) can be applied for infrequent message passing in the D-norm approach.

Appendix 8: Proof of Proposition 2.5

The Lagrange dual function associated with (2.45) is

$$L(\mathbf{p}, \mathbf{y}_q, \lambda, \hat{\nu}_q, \mu_q^k) = \sum_{k=1}^{K} \log\left(1 + \frac{p^k h^k}{\sigma^2 + f_{\mathscr{Q}}^k}\right) - \lambda\left(\sum_{k=1}^{K} p^k - p^{\max}\right)$$
$$- \sum_{q=1}^{Q} \hat{\nu}_q \left(\sum_{k=1}^{K} \varepsilon_{\mathbf{g}_q}^k y_q^k - IT_q\right) - \sum_{q=1}^{Q}\sum_{k=1}^{K} \mu_q^k \left(p^k - \sum_{i=1}^{K} [M_q^{\mathrm{T}}]^{ki} y_q^i\right),$$

where λ, $\hat{\nu}_q$, and μ_q^k are the dual variables corresponding to C_1, C_2, and C_3 in (2.45), respectively. For the convex optimization problem considered here, the duality gap is zero, and solving the dual problem is equivalent to solving the original problem [19]. The optimal transmit power vectors $\tilde{\mathbf{p}}^* = [\tilde{p}^{*1}, \ldots, \tilde{p}^{*K}]$ and $\mathbf{y}_q$ can be obtained from the saddle point of the following optimization problem:

$$\max_{\mathbf{p},\, \mathbf{y}_q} \min_{\lambda,\, \hat{\nu}_q,\, \mu_q^k} L(\mathbf{p}, \mathbf{y}_q, \lambda, \hat{\nu}_q, \mu_q^k). \tag{2.87}$$

For any given values of λ, $\hat{\nu}_q$, and μ_q^k, differentiating (2.87) with respect to p^k and y_q^k and setting the derivatives to zero leads to the maximization of (2.87), that is,

$$\frac{\partial L}{\partial p^k} = 0 \quad \Rightarrow \quad \tilde{p}^{*k} = \left[\frac{1}{\lambda + \sum_{q \in \mathscr{Q}} \mu_q^k} - \frac{\sigma^2 + f_{\mathscr{Q}}^k}{h^k}\right]^+,$$
$$\frac{\partial L}{\partial y_q^k} = 0 \quad \Rightarrow \quad \sum_{i=1}^{K} \mu_q^i [M_q^{\mathrm{T}}]^{ki} = \hat{\nu}_q \varepsilon_{\mathbf{g}_q}^k.$$

Therefore, the optimal transmit power, that is, $\tilde{\mathbf{p}}^*$, is obtained by (2.46). Also, since each optimal solution to a convex problem satisfies the KKT conditions [19], the values of λ, $\hat{\nu}_q$, and $\tilde{p}^{*k}$ are related via $\lambda \times (\sum_{k=1}^{K} \tilde{p}^{*k} - p^{\max}) = 0$, $\hat{\nu}_q \times (\sum_{k=1}^{K} \varepsilon_{\mathbf{g}_q}^k y_q^k - IT_q) = 0$, and $\mu_q^k \times (\tilde{p}^{*k} - \sum_{i=1}^{K} [M_q^{\mathrm{T}}]^{ki} y_q^i) = 0$, respectively, $\forall q \in \mathscr{Q}$, $\forall k \in \mathscr{K}$.

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Chapter 3
Robust Noncooperative Resource Allocation

Game theory is a branch of applied mathematics that can describe and analyze interactive decisions in resource allocation problems in wireless networks. Noncooperative game theory is suitable for solving and analyzing such problems when each user in the network is a greedy (noncooperative) and rational player that aims to maximize its own utility by utilizing the available side information. In practice, however, side information and measurements are noisy and uncertain, which calls for introducing in noncooperative games robustness against such uncertainties. In this chapter, we explain how to use game theory to allocate resources in networks and how to utilize worst-case optimization in robust games. Specifically, we analyze such games, study the conditions for the existence and uniqueness of their equilibria, derive the gap between the utility values of nominal (nonrobust) and robust games, and present distributed algorithms for solving such games. We also evaluate a robust game's performance vis-a-vis that of a nominal (i.e., no uncertainty) game and compare the conditions for the existence and uniqueness of a robust game's equilibrium with those of a nominal game's equilibrium.

3.1 Introduction

There exist at least three main reasons for the extensive use of game theory in resource allocation problems in networks: (1) the inherent competitive nature of multi-user wireless networks in utilizing the available resources; (2) the ability of users, enhanced by recent advances in hardware and signal processing techniques, to make independent decisions based on their own measurements and other available information; and (3) the exponential growth in the number of users, which contributes to the ever-increasing importance of scalability, decentralized decision making, and self-organization in networks.

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The objective is to develop a scheme for selfish and independent users to adjust their network access by choosing their strategies in a distributed manner with a view to maximizing each user's goodput/utility, for example, its throughput, subject to the prevailing constraints, for example, its maximum transmit power or its induced interference, and taking into account other users' actions (strategies) [1]. In practice, users may be different (heterogeneous) in their measurement and processing capabilities. When users' strategies are independent of one another, the strategic noncooperative game theory is a useful mathematical tool to analyze their interactions.

The main task is to analyze equilibrium point(s) in game-theoretic resource allocation and study performance. In doing so, the notion of Nash equilibrium (NE), at which no user can attain a higher utility by unilaterally changing its strategy, is frequently used [1, 2]. However, there is no guarantee that any game will have NE. Hence, it is essential to derive the conditions for the existence and uniqueness of the NE. There is also a need to develop efficient distributed algorithms for allocating resources and to obtain the conditions for their convergence to NE [3–7]. Moreover, it is important to compare the performance of a noncooperative distributed algorithm with that of its corresponding cooperative centralized scheme.

During the past two decades, the aforementioned topics have been extensively studied [8–17]. Our focus in this chapter is on uncertainty in user measurements and side information and on robust noncooperative strategic game-theoretic schemes. There are two different approaches to introducing robust games: the stochastic (Bayesian) approach and the worst-case approach. Ours is the latter (worst-case), where for all instances of error confined to the uncertainty region the solution satisfies the pertinent constraints [5–7, 18–23]. We begin with the nominal (i.e., no uncertainty) noncooperative strategic game and study its NE. We then introduce robustness in the game and investigate its robust NE (RNE).

In the worst-case robust game, analyzing the NE is not simple, that is, a closed form for each user's best response cannot be easily obtained. Hence, the conventional approaches to analyzing the NE, for example, the fixed point theory and contraction mapping [1, 2, 24], cannot be applied in a straightforward manner [2]. Consequently, we present alternative approaches, namely, *variational inequalities (VI), sensitivity analysis*, and *reformulation of constraints pertaining to the uncertainty region*. We also present distributed algorithms and investigate the impact of uncertainty on a system's performance at the RNE and compare it with the performance at the nominal NE. In doing so, we consider three practical cases: wireless networks with homogeneous users, underlay cognitive radio networks (CRNs), and wireless networks with heterogeneous users.

The organization of this chapter is as follows. We first provide a short overview of noncooperative strategic game theory with a view to studying game equilibria. Then we focus on three examples of utilizing worst-case robust games in wireless networks by studying the existence and uniqueness of a robust game's equilibrium for each example and compare the social utility of robust and nominal games at their respective equilibria. We continue by presenting distributed algorithms for solving robust games, compare existing works on the application of noncooperative games in wireless networks, and, finally, present a summary of the chapter.

3.2 Overview of Nominal Noncooperative Strategic Games

In this section, we present an overview of the nominal noncooperative strategic game, its equilibrium point, its performance, and a distributed algorithm for reaching the equilibrium point. In doing so, we assume that the parameter values are exact, that is, no uncertainty. We do not attempt to cover such topics in detail, and the interested reader is referred to [1, 24–28] for a more in-depth treatment.

A nominal noncooperative strategic game is played between rational players that act independently and are greedy, with no prior knowledge of the other players' strategies.

Definition 3.1. A strategic noncooperative game is denoted by

$$\mathscr{G} = \{\mathscr{N}, (v_n)_{n\in\mathscr{N}}, \mathscr{A}\},$$

where $\mathscr{N} = \{1, \dots, N\}$ is the finite set of players, $v_n : \prod_{n\in\mathscr{N}} \mathscr{A}_n \to \mathbb{R}$ is the utility of player n whose value depends on the strategy vector of all players, $\mathscr{A} = \prod_{n\in\mathscr{N}} \mathscr{A}_n$ is the set of possible actions (strategies) of players, and $\mathscr{A}_n$ is a nonempty set of actions (strategies) of player n. The strategy of each player $n \in \mathscr{N}$ is its transmit power vector $\mathbf{p}_n \in \mathbb{R}^{1\times K}$ in K channels that belongs to $\mathscr{A}_n$. We assume that the strategy of each player is independent of other players' strategies.

We denote the strategy space of other players except player n by $\mathbf{p}_{-n} \in \mathscr{A}_{-n}$, where $\mathscr{A}_{-n} = \prod_{m\in\mathscr{N},\, m\neq n} \mathscr{A}_m$ is the strategy space of all players except player n. We also denote the feasible strategies of all players by $\mathbf{p} = [\mathbf{p}_1, \dots, \mathbf{p}_N]$.

Players in a noncooperative game are assumed to be rational, meaning that each player n competes with other players by choosing a strategy profile $\mathbf{p}_n \in \mathscr{A}_n$ that maximizes its own utility function $v_n(\mathbf{p}_n, \mathbf{p}_{-n})$, given the actions of other players $\mathbf{p}_{-n} \in \mathscr{A}_{-n}$. The strategy of each player is obtained by solving the following set of optimization problems:

$$\max_{\mathbf{p}_n\in\mathscr{A}_n} v_n(\mathbf{p}_n, \mathbf{p}_{-n}), \quad \forall n \in \mathscr{N}. \tag{3.1}$$

The optimal solution for player n in (3.1) is denoted by $\mathbf{p}_n^*$ when other players' strategies $\mathbf{p}_{-n}$ are fixed but arbitrary.

Definition 3.2. For the noncooperative game $\mathscr{G} = \{\mathscr{N}, (v_n)_{n\in\mathscr{N}}, \mathscr{A}\}$, a strategy profile $\mathbf{p}^* = [\mathbf{p}_1^*, \dots, \mathbf{p}_N^*]$ is a pure strategy at its Nash equilibrium (NE) when

$$v_n(\mathbf{p}_n^*, \mathbf{p}_{-n}^*) \geq v_n(\mathbf{p}_n, \mathbf{p}_{-n}^*), \quad \forall \mathbf{p}_n \in \mathscr{A}_n, \quad \forall n \in \mathscr{N}.$$

Note that the NE is a self-enforcing strategy profile in which no single player can individually reach a higher utility value by choosing another strategy when all other players stay with their chosen strategies. Hence, the NE characterizes the game, and its analysis is important. The following questions need to be answered in the analysis of NE:

(1) What are the conditions for the existence and uniqueness of NE?
(2) What is the performance penalty of utilizing a decentralized scheme as opposed to the Pareto-optimal centralized scheme?
(3) How can the NE be reached in a distributed resource allocation scheme?
(4) What are the conditions for the convergence of distributed algorithms?

The NE is unique when the utility function of all users is strictly convex and the feasibility region of each user's optimization problem is also convex. However, in some cases, it is not easy to establish convexity, and in general, due to mutual interference between players, the existence and uniqueness of the NE are not guaranteed in wireless networks. Hence, to answer the preceding questions, two approaches to restating the NE are commonly used in the literature, the best response and variational equalities, and the conditions for the existence and uniqueness of the NE are obtained via such restatements. When the best response restatement or the variational inequality reformulations are monotonic, the local and global stability, as well as the existence and uniqueness, of the NE are established [2, 24].

Restatement of NE by Best Response [1]: The best response function is

$$B_n(\mathbf{p}_{-n}) = \left\{\mathbf{p}_n \in \mathscr{A}_n | \upsilon_n(\mathbf{p}_n, \mathbf{p}_{-n}) \geq \upsilon_n(\mathbf{p}'_n, \mathbf{p}_{-n}), \quad \forall \mathbf{p}'_n \in \mathscr{A}_n.\right\} \tag{3.2}$$

Note that $B_n(\mathbf{p}_{-n})$ is a set of actions for player n that yields the highest possible utility value for that player when other players choose $\mathbf{p}_{-n}$. If $B_n(\mathbf{p}_{-n})$ is single-valued, that is, when it contains only one element for every action $\mathbf{p}_{-n}$ of other players, $B_n(\mathbf{p}_{-n})$ is called the best response function. An action profile $\mathbf{p}^*$ is the NE of the game when for each player $n \in \mathscr{N}$ we have

$$\mathbf{p}_n^* \in B_n(\mathbf{p}_{-n}^*).$$

The multi-function mapping $\mathbf{B}(\mathbf{p})$ is defined as $\mathbf{B}(\mathbf{p}) : \mathscr{A} \ni \mathbf{p} \to B_1(\mathbf{p}_{-1}) \times B_2(\mathbf{p}_{-2}) \times \cdots \times B_N(\mathbf{p}_{-N})$. A strategy profile $\mathbf{p}^* \in \mathscr{A}$ is a pure-strategy NE of $\mathscr{G}$ when

$$\mathbf{p}^* \in \mathbf{B}(\mathbf{p}^*).$$

If $\mathbf{B}(\mathbf{p})$ is a single-valued function, denoted by $\mathbf{B}(\mathbf{p})$, an action profile $\mathbf{p}^*$ is the NE of the game when [1, 2]

$$\mathbf{p}^* = \mathbf{B}(\mathbf{p}^*). \tag{3.3}$$

This restatement of the NE is useful for obtaining the conditions for the existence and uniqueness of the NE, as well as for a convergence analysis of the distributed algorithms.

Definition 3.3 (Variational Inequalities (VI)). Given the set $\mathscr{A}$ in the n-dimensional Euclidean space $\mathbb{R}^n$, and the mapping $\mathscr{F}$, the VI problem, denoted by $VI(\mathscr{A}, \mathscr{F})$, is to find a vector $\mathbf{x}^* \in \mathscr{A}$ such that [24, 28]

$$(\mathbf{x} - \mathbf{x}^*)\mathscr{F} \geq 0, \quad \forall \mathbf{x} \in \mathscr{A}. \tag{3.4}$$

Restatement of NE by VI [2, 24, 28]: For $\mathscr{G} = \{\mathscr{N}, (v_n)_{\forall n \in \mathscr{N}}, \mathscr{A}\}$, consider the mapping vector $\mathscr{F}(\mathbf{p}) = (\mathscr{F}_n(\mathbf{p}))_{n=1}^N$, where

$$\mathscr{F}_n(\mathbf{p}) = -\nabla_{\mathbf{p}_n} v_n(\mathbf{p}_n, \mathbf{p}_{-n}), \tag{3.5}$$

in which $\nabla_{\mathbf{p}_n} v_n(\mathbf{p}_n, \mathbf{p}_{-n})$ is the column gradient vector of $v_n(\mathbf{p}_n, \mathbf{p}_{-n})$ with respect to $\mathbf{p}_n$. The NE of $\mathscr{G}$ can be obtained by solving $VI(\mathscr{A}, \mathscr{F})$ (Proposition 1.4.2 in [24]) as $(\mathbf{p} - \mathbf{p}^*)\mathscr{F}(\mathbf{p}^*) \geq 0$, for all $\mathbf{p} \in \mathscr{A}$.

Note that the NE as defined in this chapter only pertains to the case of pure strategy in noncooperative strategic games. Other definitions of NE pertaining to other strategies, such as the mixed strategy, is not the focus in this chapter.

3.2.1 Existence and Uniqueness of NE

To obtain the conditions for the existence and uniqueness of the NE, the following approaches have been proposed in the literature:

- Considering the NE as the solution of the optimization problem;
- Considering the NE as the fixed-point solution of the best response functions;
- Reformulating the NE as the solution of the VI problem;
- Using the specifics of the game to analyze its NE.

Each of the preceding approaches leads to a different set of conditions for the existence and uniqueness of the NE. In general, a less restrictive set of conditions is more desirable [2].

3.2.1.1 Existence of NE

When the NE is considered the fixed-point solution of the best response functions, the following theorem establishes the sufficient conditions for the NE's existence.

Theorem 3.1 ([2, 24, 29]). *The game $\mathcal{G} = \{\mathcal{N}, (v_n)_{n\in\mathcal{N}}, \mathcal{A}\}$ admits a pure strategy at the NE when $\mathcal{A}$ is a finite set and*

- *Each $\mathcal{A}_n$ is a nonempty, compact, finite, and convex set; and*
- *Either $v_n(\mathbf{p}_n, \mathbf{p}_{-n})$ is continuous over $\mathcal{A}$ and is a quasi-concave function of $\mathbf{p}_n$ for any $\mathbf{p}_{-n} \in \mathcal{A}_{-n}$, or $v_n(\mathbf{p}_n, \mathbf{p}_{-n})$ is continuous over $\mathcal{A}$ and the optimization problem (3.1) admits a unique solution to any $\mathbf{p}_{-n} \in \mathcal{A}_{-n}$.*

When the NE can be considered the solution of the VI problem, the following theorem can be used to prove the existence of the NE.

Theorem 3.2 (Proposition 1.4.2 in [24]). *The NE of $\mathcal{G}=\{\mathcal{N}, (v_n)_{n\in\mathcal{N}}, \mathcal{A}\}$ is the solution to VI$(\mathcal{A}, \mathcal{F})$ when*

- *$\mathcal{A}$ is a closed convex set;*
- *$\mathcal{F}$ is continuous function.*

In such cases, the set of solutions to VI$(\mathcal{A}, \mathcal{F})$ is nonempty and compact, and, consequently, $\mathcal{G} = \{\mathcal{N}, (v_n)_{n\in\mathcal{N}}, \mathcal{A}\}$ admits a pure-strategy NE.

In addition, the NE exists when $\mathcal{G}$ belongs to one of the following categories:

- Concave games [1, 2, 24]: When $\mathcal{A}_n$ is convex and compact, and $v_n(\mathbf{p}_n, \mathbf{p}_{-n})$ is concave with respect to $\mathbf{p}_n$;
- Potential games [30, 31]: When for all n and $m \in \mathcal{N}$

$$\frac{\partial^2 (v_n(\mathbf{p}_n, \mathbf{p}_{-n}) - v_m(\mathbf{p}_m, \mathbf{p}_{-m}))}{\partial \mathbf{p}_n \partial \mathbf{p}_m} = 0, \quad \forall n \neq m;$$

- Supermodular games [1, 32]: When for all n and $m \in \mathcal{N}$, $\mathcal{A}_n$ is a lattice and

$$\frac{\partial^2 (v_n(\mathbf{p}_n, \mathbf{p}_{-n}) - v_m(\mathbf{p}_m, \mathbf{p}_{-m}))}{\partial \mathbf{p}_n \partial \mathbf{p}_m} \geq 0, \quad \forall n \neq m;$$

- Standard function [33]: When the best response strategy is a standard function of a player's action.[1]

[1] A function $f(\mathbf{p})$ is standard when $f(\mathbf{p}) > 0$ for all $\mathbf{p}$, and $f(\mathbf{p}) > f(\mathbf{p}')$ for all $\mathbf{p} > \mathbf{p}'$, and $\alpha f(\mathbf{p}) > f(\alpha \mathbf{p})$ for all $\alpha > 1$.

3.2.1.2 Uniqueness of NE

When the NE is unique, players' strategies can be adjusted to achieve optimal performance at the game's equilibrium point. In general, when the utility function of each player is strictly convex and the feasibility region is also convex, the game's NE is unique. In some cases, it is not easy to establish convexity, and other approaches need to be devised to investigate the NE's uniqueness.

Theorem 3.3 (Theorem 11.3 in [2]). *The NE in the best response approach for* $\mathscr{G} = \{\mathscr{N}, (v_n)_{n\in\mathscr{N}}, \mathscr{A}\}$ *is unique when* $\mathbf{B}(\mathbf{p}^*)$ *in (3.3) is a contraction mapping in some vector norm* $\|\cdot\|$ *with* $\alpha \in [0, 1)$ *for a closed set* $\mathscr{A}$*, that is,*

$$\|\mathbf{B}(\mathbf{p}) - \mathbf{B}(\mathbf{p}')\| \le \alpha\|\mathbf{p} - \mathbf{p}'\|, \quad \forall \mathbf{p} \in \mathscr{A}, \quad \forall \mathbf{p}' \in \mathscr{A}. \tag{3.6}$$

Alternatively, the fixed-point mapping of the best response, that is, $\mathbf{T}(\mathbf{p}) = \mathbf{p} - \mathbf{B}(\mathbf{p})$, can be used to obtain the conditions for the NE's uniqueness. For instance, the NE is unique when $\mathbf{T}(\mathbf{p})$ is continuously differentiable and its Jacobian $\mathbf{J}(\mathbf{p})$ is either a P-matrix or an N-matrix, or $\mathbf{J}(\mathbf{p}) + \mathbf{J}(\mathbf{p})^{\mathrm{T}}$ is semidefinite, and between the pair $\mathbf{p} \neq \mathbf{p}'$, there is a point $\mathbf{p}''$ such that $\mathbf{J}(\mathbf{p}'') + \mathbf{J}(\mathbf{p}'')^{\mathrm{T}}$ is positive [2].

In what follows, Theorem 3.4 establishes the conditions for the NE's uniqueness in the VI approach when the continuous mapping $\mathscr{F}$ is monotone, strictly monotone, ζ-monotone for $\zeta > 1$, or strongly monotone.[2]

Theorem 3.4 ([24]). *For a closed convex set* $\mathscr{A}$ *and a continuous mapping* $\mathscr{F}$*:*

- *VI*$(\mathscr{A}, \mathscr{F})$ *has at most one solution when* $\mathscr{F}$ *is strictly monotone on* $\mathscr{A}$*;*
- *VI*$(\mathscr{A}, \mathscr{F})$ *has a unique solution when* $\mathscr{F}$ *is* ζ*-monotone on* $\mathscr{A}$ *for* $\zeta > 1$.

---

[2] A mapping $\mathscr{F}$ is said to be [24]:

- Monotone when $(\mathscr{F}(\mathbf{p}) - \mathscr{F}(\mathbf{p}'))(\mathbf{p} - \mathbf{p}') \ge 0, \quad \forall \mathbf{p} \in \mathscr{A}, \quad \forall \mathbf{p}' \in \mathscr{A}$;
- Strictly monotone when $(\mathscr{F}(\mathbf{p}) - \mathscr{F}(\mathbf{p}'))(\mathbf{p} - \mathbf{p}') > 0, \quad \mathbf{p} \neq \mathbf{p}', \quad \forall \mathbf{p} \in \mathscr{A}, \quad \forall \mathbf{p}' \in \mathscr{A}$;
- ζ-monotone for $\zeta > 1$ when there exists a constant $c > 0$ such that $(\mathscr{F}(\mathbf{p}) - \mathscr{F}(\mathbf{p}'))(\mathbf{p} - \mathbf{p}') > c\|\mathbf{p} - \mathbf{p}'\|^{\zeta}, \quad \forall \mathbf{p} \in \mathscr{A}, \quad \forall \mathbf{p}' \in \mathscr{A}$;
- Strongly monotone when there exists a constant $c > 0$ such that $(\mathscr{F}(\mathbf{p}) - \mathscr{F}(\mathbf{p}'))(\mathbf{p} - \mathbf{p}') > c\|\mathbf{p} - \mathbf{p}'\|^2, \quad \forall \mathbf{p} \in \mathscr{A}, \quad \forall \mathbf{p}' \in \mathscr{A}$.

Note that strong monotonicity implies ζ-monotonicity, ζ-monotonicity implies strict monotonicity, and strict monotonicity implies monotonicity, but the converse is not true.

Remark 3.1. The conditions for the strong monotonicity of $\mathscr{F}$ are also the conditions for the NE's uniqueness. To obtain the conditions for the strong monotonicity of $\mathscr{F}$, we write [2]

$$\alpha_n(\mathbf{p}) \triangleq \text{smallest eigenvalue of} - \nabla^2_{\mathbf{p}_n} v_n(\mathbf{p}_n, \mathbf{p}_{-n}),$$
$$\beta_{nm}(\mathbf{p}) \triangleq \| - \nabla_{\mathbf{p}_n \mathbf{p}_m} v_n(\mathbf{p}_n, \mathbf{p}_{-n})\|_2, \quad \forall n \neq m,$$

where $\nabla^2_{\mathbf{p}_n} v_n (\mathbf{p}_n, \mathbf{p}_{-n})$ and $\nabla_{\mathbf{p}_n \mathbf{p}_m} v_n (\mathbf{p}_n, \mathbf{p}_{-n})$ are the $K \times K$ Jacobian matrices of $\mathscr{F}_n(\mathbf{p})$ with respect to $\mathbf{p}_n$ and $\mathbf{p}_m$, respectively, and $\| - \nabla_{\mathbf{p}_n \mathbf{p}_m} v_n (\mathbf{p}_n, \mathbf{p}_{-n}) \|_2$ is the l_2-norm of $v_n (\mathbf{p}_n, \mathbf{p}_{-n})$. Let

$$\alpha_n^{\texttt{min}} \triangleq \inf_{\mathbf{p} \in \mathscr{A}} \alpha_n(\mathbf{p}), \tag{3.7}$$

$$\beta_{nm}^{\texttt{max}} \triangleq \sup_{\mathbf{p} \in \mathscr{A}} \beta_{nm}(\mathbf{p}), \tag{3.8}$$

for all players, and as in Section 12 in [2], define the $N \times N$ matrix $\boldsymbol{\Upsilon}$ whose elements are

$$[\boldsymbol{\Upsilon}]_{nm} = \begin{cases} \alpha_n^{\texttt{min}}, & \text{if} \quad m = n, \\ -\beta_{nm}^{\texttt{max}}, & \text{if} \quad m \neq n. \end{cases} \tag{3.9}$$

When $\boldsymbol{\Upsilon}$ is a P-matrix, the mapping $\mathscr{F}$ is strongly monotone, and hence the nominal NE is unique (Theorem 12.5 in Section 12.4.1 in [2]). The matrix $\boldsymbol{\Upsilon}$ is a P-matrix if for any nonzero vector $\mathbf{x}$ we have

$$x_n(\boldsymbol{\Upsilon}\mathbf{x})_n > 0, \tag{3.10}$$

where x_n is the nth element of $\mathbf{x}$ [24].

From Theorem 3.2 and Remark 3.1 we can derive the conditions for the NE's uniqueness when there is no uncertainty in parameter values. Also, when the best response is a standard function or when the game belongs to potential games, it can be proved that the game has a unique NE.

3.2.1.3 Existence and Uniqueness of NE in Nominal Noncooperative Power Control Games with Homogeneous Users

Consider a wireless network with homogeneous users in the unlicensed band, in which K channels are shared between N noncooperative users where each user consists of one pair of transmitter and receiver. The system model for this set is similar to the system model in Section 1.4 of Chapter 1. The power control problem can be formulated by a strategic noncooperative game, $\mathscr{G} = \{\mathscr{N}, (v_n)_{n\in\mathscr{N}}, \mathscr{A}\}$, where v_n is the utility of user n. Generally, the utility function of user n is

$$v_n(\mathbf{p}_n, \mathbf{p}_{-n}) = \sum_{k=1}^{K} v_n^k\left(p_n^k, \mathbf{p}_{-n}\right).$$

The interference on user n caused by other users is an additive function of the latter's transmit power levels, and the game is called an additively coupled game [34–36]. This type of games can model a number of practical problems such as the downlink transmit power allocation in a digital subscriber line access multiplexer (DSLAM) and routing delay minimization in Jackson networks [37].

The utility function of user n in this game is

$$v_n(\mathbf{p}_n, \mathbf{f}_n(\mathbf{p}_{-n}, \mathbf{s}_n)) = \sum_{k=1}^{K} v_n^k\left(p_n^k, f_n^k(\mathbf{p}_{-n}, \mathbf{s}_n)\right),$$

where $\mathbf{f}_n(\mathbf{p}_{-n}, \mathbf{s}_n) = [f_n^1(\mathbf{p}_{-n}, \mathbf{s}_n), \ldots, f_n^K(\mathbf{p}_{-n}, \mathbf{s}_n)]$ is the $1 \times K$ vector of noise plus other users' interference on user n, that is, $f_n^k(\mathbf{p}_{-n}, \mathbf{s}_n) = \sum_{m\in\mathscr{N},\, m\neq n} p_m^k h_{mn}^k + \sigma^2$, where σ^2 is the noise power assumed to be equal at all receivers in all channels. Also, $\mathbf{s}_n \triangleq [\mathbf{h}_{n1}, \ldots, \mathbf{h}_{n(n-1)}, \mathbf{h}_{n(n+1)}, \ldots, \mathbf{h}_{nN}, \boldsymbol{\sigma}_n^2]$ is the vector of side (system) information (SI) for user n, where $\mathbf{h}_{nm}$ is a $1 \times K$ vector whose element h_{nm}^k is the channel gain between user m and user n, and $\boldsymbol{\sigma}_n^2$ is a $1 \times K$ vector of noise power over K channels of user n. In what follows, when there is no ambiguity, we drop the arguments $\mathbf{p}_{-n}$ and $\mathbf{s}_n$ in $\mathbf{f}_n(\mathbf{p}_{-n}, \mathbf{s}_n)$ for convenience.

We consider the case in which the strategy of user n is its transmit power defined by (1.20) in Chapter 1, its signal-to-interference-plus-noise ratio (SINR) is defined by (1.19) in Chapter 1, and its utility v_n is its throughput as defined by (1.21) in Chapter 1. For this type of utility function in wireless networks, the following assumptions are made:

A1: The utility of user n is a strictly concave and differentiable function of $\mathbf{p}_n$, and its gradient is bounded;
A2: The utility of user n is a decreasing and convex function of $f_n^k(\mathbf{p}_{-n}, \mathbf{s}_n)$;
A3: The second-order mixed partial derivatives of the utility function of user n, that is, $\frac{\partial^2 v_k^n}{\partial p_k^n \partial f_k^n}$ and $\frac{\partial^2 v_k^n}{\partial f_k^n \partial p_k^n}$, exist and are continuous.

When users are greedy and noncooperative, each user n aims to maximize its own utility subject to its strategy space via $\max_{\mathbf{p}_n\in\mathscr{A}_n} v_n(\mathbf{p})$. The NE of this game is denoted by the strategy profile $\mathbf{p}^* = [\mathbf{p}_1^*, \ldots, \mathbf{p}_N^*]$. It can be shown that the NE for $\mathscr{G}$ exists since the conditions in Theorem 3.1 hold for the set $\mathscr{A}$ in (1.20) in Chapter 1, that is, $\mathscr{A}$ is a nonempty, compact, finite, and convex set, and $v_n(\mathbf{p})$ is continuous and convex. The uniqueness condition for the NE of this game is studied extensively in [3, 4, 38–42]. In what follows, we compare the conditions for the NE's uniqueness in the best response and VI approaches.

- **NE's uniqueness in the best response approach:** Theorem 2 in [3] proves that the nominal power allocation game $\mathscr{G} = \{\mathscr{N}, (v_n)_{n\in\mathscr{N}}, \mathscr{A}\}$ has a unique NE when

$$\rho(\mathbf{S}(k)) \leq 1, \quad \forall k \in \mathscr{K}, \tag{3.11}$$

where

$$[S(k)]_{mn} = \begin{cases} 0, & \text{if} \quad m = n, \\ \frac{h_{nm}^k p_n^{\max}}{h_{nn}^k p_m^{\max}}, & \text{if} \quad m \neq n. \end{cases} \tag{3.12}$$

Note that (3.11) holds when the best response of the game is obtained via the water-filling algorithm. In [3], it is also shown that to satisfy (3.11) and guarantee the NE's uniqueness, the interference on each receiver should be below a specific threshold.

- **NE's uniqueness in the VI approach:** From Remark 3.1, the NE is unique in the VI approach when $\boldsymbol{\Upsilon}$ in (3.9) is a P-matrix [2, 37, 43, 44] and (3.7) and (3.8) are

$$\alpha_n^{\min} = \min_{k\in\mathscr{K}} \left(\frac{h_{nn}^k}{\sigma^2 + \sum_{m\in\mathscr{N}} (p_m^k)^{\max} h_{mn}^k} \right)^2,$$

$$\beta_{nm}^{\max} = \max_{k\in\mathscr{K}} \frac{h_{nn}^k h_{mn}^k}{\left(\sigma^2 + \sum_{m\in\mathscr{N}} (p_m^k)^{\min} h_{mn}^k\right)^2},$$

where $(p_n^k)^{\min}$ and $(p_n^k)^{\max}$ are defined by (1.20) in Chapter 1. The matrix $\boldsymbol{\Upsilon}$ in (3.9) is a P-matrix when (3.10) holds. For this case, (3.10) can be rewritten as [45]

$$\begin{aligned} &\min_{k\in\mathscr{K}} \frac{h_{nn}^k w_n^k}{\left(\sigma^2 + \sum_{m\in\mathscr{N}} (p_m^k)^{\max} h_{mn}^k\right)^2} \\ &\quad > \sum_{m\neq n} \max_{k\in\mathscr{K}} \frac{h_{mn}^k w_m^k}{\left(\sigma^2 + \sum_{m\in\mathscr{N}} (p_m^k)^{\min} h_{mn}^k\right)^2}, \quad \forall n \in \mathscr{N}, \quad \forall w_n^k \in \mathscr{A}_n. \end{aligned} \tag{3.13}$$

A physical interpretation of (3.13) as presented in [44] is as follows. We denote the maximum interference on the receiver of user n in channel k by $f_{n,k}^{\max} = \left(\sigma^2 + \sum_{m\in\mathscr{N}} (p_m^k)^{\max} h_{mn}^k\right)^2$ and the minimum interference on the receiver of user n in channel k by $f_{n,k}^{\min} = \left(\sigma^2 + \sum_{m\in\mathscr{N}} (p_m^k)^{\min} h_{mn}^k\right)^2$. Now, the minimum SINR of user n over all channels is $\min_{k\in\mathcal{K}} \frac{h_{nn}^k w_n^k}{f_{n,k}^{\mathtt{max}}}$, and the normalized interference of user m on user n in channel k is $\frac{h_{nm}^k w_m^k}{I_{n,k}^{\mathtt{min}}}$. From (3.13), when the minimum SINRs of all users are greater than the sum of the maximum normalized interference levels, the nominal NE is unique. Also, when interference is low and channel gains from transmitters to their intended receivers are high, the nominal NE is unique.

By comparing the uniqueness conditions in (3.12) and (3.13), we note the following points:

- The uniqueness conditions depend on channel realizations (e.g., the distance between transmitters and receivers) [3]. In both approaches, the NE's uniqueness is guaranteed when the interference is sufficiently low;
- The two approaches have different conditions for the NE's uniqueness.

The choose between the best response approach or the VI approach, the probability of satisfying the conditions for each approach should be compared with the same for the other approach. The approach for which the uniqueness condition is satisfied for a wider range of channel realizations is preferred.

3.2.2 Social Utility (Sum Rate) at NE

When the game has a unique NE, its throughput at the equilibrium may not be optimal, that is, there may be a gap between the throughput at the NE and the optimal throughput. This gap is a measure of efficiency for the solution at the NE. When the game has multiple equilibria, it may have a higher throughput at one of the equilibrium points as compared to other equilibrium points. In what follows, we focus on the aforementioned gap with a view to developing algorithms and schemes for reducing this gap and improving the performance [3, 46–48]. We begin by introducing the concepts of Pareto optimality and Pareto efficiency.

Definition 3.4 ([1, 2]). For the strategic game $\mathcal{G} = \{\mathcal{N}, (v_n)_{n\in\mathcal{N}}, \mathcal{A}\}$ and two action profiles $\mathbf{p} \in \mathcal{A}$ and $\mathbf{p}' \in \mathcal{A}$, $\mathbf{p}$ is said to be Pareto dominant on $\mathbf{p}'$ when

$$v_n(\mathbf{p}) \geq v_n(\mathbf{p}'), \quad \forall n \in \mathcal{N},$$

and for at least one user m

$$v_m(\mathbf{p}) > v_m(\mathbf{p}'), \quad \exists m \in \mathcal{N}.$$

A strategic profile $\mathbf{p}$ is Pareto efficient (optimal) when there is no other feasible strategy that Pareto dominates $\mathbf{p}$.

In general, obtaining a Pareto-efficient solution requires knowledge of the complete system information pertaining to all users, which is not the case in the game-theoretic approach of formulating resource allocation problems, where each user obtains its own solution by utilizing the locally available information. To obtain the globally optimal solution, we first obtain the maximum utility for all strategy profiles by solving the following multi-objective optimization problem (MOP) [3]:

$$\max_{\forall \mathbf{p}_n \in \mathscr{A}_n} \{v_1(\mathbf{p}_1, \mathbf{p}_{-1}) \cdots v_N(\mathbf{p}_N, \mathbf{p}_{-N})\}. \tag{3.14}$$

Now, the Pareto-optimal solution is obtained by solving the following optimization problem [49, 50]:

$$\max_{\forall \mathbf{p}_n \in \mathscr{A}_n} \sum_{n \in \mathscr{N}} \varsigma_n v_n(\mathbf{p}_n, \mathbf{p}_{-n}), \tag{3.15}$$

where ς_n is a positive weight for user n, which can be regarded as its priority. In general, the NE is not Pareto efficient.

An interesting question is whether one can modify the utility function of players with a view to matching the NE of the modified game with the Pareto-optimal solution. To answer this question, let us consider a modified game $\mathscr{G}'' = \{\mathscr{N}, (v''_n)_{n \in \mathscr{N}}, \mathscr{A}\}$, in which the utility function of player n is

$$v''_n(\mathbf{p}_n, \mathbf{p}_{-n}) = v_n(\mathbf{p}_n, \mathbf{p}_{-n}) + \frac{1}{\varsigma_n} \sum_{m \neq n} \varsigma_m v_m(\mathbf{p}_m, \mathbf{p}_{-m}), \qquad \forall n \in \mathscr{N}. \tag{3.16}$$

It can be shown that for any given $\varsigma_n > 0$ for all $n \in \mathscr{N}$, the solution set of $\mathscr{G}''$ is not empty and contains the Pareto-optimal solution to (3.15), which is globally optimal. Note that the NE of $\mathscr{G}''$ is Pareto-optimal due to the fact that the utility function of each player is modified to take into account the strategies of other players. However, each player needs to know the utility values and strategies of other players in each iteration, which means a significant increase in message passing between players. Hence, although a Pareto-optimal solution is desirable and can be obtained, the significant cost of additional signaling and coordination among players is prohibitive and not desirable.

Another approach to improving performance and achieving Pareto optimality is to introduce pricing to each player proportional to its use of resources. The pricing-based utility function for player n is

$$v_n^{\text{pricing}}(\mathbf{p}_n, \mathbf{p}_{-n}) = v_n(\mathbf{p}_n, \mathbf{p}_{-n}) - \theta_n(\mathbf{p}_n), \quad \forall n \in \mathcal{N}, \tag{3.17}$$

where $\theta_n(\mathbf{p}_n)$ is a pricing function that depends only on $\mathbf{p}_n$ for user $n \in \mathcal{N}$. Different pricing functions can be considered for wireless networks [51], for example:

- A linear function of the transmit power, that is, $\theta_n(\mathbf{p}_n) = c \sum_{k \in \mathcal{K}} p_n^k$ for the uplink or downlink [52–54];
- A linear function of interference, that is, $\theta_n(\mathbf{p}_n) = c \sum_{q \in \mathcal{Q}} \sum_{k \in \mathcal{K}} p_n^k g_{nq}$ caused by secondary users (SUs) on primary users (PUs) [45, 55].

Note that pricing-based distributed schemes require neither a central coordinator nor extra message passing between players. The latter is due to the fact that $\theta_n(\mathbf{p}_n)$ in (3.17) is only a function of $\mathbf{p}_n$ for user $n \in \mathcal{N}$ and does not depend on other users' parameters, for example, $\mathbf{p}_{-n}$. Although the solution may not be globally optimal, via (3.16), the global optimum can be achieved by more message passing to provide other users with information pertaining to the second term in (3.16). Note also that the second term in the modified utility function $v_n''(\mathbf{p}_n, \mathbf{p}_{-n})$ in (3.16) can be considered the optimal pricing [3].

3.2.3 Distributed Algorithms

Distributed algorithms are needed to implement game-theoretic formulations of resource allocation problems and attain the NE. When users are noncooperative and there is no central entity to coordinate users, the general class of totally asynchronous algorithms is of interest [2, 56], where there is no specific sequence for updating user strategies. To mathematically describe this general framework, let the iteration number t in the discrete set $\mathcal{T} = 0, 1, 2, \ldots$ be the updating time for the strategy $\mathbf{p}_n(t)$ of user n. In a totally asynchronous algorithm, each user n has its own set $\mathcal{T}_n \subset \mathcal{T}$ containing all instances of updating its transmit power (strategy). In other words, at $t \in \mathcal{T}_n$, the transmit power of user n is updated by solving its own optimization problem (3.1) or is kept unchanged. Let τ_n^m be the most recent time at which the strategy profile of user m is observed by user n and $0 \leq \tau_n^m \leq t$. When user n updates its strategy at t, it maximizes its utility via (3.1) by considering the last updated strategy profile of other users at τ_n^m for all $m \in \mathcal{N}$ and $m \neq n$. We denote this strategy by $\mathbf{p}_n^*(\tau_n^{-n})$, where τ_n^{-n} is the last instance at which the strategy profiles of all users except user n were updated. The following three assumptions hold in a totally asynchronous algorithm: (1) $0 \leq \tau_n^m \leq t$, (2) $\lim_{t\to\infty} \tau_n^m = \infty$, and (3) $|\mathcal{T}_n| = \infty$ [2, 56].

A totally asynchronous algorithm is described in Table 3.1 that covers sequential (Gauss–Seidel) and simultaneous (Jacobi) schemes for updating users' strategies, as explained further in the sequel.

Table 3.1 Totally asynchronous distributed algorithm

Set any $\mathbf{p}_n(t=0) \in \mathscr{A}_n$ for all $n \in \mathscr{N}$
Set $t = 0$
Repeat
For $n = 1, 2, \ldots, N$

$$\textbf{Step 1: } \mathbf{p}_n(t+1) = \begin{cases} \mathbf{p}_n^*(\tau_n^{-n}), & \text{if} \quad t \in \mathscr{T}_n, \\ \mathbf{p}_n(t), & \text{otherwise.} \end{cases}$$

End if the condition for convergence is satisfied
otherwise $t = t + 1$

Table 3.2 Sequential distributed algorithm

Set any $\mathbf{p}_n(t=0) \in \mathscr{A}_n$ for all $n \in \mathscr{N}$
Set $t = 0$
Repeat
For $n = 1, 2, \ldots, N$

$$\textbf{Step 1: } \mathbf{p}_n(t+1) = \begin{cases} \mathbf{p}_n^*(\tau_n^{-n}), & \text{if} \quad (t+1) \bmod N = n, \\ \mathbf{p}_n(t), & \text{otherwise.} \end{cases}$$

End if the condition for convergence is satisfied
otherwise $t = t + 1$

Table 3.3 Simultaneous distributed algorithm

Set $\mathbf{p}_n(t=0) \in \mathscr{A}_n$ for all $n \in \mathscr{N}$
Set $t = 0$
Repeat
Step 1: $\mathbf{p}_n(t+1) = \mathbf{p}_n^*(\tau_n^{-n}), \quad \forall n \in \mathscr{N}$
End if the condition for convergence is satisfied
otherwise $t = t + 1$ and go to **Step 1**

- **Sequential Distributed Algorithm:** A sequential iterative algorithm based on the Gauss–Seidel scheme [4] is shown in Table 3.2, where each user sequentially updates its strategy by utilizing the best response approach.
- **Simultaneous Distributed Algorithm:** A simultaneous distributed algorithm based on the Jacobi scheme [56] is shown in Table 3.3, where in each iteration l, all users simultaneously update their strategies, each by taking into account the measured interference in the previous iteration [3].

The condition for the convergence of the distributed algorithms in Tables 3.1, 3.2, and 3.3 is stated as follows in Theorem 3.5.

Theorem 3.5 ([2]). *For a given game $\mathscr{G} = \{\mathscr{N}, (v_n)_{n\in\mathscr{N}}, \mathscr{A}\}$, there exists, a unique equilibrium point at which the totally asynchronous algorithm converges when $\mathbf{B}(\mathbf{p}^*)$ in (3.3) is a contraction with respect to the block-*

(continued)

Theorem 3.5 (continued)
maximum norm with modulus $\alpha \in [0, 1)$, *stated by*

$$\|\mathbf{B}(\mathbf{p}) - \mathbf{B}(\mathbf{p}')\|_{\text{block}} \le \alpha \|\mathbf{p} - \mathbf{p}'\|_{\text{block}}, \quad \forall \mathbf{p} \in \mathscr{A}, \quad \forall \mathbf{p}' \in \mathscr{A}, \tag{3.18}$$

where $\|\mathbf{B}(\mathbf{p})\|_{\text{block}} = \max_{n\in\mathscr{N}} \|\mathbf{B}_n(\mathbf{p})\|_n$, $\|\cdot\|_n$ *is the vector's norm, and* $\mathscr{A}$ *is a closed set.*

The algorithms in Tables 3.1, 3.2, and 3.3 can be extended to other cases in which users update their transmit power levels using other approaches (i.e., not via (3.1)). For instance, in memory-based updating, each user updates its transmit power by

$$\mathbf{p}_n(t+1) = \alpha_n \mathbf{p}_n(t) + (1-\alpha_n)\mathbf{p}_n^*(\tau_n^{-n}), \quad \forall n \in \mathscr{N}, \tag{3.19}$$

where $\alpha_n \in [0, 1)$ for all $n \in \mathscr{N}$ is the forgetting factor that smooths variations in channel gains in (3.19). When the channel gains are relatively stable, we set $\alpha_n \approx 1$, and when they are highly fluctuating, it is better to set $\alpha_n \ll 1$. Theorem 2 in [4] shows that the value of α_n affects the speed of the algorithm's convergence. The convergence of the memory-based distributed algorithm is guaranteed when its best response function is a contraction mapping, as stated earlier in Theorem 3.5.

3.3 Worst-Case Robust Power Control in Noncooperative Games

In this section, we consider uncertainty in users' observations in robust games for distributed power allocation in wireless networks and analyze their solutions.

3.3.1 Robust Power Control for Noncooperative Homogeneous Users

We now present a general form of worst-case robust game for the nominal game in Section 3.2.1.3 in this chapter, followed by an analysis of their respective equilibria. We will also compare the throughput of the nominal solution with its robust counterpart.

Homogeneous users are similar in their ability to measure the interference caused by other users, which we assume to be uncertain. We formulate the robust power control for each user via game theory. To obtain the sufficient condition for the existence and uniqueness of the RNE, we apply VI [18–20, 24] and show that when uncertainty is bounded and convex, the RNE always exists. We also show that when uncertainty is small, the RNE is a bounded perturbed version of the nominal NE and derive the condition for the uniqueness of the RNE, which is similar to the condition for the uniqueness of the nominal NE. When the nominal NE is unique, we show that the social utility (the sum of utilities of all users) at the RNE is always less than that at the nominal NE and derive the upper bound for the difference between users' transmit power at the RNE and at the nominal NE. When the nominal NE is not unique, we show that the social utility at the RNE may be higher than that at the nominal NE and derive a sufficient condition for this phenomenon. Finally, we use the proximal response map associated with the worst-case utility function to propose a distributed algorithm for reaching the RNE and derive the sufficient condition for its convergence.

Consider the case where the transmitter n does not know the exact value of interference caused by other transmitters on its receiver. The uncertain interference $\mathbf{f}_n$ is modeled by

$$\mathbf{f}_n = \bar{\mathbf{f}}_n + \widehat{\mathbf{f}}_n, \quad \forall n \in \mathscr{N},$$

where $\bar{\mathbf{f}}_n = [\bar{f}_n^1, \ldots, \bar{f}_n^K]$ and $\widehat{\mathbf{f}}_n = [\widehat{f}_n^1, \ldots, \widehat{f}_n^K]$ are the nominal (exact) value and the error in the measured interference on the intended receiver of transmitter n, respectively. We assume that uncertainties are bounded to the uncertainty region

$$\mathscr{R}_{\mathbf{f}_n}(\mathbf{p}_{-n}) = \{\mathbf{f}_n | \; \|\widehat{\mathbf{f}}_n\|_2 \le \varepsilon_{\mathbf{f}_n}\}, \quad \forall n \in \mathscr{N},$$

where $\varepsilon_{\mathbf{f}_n} \ge 0$ is the bound on $\widehat{\mathbf{f}}_n$, and $\| \cdot \|_2$ is the l_2-norm [22, 57, 58].

We rewrite the optimization problem (1.9) in Chapter 1 for user n as [18]

$$\widetilde{u}_n = \max_{\mathbf{p}_n \in \mathscr{A}_n} \; \min_{\mathbf{f}_n \in \mathscr{R}_{\mathbf{f}_n}(\mathbf{p}_{-n})} u_n(\mathbf{p}_n, \mathbf{f}_n), \tag{3.20}$$

where $\widetilde{u}_n$ is the expected utility of user n in the worst-case approach, and $u_n(\mathbf{p}_n, \mathbf{f}_n)$ is the utility function of user n defined as

$$u_n(\mathbf{p}_n, \mathbf{f}_n) = \sum_{k=1}^{K} u_n^k(p_n^k, f_n^k), \quad \forall n \in \mathscr{N}.$$

When the uncertainty region shrinks to zero (i.e., when there is no uncertainty), the utility functions of the nominal and robust optimization problems are the same, and we have [21, 22]

$$v_n(\mathbf{p}_n, \bar{\mathbf{f}}_n) = u_n(\mathbf{p}_n, \mathbf{f}_n)|_{\varepsilon_{\mathbf{f}_n}=0}, \quad \forall n \in \mathscr{N}. \tag{3.21}$$

The robust game is

$$\widetilde{\mathscr{G}} = \{\mathscr{N}, (u_n)_{n\in\mathscr{N}}, \widetilde{\mathscr{A}}(\mathbf{p})\},$$

where $\widetilde{\mathscr{A}}(\mathbf{p}) = \prod_{n=1}^{N} \widetilde{\mathscr{A}}_n(\mathbf{p}_{-n})$, in which the strategy set for (3.20) is

$$\widetilde{\mathscr{A}}_n(\mathbf{p}_{-n}) = \mathscr{A}_n \times \mathscr{R}_{\mathbf{f}_n}(\mathbf{p}_{-n}), \quad \forall n \in \mathscr{N}. \tag{3.22}$$

Note that $\widetilde{\mathscr{A}}_n(\mathbf{p}_{-n})$ highlights the fact that the strategy of each user in the robust scheme depends on other users' strategies, which is not the case in the nominal (nonrobust) scheme. This implies that the robust game's equilibrium and its analysis are different from the nominal game's NE and its analysis [2].

3.3.1.1 Existence and Uniqueness of RNE

In general, when the strategy of each user depends on the strategies of other users, the generalized Nash equilibrium (GNE) as defined in what follows is used instead of the NE [2].

Definition 3.5. When the strategy of each user depends on the strategies of other users, the game denoted by $\underline{\mathscr{G}} = \{\mathscr{N}, (v_n)_{n\in\mathscr{N}}, \underline{\mathscr{A}}(\mathbf{p})\}$ is called a generalized game, and the vector $\mathbf{p}^*$ is the GNE point, at which we have

$$v_n(\mathbf{p}^*) \geq v_n(\mathbf{p}_n, \mathbf{p}^*_{-n}), \quad \forall \mathbf{p}_n \in \underline{\mathscr{A}}_n(\mathbf{p}), \quad \forall n \in \mathscr{N}.$$

In [2], several important nominal (i.e., no uncertainty) generalized games are introduced. In Section 12.2.1 of [2], different aspects of multifunction and set-valued mapping that can be used to derive the conditions for the existence and uniqueness of the GNE are discussed. It is shown that the continuity of $\underline{\mathscr{A}}(\mathbf{p})$ is critical for the existence and uniqueness of the GNE for $\underline{\mathscr{G}}$. It is also shown that in many practical cases, $\underline{\mathscr{A}}(\mathbf{p})$ is

$$\underline{\mathscr{A}}(\mathbf{p}) = \left\{\mathbf{p}_n | \, \mathbf{A}_n\mathbf{p}_n^{\mathrm{T}} = b_n(\mathbf{p}_{-n}) \quad \text{and} \quad g_n(\mathbf{p}_n, \mathbf{p}_{-n}) \leq 0, \quad \forall n \in \mathscr{N}\right\}, \tag{3.23}$$

where $\mathbf{A}_n \in \mathbb{R}^{l_n \times K}$ is a given matrix, $b_n(\mathbf{p}_{-n})$ is a given vector function of the strategy vectors $\mathbf{p}_{-n} \in \mathbb{R}^{l_n}$, and $g_n(\mathbf{p}_n, \mathbf{p}_{-n}) : \mathbb{R}^N \rightarrow \mathbb{R}^{m_n}$ is a given vector function that is convex with respect to $\mathbf{p}_n$. The continuity of $\underline{\mathscr{A}}(\mathbf{p})$ discussed in Propositions 12.3 and 12.4 of [2] is used to establish the existence of the GNE for $\underline{\mathscr{G}}$.

In Appendix 1, we establish the connection between the GNE and VI and show its application for the nominal resource allocation problem of CRNs under global interference threshold constraints. In general, obtaining the GNE is not easy due to the interdependence of users' strategies. One approach to dealing with this difficulty is to reformulate the generalized optimization problem via the quasi-variational inequality (QVI) and derive the conditions for the existence and uniqueness of its solution [24, 59]. The alternative is to avoid the GNE by reformulating the optimization problem and analyze its RNE instead. In what follows, we show how to do this.

The solution to (3.20) (its saddle point) for each user in the robust game $\widetilde{\mathscr{G}}$ is a pair $(\widetilde{\mathbf{p}}'_n, \mathbf{f}'_n) \in \widetilde{\mathscr{A}}_n(\mathbf{p}_{-n})$ that satisfies [6, 37]

$$\max_{\mathbf{p}_n \in \mathscr{A}_n} u_n(\mathbf{p}_n, \mathbf{f}'_n) = u_n(\widetilde{\mathbf{p}}'_n, \mathbf{f}'_n) = \min_{\mathbf{f}_n \in \mathscr{R}_{\mathbf{f}_n}(\mathbf{p}_{-n})} u_n(\widetilde{\mathbf{p}}'_n, \mathbf{f}_n). \tag{3.24}$$

Definition 3.6. The RNE of the robust game $\widetilde{\mathscr{G}}$ corresponds to the strategy profile $\widetilde{\mathbf{p}}^* = [\widetilde{\mathbf{p}}_1^*, \ldots, \widetilde{\mathbf{p}}_N^*]$ when for any other strategy $\widetilde{\mathbf{p}}_n$ we have [18–20]

$$\min_{\mathbf{f}_n(\widetilde{\mathbf{p}}^*_{-n}, \mathbf{s}_n) \in \mathscr{R}_{\mathbf{f}_n}(\mathbf{p}^*_{-n})} u_n(\widetilde{\mathbf{p}}_n^*, \mathbf{f}_n(\widetilde{\mathbf{p}}^*_{-n}, \mathbf{s}_n)) \geq \tag{3.25}$$
$$\min_{\mathbf{f}_n(\widetilde{\mathbf{p}}^*_{-n}, \mathbf{s}_n) \in \mathscr{R}_{\mathbf{f}_n}(\mathbf{p}^*_{-n})} u_n(\widetilde{\mathbf{p}}_n, \mathbf{f}_n(\widetilde{\mathbf{p}}^*_{-n}, \mathbf{s}_n)), \quad \forall\, \widetilde{\mathbf{p}}_n \in \mathscr{A}_n,$$

where $\widetilde{\mathbf{p}}^*_{-n} = [\widetilde{\mathbf{p}}_1^*, \ldots, \widetilde{\mathbf{p}}_{n-1}^*, \widetilde{\mathbf{p}}_{n+1}^*, \ldots, \widetilde{\mathbf{p}}_N^*]$. At the RNE, the achieved utility of user n is $\widetilde{u}_n^*$, and the social utility is $\widetilde{u}^* = \sum_{n=1}^N \widetilde{u}_n^*$.

At the RNE, each user reaches its maximum utility under the worst-case uncertainty, and no user can reach a higher utility by unilaterally changing its strategy. Note the difference with the nominal game, at whose NE each user aims to maximize its utility by choosing a strategy from its strategy set without considering uncertainty in $\mathbf{f}_n$. When $\varepsilon_{\mathbf{f}_n} = 0$, the RNE and the NE are identical.

In studying the existence of the RNE, we note the following points:

- By considering uncertainty, the strategy space of user n is $\widetilde{\mathscr{A}}_n(\mathbf{p}_{-n}) = \mathscr{A}_n \times \mathscr{R}_{\mathbf{f}_n}(\mathbf{p}_{-n})$, which is a set-valued mapping that depends on the actions of other users;
- A closed-form solution for the best response map does not exist when uncertainty is considered.

The preceding points show that the convexity of each user's optimization problem is not sufficient for the existence of RNE, and we need to consider other approaches to derive the conditions for the RNE's existence. In doing so, we will utilize Proposition 12.4 in [2], and the utility function of user n proposed in [37] as

$$\Psi_n(\mathbf{p}_n, \mathbf{p}_{-n}) = \min_{\mathbf{f}_n \in \mathscr{R}_{\mathbf{f}_n}(\mathbf{p}_{-n})} u_n(\mathbf{p}_n, \mathbf{f}_n) = u_n(\mathbf{p}_n, \mathbf{f}_n^*), \tag{3.26}$$

and $\mathbf{f}_n^* = \bar{\mathbf{f}}_n - \varepsilon_{\mathbf{f}_n} \boldsymbol{\vartheta}_n$, where $\mathbf{f}_n^* = [f_n^{1*}, \ldots, f_n^{K*}]$, $\boldsymbol{\vartheta}_n = [\vartheta_n^1, \ldots, \vartheta_n^K]$, in which ϑ_n^k is

$$\vartheta_n^k = \frac{\frac{\partial u_n^k(\mathbf{p}_n, \mathbf{f}_n^*)}{\partial \widetilde{f}_n^k}}{\sqrt{\sum_{k=1}^K \left(\frac{\partial u_n^k(\mathbf{p}_n, \mathbf{f}_n^*)}{\partial \widetilde{f}_n^k}\right)^2}}. \tag{3.27}$$

In [37], it is shown that $\Psi_n(\mathbf{p}_n, \mathbf{p}_{-n})$ is a concave, continuous, and differentiable function of $\mathbf{p}_n$ for every $\mathbf{p}_{-n}$. For proof, see Appendix 2 in this chapter.

Note that the optimization problem for user n is $\widetilde{u}_n = \max_{\mathbf{p}_n \in \mathscr{A}_n} \Psi_n(\mathbf{p}_n, \mathbf{p}_{-n})$, and the robust game is reformulated as $\widetilde{\mathscr{G}} = \{\mathscr{N}, (\Psi_n)_{n \in \mathscr{N}}, \mathscr{A}\}$, whose RNE can be considered as the solution to the VI problem via the mapping $\widetilde{\mathscr{F}}(\mathbf{p}) = \left(\widetilde{\mathscr{F}}_n(\mathbf{p})\right)_{n=1}^N$, where $\widetilde{\mathscr{F}}_n(\mathbf{p}) = -\frac{\partial \Psi_n(\mathbf{p}_n, \mathbf{p}_{-n})}{\partial \mathbf{p}_n}$. From (3.27), $\widetilde{\mathscr{F}}(\mathbf{p})$ is a set-valued mapping, and the RNE can be obtained via generalized variational inequality (GVI) [20], that is, $\widetilde{\mathbf{p}}^*$ is the RNE when it is a solution to $GVI(\mathscr{A}, \widetilde{\mathscr{F}})$. Note that by rewriting the utility function as $\Psi_n(\mathbf{p}_n, \mathbf{p}_{-n})$, the strategy space of each user n is $\mathscr{A}_n$. The following theorem states the conditions for the existence of the solution to $GVI(\mathscr{A}, \widetilde{\mathscr{F}})$.

Theorem 3.6. *For any set of parameter values, users' actions, and the bound on the uncertainty region, there always exists a RNE for $\widetilde{\mathscr{G}}$.*

Proof. From Assumptions A1–A3 in Section 3.2.1.3 in this chapter, it is easy to show that all the assumptions in Lemma 3.1 and Theorem 3.2 in [20] hold, and $GVI(\mathscr{A}, \widetilde{\mathscr{F}})$ has a solution. Hence, $\widetilde{\mathscr{G}}$ has a RNE.

Since a closed-form solution to (3.20) cannot be obtained, the fixed-point algorithm and the contraction mapping cannot be applied as in [3, 34] to derive the sufficient conditions for the RNE's uniqueness. However, from the definition of the mapping $\widetilde{\mathscr{F}}_n$ that is based on $\Psi_n(\mathbf{p}_n, \mathbf{p}_{-n})$, it is obvious that there exists a relationship between $\widetilde{\mathscr{F}}_n$ and $\mathscr{F}_n$. Now, the question is how to exploit this relationship in analyzing the RNE when there is no closed-form expression for the RNE. In [37], it is shown that when uncertainty is small, the RNE is the bounded perturbed version of the NE of the nominal game, and the condition for the RNE's uniqueness can be derived without a closed-form solution to (3.20). This is stated in the following lemma.

Lemma 3.1. *For small values of $\varepsilon_{\mathbf{f}_n}$ compared to the nominal parameter values for all $n \in \mathscr{N}$, the mapping $\widetilde{\mathscr{F}}(\mathbf{p})$ is a bounded perturbed version of the mapping $\mathscr{F}(\mathbf{p})$, that is, there exists a $0 < \wp < \infty$ such that $\|\widetilde{\mathscr{F}}_n(\mathbf{p}) - \mathscr{F}_n(\mathbf{p})\|_2 \leq \wp$.*

Proof. See Appendix 3.

From Lemma 3.1, when the mapping $\widetilde{\mathscr{F}}(\mathbf{p})$ is a bounded perturbed version of $\mathscr{F}(\mathbf{p})$, the robust game $\widetilde{\mathscr{G}} = \{\mathscr{N}, (\Psi_n)_{n\in\mathscr{N}}, \mathscr{A}\}$ can be reformulated, and the following Theorem 3.7 establishes a sufficient condition for the RNE's uniqueness.

Theorem 3.7. *When $\boldsymbol{\Upsilon}$ in (3.9) is a P-matrix, for any small $\boldsymbol{\varepsilon} = [\varepsilon_{\mathbf{f}_1}, \ldots, \varepsilon_{\mathbf{f}_N}]$, the robust game $\widetilde{\mathscr{G}}$ has a unique RNE.*

Proof. From Lemma 3.1, the mapping $\widetilde{\mathscr{F}}(\mathbf{p})$ in *GVI*$(\mathscr{A}, \widetilde{\mathscr{F}})$ is set-valued and a perturbed version of $\mathscr{F}(\mathbf{p})$. The perturbation in $\mathscr{F}(\mathbf{p})$, defined as

$$\mathscr{Q} = \|\mathscr{F}(\mathbf{p}) - \widetilde{\mathscr{F}}(\mathbf{p})\|_2 \quad \forall \mathbf{p} \in \mathscr{A},$$

is bounded because the users' strategy space is bounded, and the uncertainty region is bounded and convex. In other words,

$$\widetilde{\mathscr{F}}(\mathbf{p}) \approx \mathscr{F}(\mathbf{p}) + \mathbf{q}, \tag{3.28}$$

where $\mathbf{q} = (\mathbf{q}_n)_{n=1}^N$, $\mathbf{q}_n = [q_n^1, \ldots, q_n^K]^{\mathrm{T}}$, and each q_n^k is bounded because perturbation is assumed to be bounded. In the following lemma, we use Eq. (3.28) to establish the conditions for RNE's uniqueness.

Lemma 3.2. *When the mapping $\mathscr{F}(\mathbf{p})$ is strongly monotone and $\widetilde{\mathscr{F}}(\mathbf{p})$ is a bounded perturbed version of $\mathscr{F}(\mathbf{p})$, $\widetilde{\mathscr{F}}(\mathbf{p})$ is strongly monotone.*

Proof. To establish the strong monotonicity of $\widetilde{\mathscr{F}}(\mathbf{p})$, we need to show that there exists a $c > 0$ such that

$$(\mathbf{p} - \mathbf{p}')\left(\widetilde{\mathscr{F}}(\mathbf{p}) - \widetilde{\mathscr{F}}(\mathbf{p}')\right) \geq c\|\mathbf{p} - \mathbf{p}'\|_2, \quad \forall \mathbf{p} \in \mathscr{A}, \quad \forall \mathbf{p}' \in \mathscr{A}.$$

To do so, when $\widetilde{\mathscr{F}}(\mathbf{p})$ is a bounded perturbed version of $\mathscr{F}(\mathbf{p})$, we use (3.28) and write

$$\begin{aligned}&(\mathbf{p} - \mathbf{p}')\left(\widetilde{\mathscr{F}}(\mathbf{p}) - \widetilde{\mathscr{F}}(\mathbf{p}')\right) \\ &\quad \approx (\mathbf{p} - \mathbf{p}')\Big(\mathscr{F}(\mathbf{p}) + \mathbf{q} - \mathscr{F}(\mathbf{p}') - \mathbf{q}\Big) = (\mathbf{p} - \mathbf{p}')\Big(\mathscr{F}(\mathbf{p}) - \mathscr{F}(\mathbf{p}')\Big).\end{aligned} \tag{3.29}$$

(continued)

Lemma 3.2 (continued)
When $\mathscr{F}(\mathbf{p})$ is strongly monotone, there exists a $c_1 > 0$ such that

$$(\mathbf{p}-\mathbf{p}')\Big(\mathscr{F}(\mathbf{p})-\mathscr{F}(\mathbf{p}')\Big) \geq c_1\|\mathbf{p}-\mathbf{p}'\|_2. \tag{3.30}$$

We use (3.30) to rewrite (3.29) as

$$(\mathbf{p}-\mathbf{p}')\Big(\widetilde{\mathscr{F}}(\mathbf{p})-\widetilde{\mathscr{F}}(\mathbf{p}')\Big) \geq c_1\|\mathbf{p}-\mathbf{p}'\|_2,$$

which means $\widetilde{\mathscr{F}}(\mathbf{p})$ is strongly monotone.

When (3.9) is a P-matrix, $v_n(\mathbf{p}_n, \mathbf{p}_{-n})$ is uniformly strongly convex as per Section 12.4.1 in [2], and its gradient vector $\mathscr{F}(\mathbf{p})$ in (3.5) is strongly monotone [24]. Hence, from the preceding lemma, the mapping $\widetilde{\mathscr{F}}(\mathbf{p})$ is strongly monotone. In addition, $\widetilde{\mathscr{F}}(\mathbf{p})$ is continuous since $\Psi_n(\mathbf{p}_n, \mathbf{p}_{-n})$ is continuous and convex. Thus, the assumptions of Theorem 4.3 in [20] hold, and $GVI(\mathscr{A}, \widetilde{\mathscr{F}})$ has a unique solution.

Since $\boldsymbol{\Upsilon}$ in (3.9) is a P-matrix, the mapping $\mathscr{F}(\mathbf{p})$ is strongly monotone, which is a sufficient condition for the uniqueness of the nominal NE as per Theorem 12.5 in Section 12.4.1 in [2]. When this condition holds and the mapping $\widetilde{\mathscr{F}}(\mathbf{p})$ is a bounded perturbed version of the mapping $\mathscr{F}(\mathbf{p})$, as per Lemma 3.2, $\widetilde{\mathscr{F}}(\mathbf{p})$ is also strongly monotone, which is a sufficient condition for the RNE's uniqueness as per Theorem 4.3 in [20]. Note that one can derive other sufficient conditions for the uniqueness of NE in the nominal game $\mathscr{G}$ by utilizing other approaches, but such conditions may or may not establish the uniqueness or even existence of RNE in the robust game $\widetilde{\mathscr{G}}$. However, it is interesting to note that the condition for the RNE's uniqueness in the previously given Theorem 3.7 is the same as the condition for the NE's uniqueness.

Now we consider a special case in which the channel gain from user n to user m on channel k is uncertain, modeled by $h^k_{nm} = \overline{h}^k_{nm} + \widehat{h}^k_{nm}$, and the utility function for user n on channel k is θ-proportionally fair, stated by [60]

$$v^k_n(p^k_n, f^k_n) = \begin{cases} \log(c^k_n + \frac{p^k_n}{f^k_n}), & \text{if} \quad \theta = -1, \\ \frac{(c^k_n + \frac{p^k_n}{f^k_n})^{\theta+1}}{\theta+1}, & \text{if} \quad -1 < \theta < 0, \end{cases} \tag{3.31}$$

where θ is the fairness factor. In this case, the robust game $\widetilde{\mathscr{G}}$ can be simplified to the robust game introduced in [5], and the requirement for strong monotonicity is relaxed to the positive definiteness of the affine mapping. The uncertainty region for each user is

$$\mathscr{R}_n^k = \left\{ \sqrt{\sum_{m=1, m\neq n}^{N} \left(\frac{\widehat{h}_{mn}^k}{\overline{h}_{nn}^k} \right)^2} \leq \epsilon_n^k \right\}, \quad \forall k \in \mathscr{K}, \tag{3.32}$$

where ϵ_n^k is the bound on the channel gains' uncertainty for user n in channel k.

The NE of the nominal game $\mathscr{G}$ can also be obtained by solving the affine variational inequality (AVI) problem, denoted by *AVI*$(\mathscr{A}, \mathscr{M})$, where $\mathscr{M}(\mathbf{p}) = (\mathscr{M}_n(\mathbf{p}))_{n=1}^N$, and

$$\mathscr{M}_n(\mathbf{p}) = \boldsymbol{\varpi}_n + \sum_{m=1}^{N} \mathbf{M}_{nm}\mathbf{p}_m^{\mathrm{T}}, \tag{3.33}$$

in which $\boldsymbol{\varpi}_n = (\varpi_n^k)_{k=1}^K$, $\varpi_n^k = \frac{\sigma_n^k + c_n^k}{\overline{h}_{nn}^k}$, and $\mathbf{M}_{mn} = \mathrm{diag}(\frac{\overline{h}_{mn}^k}{\overline{h}_{nn}^k})_{k=1}^K$. The nominal NE is unique when

$$\max_{n\in\mathscr{N}} \|\mathbf{p}_n\|_2 > \sum_{m\neq n} M_{nm}^{\max} \|\mathbf{p}_m\|_2, \quad \forall \mathbf{p}_n \in \mathscr{A}_n, \quad \forall n \in \mathscr{N}, \tag{3.34}$$

where $M_{nm}^{\max} = \max_{k\in\mathscr{K}} \frac{\overline{h}_{mn}^k}{\overline{h}_{nn}^k}$ when $m \neq n$, and $M_{nm}^{\max} = 0$ otherwise, in the $N \times N$ matrix $\mathbf{M}^{\max}$. From the preceding points, it is straightforward to derive the condition for the uniqueness of nominal NE when the utility function is either the throughput or (3.31). In such cases, the best response of the game is

$$p_n^k = \left[\mu_n^{\frac{1}{\theta}} - \varpi_n^k - \sum_{m\neq n} \frac{\overline{h}_{mn}^k}{\overline{h}_{nn}^k} p_m^k \right]_{(p_n^k)^{\min}}^{(p_n^k)^{\max}},$$

where the Lagrange multiplier μ_n for user n is chosen so as to satisfy

$$\mu_n \times \left(\sum_{k=1}^{K} p_n^k - p_n^{\max} \right) = 0, \quad \forall n \in \mathscr{N}. \tag{3.35}$$

The game has a unique NE when $\mathscr{M}(\mathbf{p})$ is strongly monotone. From Proposition 1 in [61], when (3.34) holds, $\mathscr{M}(\mathbf{p})$ is strongly monotone, and hence, the nominal NE is unique.

A robust game whose uncertainty region is (3.32) can be analyzed by its AVI mapping as per the following theorem.

Theorem 3.8. *When the uncertainty region is (3.32), the RNE is the solution of AVI($\widetilde{\mathscr{M}}(\mathbf{p}), \mathscr{A}$), where $\widetilde{\mathscr{M}}(\mathbf{p}) = (\widetilde{\mathscr{M}}_n(\mathbf{p}))_{n=1}^N$, and*

$$\widetilde{\mathscr{M}}_n(\mathbf{p}) \leq \mathscr{M}_n(\mathbf{p}) + \widehat{\mathscr{M}}_n(\mathbf{p}), \quad \forall n \in \mathscr{N}, \tag{3.36}$$

where $\widehat{\mathscr{M}}_n(\mathbf{p}_{-n}) = \left(\epsilon_n^k \|\mathbf{p}_{-n}^k\|\right)_{k=1}^K$ *and* $\mathbf{p}_{-n}^k = \left[p_1^k, \ldots, p_{n-1}^k, p_{n+1}^k, \ldots, p_N^k\right]$. *The RNE of* $\widetilde{\mathscr{G}}$ *is unique for any small* ϵ_n^k *when (3.34) holds.*

Proof. From Lemma 3.1, the mapping for the robust game $\widetilde{\mathscr{G}}$ is the perturbed mapping of the nominal game. Since the mapping for the nominal game is linear when the utility function is (3.31), the perturbed mapping is

$$\widetilde{\mathscr{M}}_n^k(\mathbf{p}) = \varpi_n^k + \sum_{m=1}^N \frac{h_{mn}^k}{h_{nn}^k} p_m^k, \quad \forall h_{mn}^k \in \mathscr{R}_{\mathbf{f}_n}^k, \tag{3.37}$$

where $\widetilde{\mathscr{M}}_n^k(\mathbf{p})$ and ϖ_n^k are the kth elements of $\widetilde{\mathscr{M}}_n(\mathbf{p})$ and $\boldsymbol{\varpi}_n$, respectively. Now, (3.37) can be rewritten as $\widetilde{\mathscr{M}}_n^k(\mathbf{p}) = \varpi_n^k + \sum_{m=1}^N \left(\frac{\bar{h}_{mn}^k}{\bar{h}_{nn}^k} + \frac{h_{mn}^k - \bar{h}_{mn}^k}{\bar{h}_{nn}^k}\right) p_m^k \leq \varpi_n^k + \sum_{m=1}^N \frac{\bar{h}_{mn}^k}{\bar{h}_{nn}^k} p_m^k + \epsilon_n^k \|\mathbf{p}_{-n}^k\|$. Therefore, the mapping at the RNE is bounded as per (3.36).

Next, we focus on the condition for the RNE's uniqueness. Since p_m^k is bounded in $[(p_m^k)^{\mathtt{min}}, (p_m^k)^{\mathtt{max}}]$, and the uncertainty region is bounded and small, the value of $\boldsymbol{\epsilon}_n^k \|\mathbf{p}_{-n}^k\|_2$ is bounded. Hence, for any bounded and small uncertainty region in the robust game $\widetilde{\mathscr{G}}$, its AVI is $AVI = (\mathscr{A}, \mathscr{M} + \mathbf{m})$, where $\mathbf{m} = (\mathbf{m}_n)_{n=1}^N = (\boldsymbol{\varpi}_n + \widehat{\mathscr{M}}_n)_{n=1}^N$, $\|\mathbf{m}\|_2 < \infty$, and the RNE is the perturbed solution to $AVI = (\mathscr{A}, \mathscr{M})$. From Theorem 4.3.2 in [24], when $\mathbf{M}$ is semicopositive (matrix $\mathbf{M}$ is semicopositive if for any positive vector $\boldsymbol{\varrho}$ we have $\varrho_i(\mathbf{M}\boldsymbol{\varrho})_i > 0$, where ϱ_i is the ith element of $\boldsymbol{\varrho}$), the AVI has a unique solution for any small $\mathbf{m}$. This condition holds when (3.34) holds. Therefore, when (3.34) holds, the robust game $\widetilde{\mathscr{G}}$ has a unique solution for any small uncertainty region.

Theorems 3.7 and 3.8 establish that the solution of a robust game with bounded and small uncertainty can be obtained via VI mapping of the nominal game; and there is no need to obtain a closed-form, best response solution for the robust game. This is contingent upon strong monotonicity of the mappings $\mathscr{F}$ and $\widetilde{\mathscr{F}}$. Note that in this manner, the RNE is obtained from the nominal NE. Note also that by rearranging the AVI that corresponds to the RNE for the utility function (3.31), the best response of the robust game $\widetilde{\mathscr{G}}$ is

$$\widetilde{p}_n^k = \left[\mu_n^{\frac{1}{\theta}} - \varpi_n^k - \sum_{m \neq n} \frac{\overline{h}_{mn}^k p_m^k}{\overline{h}_{nn}^k} - \epsilon_n^k \|\mathbf{p}_{-n}^k\| \right]_{(p_n^k)^{\min}}^{(p_n^k)^{\max}}, \tag{3.38}$$

where μ_n is the Lagrange multiplier for user n, which is chosen so as to satisfy (3.35).

The best response map for the closed-form solution in (3.38) can also be used to derive the uniqueness condition similar to [5] for the power control game in spectrum sharing environments. The conditions derived from the best response map in [5] is equivalent to the condition derived by VI in Theorem 3.8.

3.3.1.2 Social Utility (Sum Rate)

Similar to the performance analysis in Chapter 2, we quantify the distance between the social utility of the robust and nominal noncooperative games in this chapter. Let $\widetilde{u}^* = \sum_{n \in \mathcal{N}} \widetilde{u}_n^*$ and $v^* = \sum_{n \in \mathcal{N}} v_n^*$ be the social utility of the robust and nominal noncooperative game, where $\widetilde{u}_n^*$ and v_n^* are the utility of user n at the RNE and the nominal NE, respectively. Also, let $d_\Delta = \widetilde{u}^* - v^*$ be the difference between the social utility of the robust and nominal games. In the following Theorem 3.9, we derive the performance gap for the nominal and robust games.

Theorem 3.9. *When Theorem 3.7 holds:*

- *The social utility at the RNE is always less than or equal to that at the nominal NE, that is,* $\widetilde{u}^* \leq v^*$;
- *The difference between the social utilities at the RNE and at the nominal NE is*

$$\|v^* - \widetilde{u}^*\|_2 \approx \|\mathbf{W}(\mathbf{p}^*)\|_2 \times \frac{\|\boldsymbol{\varepsilon}\|_2}{c_{sm}(\mathcal{F})}, \tag{3.39}$$

where $\mathbf{W}(\mathbf{p}) = (\mathbf{w}^k(\mathbf{p}))_{k=1}^K$, *in which*

$$w_{nm}^k \equiv \begin{pmatrix} \frac{\partial v_n^k(p_n^k, f_n^k)}{\partial p_n^k}, & \text{if} \quad m = n, \\ \frac{\partial v_n^k(p_n^k, f_n^k)}{\partial p_m^k} x_{nm}^k, & \text{if} \quad m \neq n \end{pmatrix}, \qquad m, n \in \mathcal{N}, \tag{3.40}$$

and $c_{sm} > 0$ *is the strong monotonicity constant for the mapping* $\mathcal{F}$, *which guarantees* $(\mathbf{p} - \mathbf{p}')(\mathcal{F}(\mathbf{p}) - \mathcal{F}(\mathbf{p}')) \geq c_{sm}\|\mathbf{p} - \mathbf{p}'\|_2^2$ *for all* $\mathbf{p}, \mathbf{p}' \in \mathcal{A}$. *In such a case,* $\mathcal{F}$ *is a strongly monotone map [24];*

(continued)

Theorem 3.9 (continued)

- *The distance between the strategy profiles at the RNE and at the nominal NE is*

$$\|\mathbf{p}^* - \widetilde{\mathbf{p}}^*\|_2 \leq \frac{\|\boldsymbol{\varepsilon}\|_2}{c_{sm}(\mathscr{F})}. \tag{3.41}$$

Proof. See Appendix 4.

Theorem 3.9 shows that the RNE can be obtained from the NE. It also shows how uncertainty affects the robust game's outcome.

It can be proved that the gap between the exact value of $\|v^* - \widetilde{u}^*\|_2$ and its approximation in (3.39) is always less than or equal to $\frac{\|\mathbf{J}(\mathscr{F})\|_2 \|\boldsymbol{\varepsilon}\|_2^2}{2}$, where $\mathbf{J}(\mathscr{F})$ is the Jacobian matrix of $\mathscr{F}$ with respect to $\mathbf{p}$ [24]. When $\mathbf{J}(\mathscr{F})$ or $\boldsymbol{\varepsilon}$ is small, (3.39) is a tight approximation for the difference between the social utilities at the RNE and at the nominal NE. In such cases, the game's social utility at its equilibrium can be approximated without calculating its robust solutions.

When $\boldsymbol{\Upsilon}$ in (3.9) is a P-matrix, $\mathbf{p}_n^*$ is the attractor for $GVI(\mathscr{A}, \widetilde{\mathscr{F}})$ (Theorem 5.4.4 in [24]), that is, $\lim_{\boldsymbol{\varepsilon}\to\mathbf{0}_N} \|\mathbf{p}^* - \widetilde{\mathbf{p}}^*\|_2 = 0$, meaning that when uncertainty approaches zero, the RNE converges to the nominal NE. We conclude that when $\boldsymbol{\Upsilon}$ in (3.9) is a P-matrix and uncertainty is small, the RNE can be obtained as the perturbed nominal NE from the estimated system parameters and the bound on uncertainty. When Theorem 3.7 holds, (3.41) can be simplified to

$$\|\mathbf{p}^* - \widetilde{\mathbf{p}}^*\|_2 \leq \frac{\|\mathbf{E}\|_2}{\lambda_{\mathtt{min}}(\mathbf{M}^{\mathtt{max}})}, \tag{3.42}$$

where $E_{nm} = \|\boldsymbol{\epsilon}_n\|_\infty$, if $m = n$, and $E_{nm} = 0$ otherwise, where $\boldsymbol{\epsilon}_n = [\epsilon_n^1, \ldots, \epsilon_n^K]$, and $\|\cdot\|_\infty$ is the maximum element of the vector; and $\lambda_{\mathtt{min}}(\mathbf{M}^{\mathtt{max}})$ is the minimum eigenvalue of matrix $\mathbf{M}^{\mathtt{max}}$ (Appendix 4).

To show the importance of introducing robustness in game-theoretic resource allocation in wireless networks, the impact of uncertainty on the performance of both $\mathscr{G}$ and $\widetilde{\mathscr{G}}$ in terms of utility variations at their equilibria is shown in Fig. 3.1. In this simulation, $N = 3$, $K = 16$, and $\varepsilon = 10\,\%$ at the RNE. Following convergence to the RNE and to the nominal NE, the channel gains between users are varied up to 80 % of their nominal values, which causes variations in the utility of each user at the nominal NE and at the RNE, as shown in Fig. 3.1. Note that variations in the social utility at the nominal NE are considerable. However, in contrast, in the robust game, the social utility at the RNE is stable. This simulation confirms that the social utility at the nominal NE is very sensitive to variations in system parameters. In contrast, the social utility at the RNE is stable, which shows the importance of introducing robustness in distributed resource allocation problems.

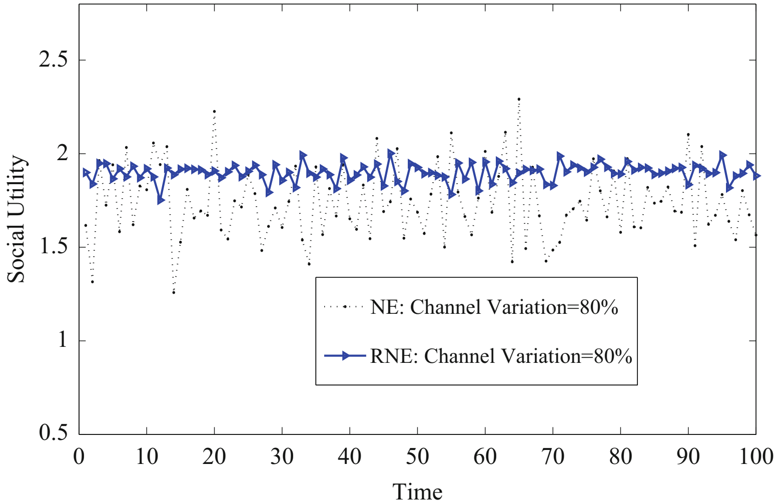

Fig. 3.1 Impact of channel variations in robust and nonrobust games

So far, we have obtained the condition for the RNE's uniqueness from the condition for the nominal NE's uniqueness. Now we focus on the case where the nominal game has multiple nominal NEs and obtain the distance between the robust and nominal equilibria. In this case, the mapping $\mathscr{F}$ for the nominal game is nonmonotone and nonsmooth [62–64], and in general, it is not easy to obtain the relations between the RNE and the nominal NE.

We begin by comparing the case of multiple nominal NEs with that of a unique nominal NE by way of an example in which $N = 2$, $K = 2$, $(p_n^k)^{\texttt{max}} = 1$, $(p_n^k)^{\texttt{min}} = 0.01$, and $\sigma^2 = 0.001$ for all users in all channels. Assume $\overline{h}_{nm}^k > 0.5\overline{h}_{nn}^k$ for all users in all channels. Hence, $\boldsymbol{\Upsilon}$ in (3.9) is not a P-matrix, and the mapping $\mathscr{F}$ is nonmonotone for both users, meaning that there exist multiple local RNEs, corresponding to multiple nominal NEs. At one nominal NE, the convergence points for users 1 and 2 are $(p_1^{*1} = 0.534, p_1^{*2} = 0.463)$ and $(p_2^{*1} = 0.417, p_2^{*2} = 0.583)$, respectively, and $v_1^* + v_2^* = 3.0176$. When $\varepsilon_{\mathbf{f}_n} < 0.8$ for all n, the RNE converges to $(\widetilde{p}_1^{*1} = 0.556, \widetilde{p}_1^{*2} = 0.444)$ and $(\widetilde{p}_2^{*1} = 0.325, \widetilde{p}_2^{*2} = 0.675)$, and $\widetilde{u}_1^* + \widetilde{u}_2^* = 3.077$.

This example shows that when there are multiple nominal NEs, the social utility in the robust game is not a monotonically decreasing function of uncertainty, which is in contrast to the case in which the nominal game has a unique NE. Also, the social utility at the RNE may be higher than that at the nominal NE, which is in line with simulation results in [5, 23]. However, due to the fact that uncertainty cannot be assumed to be the same for all instances, the increase in the social utility at the RNE cannot be counted on.

To further elaborate on Theorem 3.10, in Fig. 3.2 we compare the impact of uncertainty when Theorem 3.7 holds with that of the case where it does not, in terms of the ratio of social utilities at the RNE and at the nominal NE for different amounts

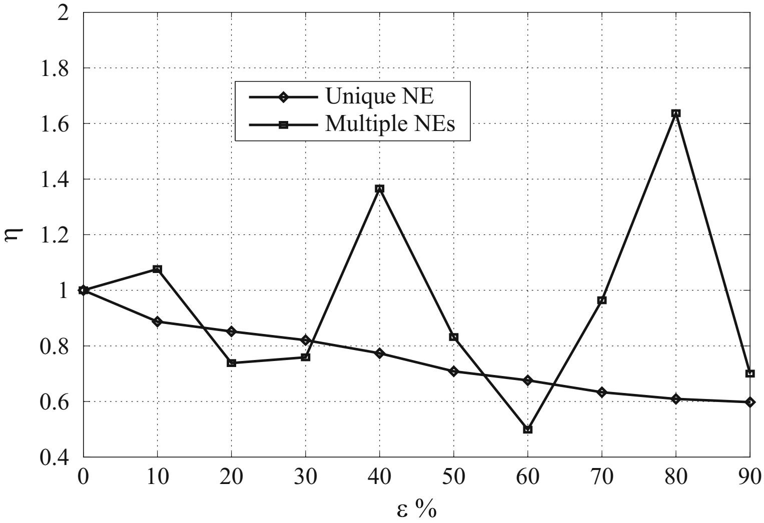

Fig. 3.2 Ratio of social utilities at RNE and at nominal NE (η) versus $\varepsilon = \varepsilon_{\mathbf{f}_n}$ for unique nominal NE and for multiple NEs

of uncertainty. Simulation parameters are the same as previously for Rayleigh fading channels and errors are bounded and uniformly distributed, modeled by $\varepsilon = \varepsilon_{\mathbf{f}_n} = \frac{\mathbf{h}_{mn} - \overline{\mathbf{h}}_{mn}}{\overline{\mathbf{h}}_{mn}}$ for all n, where $\mathbf{h}_{mn} = [h^1_{mn}, \ldots, h^K_{mn}]$ and $\overline{\mathbf{h}}_{mn} = [\overline{h}^1_{mn}, \ldots, \overline{h}^K_{mn}]$, respectively. To satisfy the condition for the uniqueness of the nominal NE [i.e., $\boldsymbol{\Upsilon}$ in (3.9) being a P-matrix], we assume $\overline{h}^k_{mn} < 0.01\overline{h}^k_{nn}$, and for multiple nominal NEs, $\overline{h}^k_{mn} > 0.5\overline{h}^k_{nn}$. Consider $\eta = \frac{u^*}{v^*}$, where u^* is the social utility at the RNE and v^* is the social utility at the nominal NE. The value of η in Fig. 3.2 is obtained by averaging over 100 channel realizations. When $\boldsymbol{\Upsilon}$ in (3.9) is a P-matrix (i.e., when the nominal NE and RNE are unique), the social utility of the robust game at its RNE is decremented when the uncertainty region is expanded, as expected from Theorem 3.10. However, for multiple nominal NEs, no uniformity in social utility is observed.

The aforementioned simulations are similar to the numerical results in Section 2.3.1.1 of Chapter 2, where it was shown when the nominal utility maximization problem is not convex, and there is a chance that via the worst-case approach the social utility of the robust problem may be higher than that of the nominal problem. Remark 5 in [37] captures this point analytically and shows that when interference is high, that is, when all interference channel gains are sufficiently greater than direct channel gains, introducing robustness can increase the social utility of the robust game as compared with that of the nominal game. This is because in such cases, introducing robustness forces users to avoid high-interference channels and instead transmit in low-interference channels, resulting in a higher throughput [5, 23, 37]. However, in general, when there are multiple nominal NEs, that is, when the nominal problem is nonconvex and the amount of uncertainty varies, such increases in the social utility cannot be assumed in all cases.

3.3.1.3 Distributed Algorithms

Since the robust game in Section 3.3.1 involves set-valued mappings, the distributed algorithm for the nominal game in Section 3.2.3 cannot be directly applied to develop a distributed algorithm for the robust game. Hence, we resort to the proximal point method to develop a distributed and efficient numerical scheme for obtaining the robust solution to $\widetilde{\mathscr{G}}$. The proximal point method is a projection method for solving problems that involve set-valued mappings, where a sequence of subproblems is iteratively solved as per Section 12 in [24] and Section 12.6.1 in [2]. The main advantages of utilizing the proximal point method for the robust game are as follows:

- The optimization problem in the proximal point method can be decomposed and solved in a distributed and efficient manner as per Section 12.6.1 in [2];
- Users' utility functions in the game, that is, Ψ_n, do not need to be strictly or strongly convex for the convergence of the distributed scheme as per Section 12.2.4 in [2];
- In some cases, utilizing the proximal point method for a robust game can lead to a closed-form solution.

Definition 3.7. Let $\mathbf{p}_n^{\text{PRM}}(\mathbf{b})$ and $\mathbf{b}_n$ be the solutions for user n in its current and previous iterations, respectively. The proximal response map of the game $\widetilde{\mathscr{G}}$ for any $\mathbf{b} = [\mathbf{b}_1, \ldots, \mathbf{b}_N] \in \mathscr{A}$, denoted by $\mathbf{p}^{\text{PRM}}(\mathbf{b}) = \left[\check{\mathbf{p}}_1(\mathbf{b}), \ldots, \mathbf{p}_N^{\text{PRM}}(\mathbf{b})\right]$ is the solution to the following optimization problem (Section 12.6.1 in [2]):

$$\mathbf{p}^{\text{PRM}}(\mathbf{b}) = \text{argmax}_{\mathbf{p}\in\mathscr{A}} \left[\sum_{n=1}^{N} \Psi_n(\mathbf{p}_n, \mathbf{b}_{-n}) - \frac{1}{2}\|\mathbf{p} - \mathbf{b}\|_2^2\right]. \tag{3.43}$$

From Proposition 12.5 in Section 12.2.4 in [2], since Ψ_n is concave (Appendix 2), the fixed point of $\mathbf{p}^{\text{PRM}}(\mathbf{b})$ is the RNE of the robust game $\widetilde{\mathscr{G}}$. Now, (3.43) can be decomposed into N subproblems (one for each user) as per Section 12.6.1 in [2]:

$$\mathbf{p}_n^{\text{PRM}}(\mathbf{b}) = \text{argmax}_{\mathbf{p}_n\in\mathscr{A}_n} \left[\Psi_n(\mathbf{p}_n, \mathbf{b}_{-n}) - \frac{1}{2}\|\mathbf{p}_n - \mathbf{b}_n\|_2^2\right], \tag{3.44}$$

for all $n \in \mathscr{N}$. A distributed iterative algorithm is developed as follows. When user n is informed about other users' actions and estimates $\mathbf{f}_n^k$ at its receiver, a solution to (3.44) can be obtained. The distributed algorithm that is based on the proximal point method is summarized in Table 3.4. In this algorithm, users update their actions at discrete instances l in $\mathscr{L} = [1, \ldots, L]$, where $\mathbf{p}_n(l)$ is the action of user n at iteration l obtained from (3.44), and $\mathbf{f}_n(l)$ is the impact of other users

Table 3.4 Proximal point distributed algorithm for robust game

Inputs for Each User
$\mathscr{L} = [1, \ldots, L]$: Users' iterations
$\varepsilon_{\mathbf{f}_n}$: Bound on the uncertainty region for user n
$0 < \zeta << 1$: Termination criteria for all users
Initialization For $l = 0$
Set $\mathbf{p}_n^{\mathtt{PRM}}(0) \in \mathscr{A}_n$ and a random $\mathbf{f}_n(0)$ for all $n \in \mathscr{N}$
Iterative Algorithm
For $l = 1, 2, \ldots$
Update $\mathbf{p}_n^{\mathtt{PRM}}(l) = \mathrm{argmax}_{\mathbf{p}_n \in \mathscr{A}_n} \Psi_n(\mathbf{p}_n, \mathbf{p}_{-n}(l-1)) - \frac{1}{2}\|\mathbf{p}_n - \mathbf{p}_n^{\mathtt{PRM}}(l-1)\|_2^2$ for all users
Each user transmits the value of $\mathbf{p}_n^{\mathtt{PRM}}(l)$ to other users and measures the aggregate impacts of other users [i.e., $\mathbf{f}_n(l)$]
If $\|\mathbf{p}_n^{\mathtt{PRM}}(l-1) - \mathbf{p}_n^{\mathtt{PRM}}(l)\|_2 \leq \zeta$, end
Otherwise $l = l + 1$, continue

on user n at iteration l, which is observed at the receiver of user n and sent to the respective transmitter.

Since the regularization term $\frac{1}{2}\|\mathbf{p}-\mathbf{b}\|_2^2$ in (3.43) guarantees the strong concavity of each user's optimization problem, (3.44) is strongly concave [2]. Hence, each $\mathbf{p}_n^{\mathtt{PRM}}$ can be obtained via efficient convex optimization algorithms. When the distributed algorithm in Table 3.4 converges, the regularization term $\|\mathbf{p}_n - \mathbf{b}_n\|_2^2$ tends to zero.

In solving (3.44), one must obtain $\widetilde{\mathscr{F}}_n(\mathbf{p}) = -\frac{\partial \Psi_n(\mathbf{p}_n, \mathbf{p}_{-n})}{\partial \mathbf{p}_n}$ [24]. When $\widetilde{\mathscr{F}}_n(\mathbf{p})$ is an affine mapping, solving (3.44) is reduced to solving an AVI, which is straightforward. For example, when the utility function is as in (3.31), solving (3.44) via the proximal point method at iteration l is similar to solving $VI\left(\mathscr{A}, \overline{\mathscr{M}}(\mathbf{p}(l-1))\right)$ (Section 12.3 in [24]), where $\overline{\mathscr{M}}_n\left(\mathbf{p}(l-1)\right) = \frac{1}{2}\left(\boldsymbol{\varpi}_n + \sum_{m=1}^{N} \mathbf{M}_{nm}\mathbf{p}_m^{\mathrm{T}}(l-1)\right) - \mathscr{I}_n$ and $\mathscr{I}_n = \left(p_n^k(l) - p_n^k(l-1)\right)_{k=1}^{K}$. The solution to (3.44) via the proximal point method is

$$p_n^k(l) = \frac{1}{2}\left[\mu_n^{\frac{1}{\theta}} - \varpi_n^k - \sum_{m \neq n} \frac{\overline{h}_{mn}^k p_m^k(l-1)}{\overline{h}_{nn}^k} - \epsilon_n^k \|\mathbf{p}_{-n}^k(l-1)\| + p_n^k(l-1)\right]_{(p_n^k)^{\mathtt{min}}}^{(p_n^k)^{\mathtt{max}}}, \tag{3.45}$$

where μ_n satisfies (3.35). In (3.45), the computational complexity to derive the robust solution is considerably reduced. However, to solve the water-filling-like formulation in (3.45) with few calculations, each user n needs to know the previous action of other users, that is, $\mathbf{p}_{-n}^k(l-1)$ for all k.

When $\widetilde{\mathscr{F}}_n(\mathbf{p})$ is not an affine mapping, from Lemma 1, $\Psi_n(\mathbf{p}_n, \mathbf{p}_{-n})$ is concave. Hence, the Lagrange function $L_n(\mathbf{p}_n, \mu_n) = u_n(\mathbf{p}_n, \mathbf{f}_n - \varepsilon_{\mathbf{f}_n}\boldsymbol{\vartheta}_n) - \frac{1}{2}\|\mathbf{p}_n(l) - \mathbf{p}_n$

$(l-1)\|_2^2 - \mu_n(\sum_{k=1}^{K} p_n^k - p_n^{\max})$ is used to iteratively solve (3.44), where μ_n is the Lagrange multiplier that satisfies (3.35) for user n.

In the following Theorem 3.10, the sufficient condition for the convergence of the iterative algorithm is stated.

Theorem 3.10. *As $L \to \infty$, the distributed algorithm in Table 3.4 converges to the unique RNE from any initial $\mathbf{p}_n(0)$, when $\boldsymbol{\Upsilon}$ in (3.9) is a P-matrix, and* $\frac{\partial^3 v_n^k(p_n^k, \bar{f}_n^k)}{\partial^2 p_n^k \partial \bar{f}_n^k} = \frac{\partial^3 v_n^k(p_n^k, \bar{f}_n^k)}{\partial p_n^k \partial^2 \bar{f}_n^k} = 0.$

Proof. See Appendix 5.

Note that Theorem 3.10 does not add any new constraint for the power control game. This is because when $\boldsymbol{\Upsilon}$ in (3.9) is a P-matrix, the condition of the preceding Theorem holds for the power control game. In this case, interference in the system is very low, and consequently, the SINR of each user is high, that is, $\frac{\bar{h}_{nn}^k p_n^k}{\bar{f}_n^k} >> 1$, and the utility function of each user is $v_n(\mathbf{p}_n, \mathbf{f}_n) \approx \sum_{k=1}^{K} \log(\frac{\bar{h}_{nn}^k p_n^k}{\bar{f}_n^k})$, which satisfies $\frac{\partial^3 v_n^k(p_n^k, \bar{f}_n^k)}{\partial^2 p_n^k \partial \bar{f}_n^k} = 0$. From Theorem 3.10, the distributed algorithm converges to a unique RNE when the uncertainty region is small, closed, and convex; and $\boldsymbol{\Upsilon}$ in (3.9) is a P-matrix.

3.3.1.4 Overview of Other Works on Robust Noncooperative Games with Homogeneous Users

Robust power control in noncooperative strategic games was introduced in [7] where the interference level is uncertain. Moreover, the computational complexity of solving the robust optimization problem by each user was reduced, and the equilibrium of the nominal game and its sensitivity on system parameters was studied via the VI. However, the social utility of the robust game at its RNE was not compared with that of the nominal game at its NE, and a distributed algorithm for reaching the RNE was not developed in [7].

Another work related to robust noncooperative games with homogeneous users is [5], in which the interference channel gains are uncertain. By assuming an ellipsoid uncertainty region, [5] rewrites the best response map for each user in the robust game by a water-filling-like closed-form solution and obtains the conditions for the RNE's existence and uniqueness via contraction mapping. In addition, the robust game's performance for the two-user scenario is studied for the high-interference and low-interference cases. It is shown that for the high-interference case, the sum rate (social utility) increases and the price of anarchy (PoA) decreases as the

Table 3.5 Overview of existing works on robust noncooperative games with homogeneous users

Reference	Equilibrium analysis	Performance comparison	Distributed algorithm
[5]	Best response	√	√
[7]	VI	–	–
[37]	VI	√	√

uncertainty region for channel gains shrinks.[3] In [5], PoA is defined as the ratio of the maximum social utility at the nominal games' NE over the minimum social utility at any RNE. In [5], it is also shown that in the low-interference case, the sum rate of users (social utility) decreases and the PoA increases by expanding the uncertainty region for channel gains. In Table 3.5, we compare the scope of existing works on robust noncooperative games with homogeneous users.

3.3.2 *Robust Power Control in Noncooperative CRNs*

Consider the power allocation problem for a CRN whose system model is the same as in Section 1.4.2.1 of Chapter 1, where the set $\mathscr{S}$ consists of S pairs of SU transmitters and receivers, and each pair unilaterally chooses its transmit power levels over all K channels subject to keeping its interference on PU q below a predefined threshold. Each SU aims to maximize its own throughput subject to its maximum transmit power and the constraint on its interference on PUs. In the underlay CRN, uncertainty is in the channel side-information (CSI) between the SUs' transmitter and the PUs' receivers [43]. In this section, we analyze the RNE when CSI is uncertain by reformulating the constraint on the interference threshold (IT) via linear matrix inequalities (LMIs) [43]. We will also obtain the conditions for the RNE's existence and uniqueness by utilizing VI and explain how the protection function can be used to reformulate the uncertainty region for CRNs.

Let us begin by focusing on the nominal game in this scenario. When CSI measurements are exact, the interference constraint is

$$\mathscr{P}_s^{\text{interference}} = \left\{ \sum_{k \in \mathscr{K}} g_{sq}^k p_s^k \leq IT_{sq}, \quad g_{sq}^k p_s^k \leq IT_{sq}^k, \ \forall k \in \mathscr{K}, \ q \in \mathscr{Q} \right\}, \quad \forall s \in \mathscr{S}, \tag{3.46}$$

where IT_{sq} and IT_{sq}^k are the maximum allowable interference over all channels and per channel k, respectively. The nominal game is $\mathscr{G}^{\text{CRN}} = \{\mathscr{N}, (v_n)_{n\in\mathscr{N}}, \mathscr{A}^{\text{CRN}}\}$, where $\mathscr{A}^{\text{CRN}} = \prod_{s\in\mathscr{S}} \mathscr{A}_s^{\text{CRN}}$ and

[3]PoA is a concept in game-theoretic economics, which shows how the efficiency of a system degrades due to the selfish behavior of agents.

$$\mathscr{A}_s^{\text{CRN}} = \mathscr{A}_s \times \mathscr{P}_s^{\text{interference}}. \tag{3.47}$$

Since $\mathscr{A}_s^{\text{CRN}}$ for SU s is decoupled from those of other SUs, and $\mathscr{A}_s^{\text{CRN}}$ is convex and compact for all channel realizations, the conditions for the existence and uniqueness of the nominal NE can be obtained, and distributed algorithms for reaching the nominal NE can be developed.

To introduce robustness in this game, the uncertainty in CSIs between SU s and PU q is modeled by

$$\mathbf{g}_{sq} = \overline{\mathbf{g}}_{sq} + \hat{\mathbf{g}}_{sq},$$

where $\mathbf{g}_{sq} = [g_{sq}^1, \ldots, g_{sq}^K]$, $\overline{\mathbf{g}}_{sq} = [\overline{g}_{sq}^1, \ldots, \overline{g}_{sq}^K]$, and $\hat{\mathbf{g}}_{sq} = [\hat{g}_{sq}^1, \ldots, \hat{g}_{sq}^K]$ are respectively the measured CSI, the exact CSI, and the error in CSI between SU s and PU q over all channels. We begin by considering the elliptical uncertainty region $\mathscr{R}_{\mathbf{g}_{sq}}$ defined via the weighted Euclidean norm as [43]

$$\mathscr{R}_{\mathbf{g}_{sq}} = \left\{\mathbf{g}_{sq} | \|\hat{\mathbf{g}}_{sq}\|_{\mathbf{w}_{sq}} \leq \varepsilon_{\mathbf{g}_{sq}}\right\} = \left\{\mathbf{g}_{sq} | \sqrt{\sum_{k\in\mathscr{K}} w_{sq}^k |\hat{g}_{sq}|^2} \leq \varepsilon_{\mathbf{g}_{sq}}\right\}, \quad \forall s \in \mathscr{S}, \quad \forall q \in \mathscr{Q}, \tag{3.48}$$

where $\mathbf{w}_{sq} = [w_{sq}^1, \ldots, w_{sq}^K]$ are positive weights and $\varepsilon_{\mathbf{g}_{sq}}$ is the uncertainty region's bound for channel gains between SU s and PU q. The robust game is

$$\widetilde{\mathscr{G}}^{\text{CRN}} = \left\{\mathscr{N}, (v_n)_{n\in\mathscr{N}}, \widetilde{\mathscr{A}}^{\text{CRN}}\right\},$$

in which

$$\widetilde{\mathscr{A}}^{\text{CRN}} = \prod_{s\in\mathscr{S}} \widetilde{\mathscr{A}}_s^{\text{CRN}} = \mathscr{A}_s \times \widetilde{\mathscr{P}}_s^{\text{interference}},$$

where

$$\widetilde{\mathscr{P}}_s^{\text{interference}} = \left\{ \sum_{k\in\mathscr{K}} g_{sq}^k p_s^k \leq IT_{sq}, \quad \forall g_{sq}^k \in \mathscr{R}_{\mathbf{g}_{sq}} \right.$$
$$\left. g_{sq}^k p_s^k \leq IT_{sq}^k, \quad \forall g_{sq}^k \in \mathscr{R}_{\mathbf{g}_{sq}} \quad \forall k \in \mathscr{K}, \quad q \in \mathscr{Q} \right\}, \quad \forall s \in \mathscr{S}. \tag{3.49}$$

The utility functions in $\widetilde{\mathscr{G}}^{\text{CRN}}$ and $\mathscr{G}^{\text{CRN}}$ are similar, but their strategy spaces are different because the constraints in the robust game include uncertain parameters. The interference constraints are given in the form of the intersection of an infinite number of convex sets, which means that the analysis of $\widetilde{\mathscr{G}}^{\text{CRN}}$ is not trivial.

3.3.2.1 Existence and Uniqueness of RNE

To simplify the analysis of RNE, in [43], $\widetilde{\mathscr{P}}_s^{\text{interference}}$ is rewritten as a set of LMIs, and the conditions for the RNE's existence and uniqueness via VI are derived. Let

$$[\Upsilon^{\text{CRN}}]_{ss'} \equiv \begin{pmatrix} 0, & \text{if} \quad s = s' \\ \max_{k \in \mathscr{K}} \frac{h^k_{s's}}{h^k_{ss}} \text{innr}^k_{s's} & \text{if} \quad s \neq s' \end{pmatrix}, \qquad s, s' \in \mathscr{S}, \tag{3.50}$$

in which

$$\text{innr}^k_{s's} = \frac{\sigma^2 + \sum_{s' \in \mathscr{S}} h_{s's} \hat{p}^{\max}_{s'}}{\sigma^2}, \qquad \hat{p}^{\max}_{s'} = \min\{p^{\max}_s, \widehat{IT}^k_s, \overline{IT}^k_s\},$$

$$\breve{IT}^k_s = \min_{q \in \mathscr{Q}} IT_{sq} \min_{k \in \mathscr{K}} \left\{ \frac{w^k_{sq}}{\varepsilon_{\mathbf{g}_{sq}}}, \frac{1}{\hat{g}_{sq}} \right\}, \quad \text{and} \quad \overline{IT}^k_s = \min_{q \in \mathscr{Q}} \left\{ \frac{IT^k_{sq}}{\left(\hat{g}^k_{sq} + \varepsilon_{\mathbf{g}_{sq}} / \sqrt{w_{sq}}\right)^2} \right\}.$$

By rewriting the uncertainty region, the RNE's existence can be proved. Also, when $\rho(\Upsilon^{\text{CRN}}) < 1$, the game $\widetilde{\mathscr{G}}^{\text{CRN}}$ has a unique equilibrium and the distributed algorithm (both synchronous and sequential) converges to its unique equilibrium (Theorems 1 and 2 in [43]). From Υ^{CRN}, we note that increasing the uncertainty in $\mathbf{g}_{sq}$ increases the chance of having a unique NE because when the uncertainty region for one PU channel is expanded, the worst-case robust transmit strategy of one SU becomes more conservative, which causes less interference on other SUs. When $\varepsilon_{\mathbf{g}_{sq}} \to 0$ for all $s \in \mathscr{S}$ and $q \in \mathscr{Q}$, the conditions for the uniqueness and convergence of $\widetilde{\mathscr{G}}^{\text{CRN}}$ coincide with those of $\mathscr{G}^{\text{CRN}}$.

3.3.2.2 Social Utility (Sum Rate)

When uncertainty is in CSIs between SUs and PUs, and the RNE is unique, expanding the uncertainty region forces SUs to reduce their transmit power levels, which in turn reduces the SUs' social utility (sum rate) compared to when CSIs are exact. The higher social utility in the latter case comes at the cost of violating the interference threshold of PUs.

3.3.2.3 Distributed Algorithms

To develop a distributed algorithm for the robust game in CRNs, [43] proposes a totally asynchronous algorithm in Table 3.1 for two cases: (1) when SUs have one single antenna for both transmit and receive and (2) when SUs are equipped with multiple antennas. The optimization problem of each user does not have a closed-form solution. However, [43] shows that by applying LMIs, the best response map can be obtained for both the aforementioned cases. When the condition in

Theorem 3.5 holds, the totally asynchronous algorithm in Table 3.1 for these two cases converges. In [43], it is also observed that more uncertainty in CSIs increases the chance of convergence of distributed algorithms to a unique RNE. The reason is similar to that stated in Section 3.3.2.1 in this chapter.

3.3.2.4 Overview of Other Works on Robust Noncooperative CRNs

Due to the coupling between user strategies that emanates from the need to satisfy interference thresholds, and, as shown in Appendix 1, the analysis of the nominal game is not easy [2]. When robustness is introduced, the analysis of the RNE is further aggravated. As a result, except for [43], to the best of our knowledge, there is no other existing work on robust games for CRNs.

3.3.3 Robust Power Control for Noncooperative Heterogeneous Users

We now focus on wireless networks in which there exist two types of users with different capabilities to extract side information, where one type (called leaders) can extract and process side information pertaining to other users, but the other type (called followers) are limited to simple measurements and observations. Interactions between these two types of users are modeled in a two-level Stackelberg game [35, 65–67], where the leaders and the followers utilize their measurements, observations, and processing capabilities to determine their respective optimal constrained transmit power with a view to maximizing their own utility value (e.g., throughput). In practice, users' side information and measurements may be uncertain, meaning that there is a need to utilize robust Stackelberg games (RSGs). In this section, we study the performance of RSGs and analyze the robust Stackelberg equilibrium (RSE) via sensitivity analysis.

Consider the system model introduced in Section 1.4.2 in Chapter 1, where the set of leaders is $\mathcal{N}_{\mathrm{L}} = \{0, 1, \ldots, N_{\mathrm{L}} - 1\}$, and the set of followers is $\mathcal{N}_{\mathrm{F}} = \{1, \ldots, N_{\mathrm{F}}\}$. In this setup, $\mathcal{N} = \mathcal{N}_{\mathrm{L}} \cup \mathcal{N}_{\mathrm{F}}$ is the set of all users. The leaders' side information is as in Section 1.4.2.2 in Chapter 1. Stackelberg games have been widely used to obtain optimal strategies for users by solving a bilevel optimization problem [68–74]. Stackelberg games have also been used to model spectrum sharing by PUs and SUs in licensed bands. For example, in [75–80], the spectrum owner authorizes SUs to utilize its licensed band subject to a payoff by each SU, determined in such a way as to maximize the spectrum owner's profit provided that the interference caused by SUs is below a given threshold. The system model in this chapter is different from those of the aforementioned works, as here we are concerned only with spectrum sharing in unlicensed bands. The use of Stackelberg games to formalize interactions among heterogeneous users does not necessarily entail a prioritization of transceivers. Rather, the formalism in this chapter takes into account the asymmetry of side information in heterogeneous users as in [35, 65, 66].

The equilibrium in a nominal Stackelberg game, called the nominal Stackelberg equilibrium (NSE), prescribes the optimal strategy set for the leaders when the followers play at their NE and is derived via backward induction. For instance, for the one-leader/one-follower Stackelberg game, where user 0 is the leader and user 1 is the follower, the leader knows that when it transmits with $\mathbf{p}_0$, the follower's transmit power (its best response) is $\mathbf{p}_1^*(\mathbf{p}_0)$. Hence, the leader takes this into account in choosing its strategy. The leader's strategy at the nominal Stackelberg equilibrium is $\mathbf{p}_0^{*\mathrm{NSE}}$ when for any $\mathbf{p}_0 \in \mathscr{A}_0$ we have

$$v_0\left(\mathbf{p}_0^{*\mathrm{NSE}}, \mathbf{f}_0\left(\mathbf{p}_1^*(\mathbf{p}_0^{*\mathrm{NSE}}), \mathbf{s}_0\right)\right) \geq v_0\Big(\mathbf{p}_0, \mathbf{f}_0\big(\mathbf{p}_1^*(\mathbf{p}_0), \mathbf{s}_0\big)\Big),$$

where v_0 is the leader's utility value, $\mathbf{f}_0$ is its interference on other users, and $\mathbf{s}_0$ is its side information. The Stackelberg equilibrium for the leader is the solution to the following bilevel optimization problem:

$$\begin{aligned} &\max\nolimits_{\mathbf{p}_0 \in \mathscr{A}_0} v_0\left(\mathbf{p}_0, \mathbf{f}_0(\mathbf{p}_1, \mathbf{s}_0)\right), \\ &\text{subject to: } \max\nolimits_{\mathbf{p}_1 \in \mathscr{A}_1} v_1\left(\mathbf{p}_1, \mathbf{f}_1(\mathbf{p}_0, \mathbf{s}_1)\right), \end{aligned} \tag{3.51}$$

where v_1 is the follower's utility value, $\mathbf{f}_1$ is its interference on other users, and $\mathbf{s}_1$ is its side-information. For the one-leader/multifollower scenario, the preceding backward procedure is applicable as well. Let $\mathbf{p}_{-0}^*(\mathbf{p}_0) \triangleq [\mathbf{p}_1^*, \ldots, \mathbf{p}_{N_\mathrm{F}}^*]$ be the followers' strategies at their nominal NE when the leader's strategy is $\mathbf{p}_0$. The strategy profile $(\mathbf{p}_0^{*\mathrm{NSE}}, \mathbf{p}_{-0}^{*\mathrm{NSE}}(\mathbf{p}_0^{*\mathrm{NSE}}))$ is the equilibrium of the nominal Stackelberg game when

$$v_0\left(\mathbf{p}_0^{*\mathrm{NSE}}, \mathbf{f}_0\big(\mathbf{p}_{-0}^{*\mathrm{NSE}}(\mathbf{p}_0^{*\mathrm{NSE}}), \mathbf{s}_0\big)\right) \geq v_0\Big(\mathbf{p}_0, \mathbf{f}\big(\mathbf{p}_{-0}^*(\mathbf{p}_0), \mathbf{s}_0\big)\Big),$$

for any $\mathbf{p}_0 \in \mathscr{A}_0$, where $\mathbf{p}_{-0}^{*\mathrm{NSE}}(\mathbf{p}_0) = [\mathbf{p}_1^{*\mathrm{NSE}}, \ldots, \mathbf{p}_{N_\mathrm{F}}^{*\mathrm{NSE}}]$. At the NSE, the utility of user n is $\omega_n^{*\mathrm{NSE}}$ and the social utility of the game is $\omega^{*\mathrm{NSE}} = \sum_{n \in \mathscr{N}} \omega_n^{*\mathrm{NSE}}$. When the followers' game has multiple NEs, its analysis is very complicated [81, 82]. We restrict our study to a Stackelberg game with a unique NE in the followers' game. The conditions for the NE's uniqueness in this case were presented in Section 3.2.1.3 in this chapter.

3.3.3.1 Robust Stackelberg Games

To introduce robustness, uncertainty in the parameter values of both followers and leaders is assumed. In the sequel, we consider the following two cases:

- **Case 1** The leader's side information is accurate, and the followers' observations are noisy;
- **Case 2** The leader's side information is uncertain, and the followers' observations are noisy.

The uncertainty in the followers' observations is modeled as in Section 3.1 in this chapter. Hence, the conditions for the existence and uniqueness of RSE for the followers' game are the same as those derived therein. Let $\mathbf{H}_{n_\mathrm{F}n_\mathrm{L}}$ denote the uncertain channel gain between the transmitter of follower $n_\mathrm{F} \in \mathscr{N}_\mathrm{F}$ and the receiver of leader $n_\mathrm{L} \in \mathscr{N}_\mathrm{L}$, where the uncertainty region is

$$\mathscr{R}_{\mathbf{H}_{n_\mathrm{F}n_\mathrm{L}}} = \{\mathbf{H}_{n_\mathrm{F}n_\mathrm{L}} \,|\, \|\widehat{\mathbf{H}}_{n_\mathrm{F}n_\mathrm{L}}\| = \|\mathbf{H}_{n_\mathrm{F}n_\mathrm{L}} - \overline{\mathbf{H}}_{n_\mathrm{F}n_\mathrm{L}}\|_2 \leq \delta_{n_\mathrm{F}n_\mathrm{L}}\}, \tag{3.52}$$

and $\delta_{n_\mathrm{F}n_\mathrm{L}}$ is the bound on the uncertainty region $\mathscr{R}_{\mathbf{H}_{n_\mathrm{F}n_\mathrm{L}}}$. When the followers' receivers do not provide any feedback, obtaining the exact value of $\delta_{n_\mathrm{F}n_\mathrm{L}}$ is not possible.

3.3.3.2 Single-Leader/Single-Follower Robust Stackelberg Games

At the RSE for Case 1 (RSE1), the follower's optimization problem is

$$\max_{\mathbf{p}_1 \in \mathscr{A}_1} \min\nolimits_{\mathbf{f}_1(\mathbf{p}_0,\mathbf{s}_1) \in \mathscr{R}_{\mathbf{f}_1}(\mathbf{p}_0)} \; u_1\left(\mathbf{p}_1, \mathbf{f}_1(\mathbf{p}_0, \mathbf{s}_1)\right), \tag{3.53}$$

the leader's side-information set is $\mathscr{I}_0^{\mathrm{RSE1}} = \{\mathscr{A}_1, v_1, \mathbf{H}_{10}, \mathbf{H}_{11}, \mathbf{H}_{01}, \mathscr{R}_{\mathbf{f}_1}(\mathbf{p}_0)\}$, and its bilevel optimization problem is

$$\max\nolimits_{\mathbf{p}_0 \in \mathscr{A}_0} \; v_0\left(\mathbf{p}_0, \mathbf{f}_0(\mathbf{p}_1\mathbf{s}_0)\right), \tag{3.54}$$

$$\text{Subject to: } \max\nolimits_{\mathbf{p}_1 \in \mathscr{A}_1} \min_{\mathbf{f}_1(\mathbf{p}_0,\mathbf{s}_1) \in \mathscr{R}_{\mathbf{f}_1}(\mathbf{p}_0)} u_1\left(\mathbf{p}_1, \mathbf{f}_1(\mathbf{p}_0, \mathbf{s}_1)\right).$$

The best response of (3.53) to the leader's action is denoted by $\mathbf{p}_1^*(\mathbf{p}_0)$, and the leader's transmit power at RSE1, denoted by $\mathbf{p}_0^{*\mathrm{RSE1}}$, satisfies

$$v_0(\mathbf{p}_0^{*\mathrm{RSE1}}, \mathbf{f}_0(\mathbf{p}_1^*(\mathbf{p}_0^{*\mathrm{RSE1}}), \mathbf{s}_0)) \geq v_0(\mathbf{p}_0, \mathbf{f}_0(\mathbf{p}_1^*(\mathbf{p}_0), \mathbf{s}_0))$$

for any $\mathbf{p}_0 \in \mathscr{A}_0$. In what follows, for notational convenience, we omit the arguments $\mathbf{p}_{-n}$ and $\mathbf{s}_n$ in $\mathbf{f}_n(\mathbf{p}_{-n}, \mathbf{s}_n)$.

It can be shown that RSE1 exists since:

- (3.53) is concave with respect to $\mathbf{p}_1(\mathbf{p}_0)$ for any fixed $\mathbf{p}_0$ and is a decreasing function of $\mathbf{f}_1$; and
- $\mathscr{A}_1$ and $\mathscr{R}_{\mathbf{f}_1}(\mathbf{p}_0)$ are convex, bounded, and disjoint.

Consequently, there always exists a solution to (3.53) [83], and optimization problem (3.54) has a nonempty feasible set. Hence, RSE1 exists.

To simplify the constraint in (3.54), we utilize (3.26) and (3.27) and rewrite (3.54) as

$$\max_{\mathbf{p}_0 \in \mathscr{A}_0} v_0(\mathbf{p}_0, \mathbf{f}_0) \quad \text{Subject to:} \quad \max_{\mathbf{p}_1 \in \mathscr{A}_1} v_1(\mathbf{p}_1, \mathbf{f}_1^*), \tag{3.55}$$

where $\mathbf{f}_1^* = \bar{\mathbf{f}}_1 - \varepsilon_{\mathbf{f}_1} \boldsymbol{\vartheta}_1$, in which $\mathbf{f}_1^* = [f_1^{1*}, \ldots, f_1^{K*}]$, $\boldsymbol{\vartheta}_1 = [\vartheta_1^1, \ldots, \vartheta_1^K]$, where ϑ_1^k is

$$\vartheta_1^k = \frac{\frac{\partial u_1^k(\mathbf{p}_1, \mathbf{f}_1^*)}{\partial f_1^k}}{\sqrt{\sum_{k=1}^K \left(\frac{\partial u_1^k(\mathbf{p}_1, \mathbf{f}_1^*)}{\partial f_1^k}\right)^2}}. \tag{3.56}$$

Note that in (3.55), the uncertainty region is removed from the leader's optimization problem, and the leader's and the follower's strategies as well as their respective utilities at RSE1 can be obtained and compared with those at NSE.

At RSE for Case 2 (RSE2), the leader's uncertain side-information set is $\mathscr{I}_0^{\text{RSE2}} = \{\mathscr{A}_1, v_1, \mathbf{H}_{10}, \overline{\mathbf{H}}_{11}, \overline{\mathbf{H}}_{01}, \mathscr{R}_{\mathbf{f}_1}(\mathbf{p}_0)\}$, in which $\mathbf{H}_{10}$ is the uncertain parameter in (3.52). The leader's bilevel worst-case optimization problem is

$$\max_{\mathbf{p}_0 \in \mathscr{A}_0} \min_{\mathbf{H}_{10} \in \mathscr{R}_{\mathbf{H}_{10}}} v_0(\mathbf{p}_0, \mathbf{f}_0) \quad , \quad \text{subject to:} \quad \max_{\mathbf{p}_1 \in \mathscr{A}_1} \min_{\mathbf{f}_1 \in \mathscr{R}_{\mathbf{f}_1}(\mathbf{p}_0)} u_1(\mathbf{p}_1, \mathbf{f}_1). \tag{3.57}$$

Note that in (3.57), the leader assumes worst-case uncertainty in its side information (i.e., $\mathbf{H}_{10}$), which reduces its transmit power, which in turn reduces its interference on the follower. In other words, $\mathbf{f}_1$ is a decreasing function of δ_{10}.

It can be shown that RSE2 always exists, because:

- $\mathscr{R}_{\mathbf{H}_{10}}$, $\mathscr{R}_1(\mathbf{p}_0)$, $\mathscr{A}_0$, and $\mathscr{A}_1$ are compact and closed sets; and
- For any realization of $\mathbf{H}_{10} \in \mathscr{R}_{\mathbf{H}_{10}}$, the uncertainty region $\mathscr{R}_1(\mathbf{p}_0)$ is closed and convex.

Hence, the follower has a feasible strategy; and from the preceding discussion, RSE2 always exists.

While the condition for the existence of RSE2 can be easily derived, solving (3.57) is significantly more complex than solving (3.54). This is because (3.57) has two uncertain parameters, $\mathbf{H}_{10}$ and $\mathbf{f}_1$, and RSE2 is a function of both $\varepsilon_{\mathbf{f}_1}$ and δ_{10}, while RSE1 is a function of $\varepsilon_{\mathbf{f}_1}$ only. To compare RSE1 and RSE2, the value of $\varepsilon_{\mathbf{f}_1}$ should be the same for both of these cases. Next, we study the relationship between RSE1 and RSE2.

The performance at RSE1 and RSE2 are compared in Table 3.6 using Propositions 1 and 2 in [44]. Note that the leader's utility at RSE1, that is, $\omega_0^{*\text{RSE1}}$, is

Table 3.6 Social utility at RSE1 and RSE2 for single-leader/single-follower RSG

Fixed δ_{10}	Fixed $\varepsilon_{\mathbf{f}_1}$
Follower's strategy is a decreasing function of $\varepsilon_{\mathbf{f}_1}$	Follower's strategy is an increasing function of δ_{10}
Leader's strategy is an increasing function of $\varepsilon_{\mathbf{f}_1}$	Leader's strategy is a decreasing function of δ_{10}
$\omega^{*\mathrm{RSE1}} > \omega^{*\mathrm{NSE}}$ when C1 : $\lvert\mathbf{D}_{10}\rvert < \lvert\mathbf{J}_{\mathbf{p}_0}\rvert$ and C2 : $\lvert\mathbf{J}_{\mathbf{p}_1}\rvert < \lvert\mathbf{D}_{01}\rvert$	$\omega^{*\mathrm{RSE2}} > \omega^{*\mathrm{NSE}}$ when C3 : $\lvert\mathbf{D}_{10}\rvert > \lvert\mathbf{J}_{\mathbf{p}_0}\rvert$ and C4 : $\lvert\mathbf{J}_{\mathbf{p}_1}\rvert > \lvert\mathbf{D}_{01}\rvert$

higher than that at NSE, that is, $\omega_0^{*\mathrm{NSE1}}$, while the follower's utility at RSE1, that is, $\omega_1^{*\mathrm{RSE1}}$, is less than that at NSE, that is, $\omega_1^{*\mathrm{NSE}}$. When C1 and C2 hold, the social utility at RSE1, that is, $\omega^{*\mathrm{RSE1}}$, is higher than that at NSE, that is, $\omega^{*\mathrm{NSE}}$. In contrast, the leader's utility at RSE2, that is, $\omega_0^{*\mathrm{RSE2}}$, is less than that at NSE, while the follower's utility at RSE2, that is, $\omega_1^{*\mathrm{RSE2}}$, is higher than that at NSE. In this case, when C3 and C4 hold, the social utility at RSE2, that is, $\omega^{*\mathrm{RSE2}}$, is higher than that at NSE. In Table 3.6, $\mathbf{J}_{\mathbf{p}_n} \triangleq \nabla_{\mathbf{p}_n} v_n(\mathbf{p}_n, \mathbf{f}_n)$ is the column gradient vector of v_n for user n and is called the direct rate of user n, where the kth element of this vector is $J^k_{\mathbf{p}_n} = \frac{\partial v_n^k(p_n^k, f_n^k(\mathbf{p}_{-n}, \mathbf{s}_n))}{\partial p_n^k}$. Also, $\mathbf{D}_{nm} \triangleq \overline{\mathbf{H}}_{mn}\mathbf{J}_{\mathbf{f}_n}$, where $\mathbf{J}_{\mathbf{f}_n} \triangleq \nabla_{\mathbf{f}_n} v_n(\mathbf{p}_n, \mathbf{f}_n)$, $\overline{\mathbf{H}}_{mn} \triangleq \mathrm{diag}\{(\overline{h}^k_{mn})_{k=1}^K\}$, and $\mathbf{D}_{nm}$ is the rate of decrease in the utility of user n caused by an increase in the strategy of user m. Hence, $\mathbf{D}_{nm}$ is the negative impact of user m on user n for $m \neq n$. When the utility of user n is its throughput, we have $D^k_{nm} = -\frac{\overline{h}^k_{nn}\overline{h}^k_{mn}p^k_n}{\overline{f}^k_n(\overline{f}^k_n + \overline{h}^k_{nn}p^k_n)}$, where D^k_{nm} is the kth element of $\mathbf{D}_{nm}$.

A higher social utility can be achieved at RSE1 when the increase in the leader's utility is more than the decrease in the follower's utility, that is, when C1 and C2 hold; whereas a higher social utility can be achieved at RSE2 when the increase in the follower's utility is more than the decrease in the leader's utility, that is, when C3 and C4 hold. In Appendix 7, we prove that the foregoing statements are true. Note that C1 is the dual of C3, and C2 is the dual of C4.

From Proposition 1 in [44], when uncertainty is small, the strategies of the leader and the follower in Case 1 can be obtained from their respective strategies at the NSE as

$$\mathbf{p}_0^{*\mathrm{RSE1}} = \mathbf{p}_0^{*\mathrm{NSE}} + \varepsilon_{\mathbf{f}_1} \times \left((\mathbf{J}_{\mathbf{p}_0\mathbf{p}_0})^{-1}\mathbf{J}_{\mathbf{f}_0\mathbf{p}_0}\overline{\mathbf{H}}_{10}(\mathbf{J}_{\mathbf{p}_1\mathbf{p}_1})^{-1}\mathbf{J}_{\mathbf{f}_1\mathbf{p}_1}\vartheta_1^{\mathrm{T}} \right)^{\mathrm{T}}, \tag{3.58}$$

$$\mathbf{p}_1^{*\mathrm{RSE1}} = \mathbf{p}_1^{*\mathrm{NSE}} - \varepsilon_{\mathbf{f}_1} \times \left((\mathbf{J}_{\mathbf{p}_1\mathbf{p}_1})^{-1}\mathbf{J}_{\mathbf{f}_1\mathbf{p}_1}\vartheta_1^{\mathrm{T}} \right)^{\mathrm{T}}, \tag{3.59}$$

where $\mathbf{J}_{\mathbf{f}_n\mathbf{p}_n} \triangleq \mathrm{diag}\{(\frac{\partial^2 v_n(\mathbf{p}_n,\mathbf{f}_n)}{\partial f_n^k \partial p_n^k})_{k=1}^K\}$, and $\mathbf{J}_{\mathbf{p}_n\mathbf{p}_n} \triangleq \mathrm{diag}\{(\frac{\partial^2 v_n(\mathbf{p}_n,\mathbf{f}_n)}{\partial^2 p_n^k})_{k=1}^K\}$.

To solve (3.57), the leader calculates the follower's transmit power from (3.53) via numerical methods in [83, 84] or via semidefinite programming (SDP) reformulation in [83] for all $\mathbf{p}_0 \in \mathscr{A}_0$. The leader also calculates its minimum utility corresponding to $\mathbf{p}_0$, subject to $\mathbf{H}_{10} \in \mathscr{R}_{\mathbf{H}_{10}}$, and chooses its transmit power $\mathbf{p}_0$ that maximizes its utility (its throughput) among all minimum utility values obtained for all $\mathbf{p}_0 \in \mathscr{A}_0$.

Since C1, C2, C3, and C4 affect the performance of RSG at RSE1 and RSE2, it is useful to give their physical interpretations in wireless networks. We consider the following three scenarios, depending on the SINRs of the leader and the follower.

- **Scenario 1.** High SINR, that is, $\overline{h}_{00}^k p_0^k \gg \overline{h}_{10}^k p_1^k + \sigma^2$ and $\overline{h}_{11}^k p_1^k \gg \overline{h}_{01}^k p_0^k + \sigma^2$. In this scenario, when both C1 and C2 hold, we have

$$\overline{h}_{10}^k > \overline{h}_{01}^k. \tag{3.60}$$

- **Scenario 2.** Low SINR, that is, $\overline{h}_{00}^k p_0^k \ll \overline{h}_{10}^k p_1^k + \sigma^2$ and $\overline{h}_{11}^k p_1^k \ll \overline{h}_{01}^k p_0^k + \sigma^2$. In this scenario, when both C1 and C2 hold, we have

$$\overline{h}_{00}^k > \overline{h}_{01}^k \qquad \text{and} \qquad \overline{h}_{10}^k > \overline{h}_{11}^k. \tag{3.61}$$

- **Scenario 3.** Moderate SINR, that is, $\overline{h}_{00}^k p_0^k \approx \overline{f}_0^k$ and $\overline{h}_{11}^k p_1^k \approx \overline{f}_1^k$, when the interference levels of the leader and the follower on each other are close, that is, $\overline{f}_1^k \approx \overline{f}_0^k$. In this scenario, when both C1 and C2 hold, we have

$$\overline{h}_{00}^k \overline{h}_{10}^k > \overline{h}_{11}^k \overline{h}_{01}^k, \qquad \forall k \in K. \tag{3.62}$$

One can use (3.60)–(3.62) to obtain the probability of each scenario from the distribution of channel gains. As an example, from (3.60), when channels are Rayleigh fading and channel gains are independent and identically distributed random variables, the probability density function of channel gains is exponential, that is, $\phi(\overline{h}_{10}^k) = \lambda_1 \exp^{\lambda_1 \overline{h}_{10}^k}$ and $\phi(\overline{h}_{01}^k) = \lambda_2 \exp^{\lambda_2 \overline{h}_{01}^k}$. The probability of $\overline{h}_{10}^k > \overline{h}_{01}^k$ is

$$\int_0^\infty \int_0^{\overline{h}_{10}^k} \phi(\overline{h}_{10}^k)\phi(\overline{h}_{01}^k) d\overline{h}_{10}^k d\overline{h}_{01}^k = \int_0^\infty \phi(\overline{h}_{10}^k) d\overline{h}_{10}^k (1 - \exp^{\lambda_2 \overline{h}_{10}^k}) = \frac{\lambda_2}{\lambda_1 + \lambda_2}.$$

One can also use (3.60)–(3.62) to predict how the social utility would change for any given channel condition in Case 1.

When C3 and C4 in Table 3.6 hold, we have

$$\text{Scenario 1:} \Longrightarrow \overline{h}_{01}^k > \overline{h}_{10}^k; \tag{3.63}$$

$$\text{Scenario 2:} \Longrightarrow \overline{h}_{01}^k > \overline{h}_{00}^k, \quad \text{and} \quad \overline{h}_{11}^k > \overline{h}_{10}^k; \tag{3.64}$$

$$\text{Scenario 3:} \Longrightarrow \overline{h}_{11}^k \overline{h}_{01}^k > \overline{h}_{00}^k \overline{h}_{10}^k, \quad \forall k. \tag{3.65}$$

3.3.3.3 Multi-user Stackelberg Games

When the sets of leaders and followers each consist of more than one user, the Stackelberg game is called a multi-user Stackelberg game. We first consider the one-leader/multi-follower game, that is, when $N_{\mathrm{L}} = 1$ and $N_{\mathrm{F}} > 1$. When the followers' observations are uncertain, modeled by (3.20), and the leader's side information is exact (Case 1), the robust game between the followers is as $\widetilde{\mathscr{G}}$ in Section 3.1, and the results therein for the existence and uniqueness of the RNE, as well as the distributed algorithm in Table 3.4, can be readily applied.

From Assumptions A1–A3 in Section 3.2.1.3 in this chapter and on $\mathscr{A}_0$, the RNE of the followers' game exists when uncertainty is small, meaning that a solution to the leader's optimization problem exists. Hence, RSE1 exists when uncertainty is small.

Case 2 assumes that the information set of the leader is uncertain, for example, when channel gains $\mathbf{H}_{n_{\mathrm{F}}0}$ between follower n_{F} and leader 0 are modeled by

$$\mathscr{R}_{\mathbf{H}_{n_{\mathrm{F}}0}} = \{\mathbf{H}_{n_{\mathrm{F}}0} \mid \|\widehat{\mathbf{H}}_{n_{\mathrm{F}}0}\|_2 = \|\mathbf{H}_{n_{\mathrm{F}}0} - \overline{\mathbf{H}}_{n_{\mathrm{F}}0}\|_2 \leq \delta_{n_{\mathrm{F}}0}\}, \tag{3.66}$$

where $\delta_{n_{\mathrm{F}}0}$ is the bound on the uncertainty region for $\mathbf{H}_{n_{\mathrm{F}}0}$. From Assumptions A1–A3 in Section 3.2.1.3 in this chapter, Statement 1 in [44], and (3.66), the RNE of the followers' game exists when uncertainty is small, meaning that a solution to the leader's optimization problem exists. Hence, RSE2 exists when the uncertainty in channel gains is small.

We now study the utility values at RSE1 and RSE2 for the multiuser RSG. Let

$$\mathrm{C5}: \mathbf{J}_{\mathbf{p}_0} > \sum_{n_{\mathrm{F}} \in \mathscr{N}_{\mathrm{F}}} \mathbf{D}_{n_{\mathrm{F}}0}, \qquad \mathrm{C6}: \mathbf{J}_{\mathbf{p}_{n_{\mathrm{F}}}} < \mathbf{D}_{0 n_{\mathrm{F}}} + \sum_{m_{\mathrm{F}} \neq n_{\mathrm{F}}, m_{\mathrm{F}} \in \mathscr{N}_{\mathrm{F}}} \mathbf{D}_{m_{\mathrm{F}} n_{\mathrm{F}}}, \quad \forall n_{\mathrm{F}} \in \mathscr{N}_{\mathrm{F}},$$

$$\mathrm{C7}: \mathbf{J}_{\mathbf{p}_0} < \sum_{n_{\mathrm{F}} \in \mathscr{N}_{\mathrm{F}}} \mathbf{D}_{n_{\mathrm{F}}0}, \quad \text{and} \quad \mathrm{C8}: \mathbf{J}_{\mathbf{p}_{n_{\mathrm{F}}}} > \mathbf{D}_{0 n_{\mathrm{F}}} + \sum_{m_{\mathrm{F}} \neq n_{\mathrm{F}}, m_{\mathrm{F}} \in \mathscr{N}_{\mathrm{F}}} \mathbf{D}_{m_{\mathrm{F}} n_{\mathrm{F}}}, \quad \forall n_{\mathrm{F}} \in \mathscr{N}_{\mathrm{F}}.$$

Table 3.7 compares the utility values at RSE1 and RSE2 with the utility value at NSE, where $\boldsymbol{\varepsilon} = [\varepsilon_{\mathbf{f}_0}, \ldots, \varepsilon_{\mathbf{f}_{N_f}}]$ and $\boldsymbol{\delta}_0 = [\delta_{10}, \ldots, \delta_{N_f 0}]$. Similar to the single-leader/single-follower RSG, the uncertainty in the followers' observations increments the leader's utility but reduces the followers' utilities. As stated in Table 3.7, when the leader's direct rate is higher than the sum of its negative impacts on all followers, that is, when C5 holds, *and* the sum of each follower's negative impacts on other followers and on the leader is greater than its direct rate, that is, when C6 holds, the social utility at RSE1 is higher than that at the NSE, that is, $\omega^{*\mathrm{RSE1}} > \omega^{*\mathrm{NSE}}$. In addition, when the leader's direct rate is less than the sum of its negative impacts on the followers, that is, when C7 holds, *and* the sum of negative impacts of each follower on other followers and on the leader is less than its direct rate, that is, when C8 holds, the social utility at RSE2 is higher than that at the NSE, that is, $\omega^{*\mathrm{RSE2}} > \omega^{*\mathrm{NSE}}$. In Appendix 8, we prove that the preceding statements are

Table 3.7 Social utility at RSE1 and RSE2 for single-leader/multifollower RSG

Case 1 ($\boldsymbol{\Upsilon}$ is a P-matrix)	Case 2 ($\boldsymbol{\Upsilon}$ is a P-matrix)
Followers' strategies decrease with $\boldsymbol{\varepsilon}$	Followers' strategies increase with $\boldsymbol{\delta}_0$
Leader's utility at RSE1 is $>$ that at NSE	Leader's utility at RSE2 is $<$ that at NSE
Followers' social utility at RSE1 is $<$ that at NSE	Followers' social utility at RSE2 is $>$ that at NSE
$\omega^{*\text{RSE1}} > \omega^{*\text{NSE}}$ when both C5 and C6 hold	$\omega^{*\text{RSE2}} > \omega^{*\text{NSE}}$ when both C7 and C8 hold

true. The leader can obtain its optimal strategy via an exhaustive search [44]. Note that C5 is the dual of C7, and C6 is the dual of C8.

In the multi-leader/multi-follower game, that is, when $N_{\text{L}} > 1$ and $N_{\text{F}} > 1$, one can consider either competition or cooperation between the leaders [65]. For instance, when cooperation is assumed, the nominal game is

$$\begin{aligned} &\max\nolimits_{\mathbf{p}_n \in \mathscr{A}_n} \sum\nolimits_{n \in \mathscr{N}_{\text{L}}} \upsilon_n(\mathbf{p}_n, \mathbf{f}_n), \\ &\text{subject to: } \max_{\mathbf{p}_m \in \mathscr{A}_m} \upsilon_m(\mathbf{p}_m, \mathbf{f}_m), \quad \forall m \in \mathscr{N}_{\text{F}}. \end{aligned} \tag{3.67}$$

Note that (3.67) is a bilevel and nonconvex optimization problem whose constraint involves the followers' game [85–88]. Hence, it belongs to the class of mathematical programs with equilibrium constraints (MPEC). In general, it is not easy to obtain a closed-form solution to (3.67). Instead, in [65, 67], numerical algorithms for solving (3.67) are proposed. Our approach to solving (3.67) and its robust counterpart is to randomly choose a leader tasked with obtaining the optimal transmit power vectors of all leaders via an exhaustive search. We assume that all leaders cooperate and provide their side information to the chosen leader, which would use such side information in its exhaustive search for optimal transmit power vectors for all the leaders. Next, all the leaders transmit at their optimal power levels, and subsequently the followers play their strategic noncooperative game to obtain their own transmit power vectors.

For the robust multi-leader/multi-follower game RSG1, the optimization problem of each follower is similar to (3.53). In this case, when $\boldsymbol{\Upsilon}$ is a P-matrix, and the uncertainty is small, the followers' strategies at RSE1 are decreasing functions of $\varepsilon_{\mathbf{f}_n}$ (Lemma 3.4 in Appendix 8), which means that when the followers' transmit power levels are reduced, their interference on the leaders is reduced, resulting in higher utilities for the leaders.

For the robust multileader/multifollower game RSG2, the leaders' optimization problem is

$$\begin{aligned} &\max\nolimits_{\mathbf{p}_n \in \mathscr{A}_n} \min\nolimits_{\mathbf{H}_{mn} \in \mathscr{R}_{\mathbf{H}_{mn}}} \sum\nolimits_{n \in \mathscr{N}_{\text{L}}} \upsilon_n(\mathbf{p}_n, \mathbf{f}_n), \\ &\text{subject to: } \max_{\mathbf{p}_m \in \mathscr{A}_m} \min_{\mathbf{f}_m \in \mathscr{R}_{\mathbf{f}_m}(\mathbf{p}_{-m})} u_m(\mathbf{p}_m, \mathbf{f}_m), \quad \forall m \in \mathscr{N}_{\text{F}}, \end{aligned} \tag{3.68}$$

which is nonconvex and MPEC. Hence, analyzing RSE2 is nontrivial. To solve this problem, the heuristic algorithm in [44] that converts the multileader/multifollower game into a single-leader/multifollower game can be used.

3.3.3.4 Overview of Other Works on RSGs

Robust down-link power control in orthogonal frequency division multiple access (OFDMA)-based networks with noncooperative macro base stations (BSs) and underlay small cells is studied in [89]. To model interactions, the RSG is used where the interference and CSI between macro BSs and small cells are uncertain, and the objective of macro BSs and small cells is to maximize their own respective throughput. The macro BSs and small cells are considered leaders and followers, respectively. To protect the quality of service (QoS) of the macro BS users, the interference of small cells with macro BS users is controlled via local and global interference constraints. The interference of each small cell with each macro BS is upper bounded, and the sum of interference levels caused by all small cells on macro-cell users is also upper bounded. The local interference constraint is reformulated so that the RSE can be analyzed. To apply the global interference constraint, the RSE is formulated as a GNE, which involves extensive calculations. By utilizing QVI [24], nonlinear complementarity problems (NCPs) [24], and restating the equilibrium point as an equilibrium program with equilibrium constraints (EPEC) [24], the RSE is studied in [89].

Other works on the RSE are [90, 91], but are not in the context of resource allocation in wireless networks. In [90], a single-leader/single-follower Stackelberg game is considered where the leader's side information is uncertain. By minimizing the second-order sensitivity function of the leader's utility with respect to the uncertain parameters, the worst-case utility for the leader is obtained. In [91], three new algorithms based on mixed-integer linear programming are proposed, namely, for cases in which the leader's side information about the follower's response is uncertain due to the follower's bounded rationality, for cases in which the follower's observations of the leader's strategy are noisy, and for cases in which the follower's reward is uncertain. Table 3.8 summarizes the key differences between other existing works on RSGs.

In this section, we analyzed the effect of uncertainty on the utilities of both leaders and followers and considered various issues associated with implementing the RSGs in wireless networks. We also presented the case of multiple leaders/multiple followers in wireless networks.

3.4 Concluding Remarks

In this chapter, we studied the impact of uncertainty in users' side information on power control games in wireless networks with noncooperative users. We began by assuming no uncertainty (the nominal game), discussed the existence

Table 3.8 Comparative overview of existing works on robust HetNets

Reference	Equilibrium analysis	Performance comparison	Distributed algorithm
[89]	VI	√	√
[90]	Best response	√	√
[91]	VI	–	–

and uniqueness of its equilibrium, presented a distributed algorithm for obtaining the game's solution, and studied its convergence. We then reformulated the game into a robust one via worst-case robust optimization and considered all of the aforementioned topics for the robust game in wireless networks with homogeneous users, in underlay CRNs, and in wireless networks with heterogeneous users. The results may be summarized as follows:

- **Robust Game-Theoretic Resource Allocation for Homogeneous Users:** We studied the RNE for a network with homogeneous users, where each user's utility depends on its action, which is additively coupled to other users' actions. When the impact of other users on each user is uncertain, we utilized the worst-case robust optimization to maximize each user's utility and analyzed the game. In doing so, we showed that the theory of finite-dimensional VI could be used to obtain the sufficient conditions for the existence and uniqueness of the RNE. We also showed that when uncertainty is small, the condition for the RNE's uniqueness is similar to that of the nominal NE, and we presented a distributed algorithm based on the proximal point method for reaching the RNE. Moreover, we showed that when the RNE is unique, the robust game's social utility is less than that of the nominal game, but when there are multiple RNEs, the robust game's social utility at one RNE may be higher than that of the nominal game. The distributed algorithm based on proximal response map was also presented for this case.
- **Robust Game-Theoretic Resource Allocation in Underlay CRNs:** In this case, channel gains between SUs and PUs are subject to uncertainty. Each SU aims to maximize its own utility, for example, its throughput, subject to its transmit power and interference constraints imposed by PUs. Even the nominal game cannot be easily analyzed due to the fact that SUs must guarantee that their interference with PUs is below a given threshold. This creates a coupling between SU strategies, which makes it difficult to satisfy the conditions for the existence and uniqueness of NE. However, we showed that it is possible to analyze such games (nominal and robust) via nonlinear complementarity problems (NCP) introduced in Appendix 1 in this chapter.
- **Robust Game-Theoretic Resource Allocation for Heterogeneous Users:** We utilized Stackelberg games to model interactions between heterogeneous users whose CSI is uncertain, and we analyzed its equilibria via sensitivity analysis (which does not require excessive calculations). We showed that when followers' observations are noisy and leaders' information is exact, the followers' utility is reduced and the leaders' utility is increased compared with those of the nominal

game. The converse is true when followers' observations are exact and leaders' side information is uncertain. We also showed the impact of uncertainty in parameter values on the sum rate of the robust game.

Appendices

Appendix 1: Complementarity Problems

When the set $\mathscr{A}$ is a cone (i.e., for $\mathbf{x} \in \mathscr{A} \rightarrow \alpha\mathbf{x} \in \mathscr{A}$ for all scalars $\alpha \geq 0$), the VI admits an equivalent form known as the complementarity problem, denoted by CP($\mathscr{A}, \mathscr{F}$), which is to find a vector $\mathbf{x}$ such that (Definition 1.1.2 in [24])

$$\mathscr{A} \ni \mathbf{x} \perp \mathscr{F}(\mathbf{x}) \in \mathscr{A}^*,$$

where the notation $\perp$ means *perpendicular*, and $\mathscr{A}^*$ is the dual cone of $\mathscr{A}$, defined as $\mathscr{A}^* = \{\mathbf{d} \in \mathbb{R}^n | \mathbf{v}^{\mathrm{T}}\mathbf{d} \geq 0,\ \forall \mathbf{v} \in \mathscr{A}\}$. When $\mathscr{A}$ is the nonnegative orthant of $\mathbb{R}^n$, CP($\mathbb{R}^n, \mathscr{F}$) is a nonlinear complementarity problem (NCP), denoted by $NCP(\mathscr{F})$. Since the dual cone of the nonnegative orthant is a nonnegative orthant itself, the task of $NCP(\mathscr{F})$ is to find a vector $\mathbf{x}$ such that (Definition 1.1.5 in [24])

$$0 \leq \mathbf{x} \perp \mathscr{F}(\mathbf{x}) \geq 0.$$

An important problem that can be solved by an NCP is a noncooperative game between SUs with a global interference constraint [45], which for each SU s is

$$\sum_{s\in\mathscr{S}} \sum_{k\in\mathscr{K}} g_{sq}^k p_s^k \leq IT_q, \quad \forall q \in \mathscr{Q}, \quad \text{or} \quad \sum_{s\in\mathscr{S}} g_{sq}^k p_s^k \leq IT_q^k, \quad \forall q \in \mathscr{Q}, \quad \forall k \in \mathscr{K},$$

where IT_q and IT_q^k are the interference thresholds of PU q over all channels and in channel k, respectively. Now, NE can be obtained via VI when Karush–Kuhn–Tucker (KKT) conditions hold for the feasibility of the SUs' strategy [2, 24].

KKT Conditions in VI: Let $\mathscr{A}$ include a finite number of differentiable inequalities and equalities, that is,

$$\mathscr{A} = \Big\{\mathbf{x} \in \mathbb{R}^n | \quad \mathbf{h}(\mathbf{x}) = 0, \quad \mathbf{g}(\mathbf{x}) \leq 0\Big\}, \tag{3.69}$$

in which $\mathbf{h}(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}^l$ and $\mathbf{g}(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are vector-valued continuously differentiable functions. For this NCP, we have that (Proposition 1.3.4 in [24])

(continued)

- The solution set of $VI(\mathscr{K}, \mathscr{F})$ is denoted by $\mathrm{SOL}(\mathscr{A}, \mathscr{F})$. For $\mathbf{x} \in \mathrm{SOL}(\mathscr{A}, \mathscr{F})$, there exist vectors $\boldsymbol{\lambda} \in \mathbb{R}^l$ and $\boldsymbol{\mu} \in \mathbb{R}^m$ such that

$$\mathscr{F}(\mathbf{x}) + \sum_j \mu_j \nabla h_j(\mathbf{x}) + \sum_i \nu_i \nabla g_i(\mathbf{x}) = \mathbf{0}, \tag{3.70}$$

$$\mathbf{h}(\mathbf{x}) = \mathbf{0}, \tag{3.71}$$

$$\mathbf{0} \le \boldsymbol{\lambda} \perp \mathbf{g}(\mathbf{x}) \ge \mathbf{0}; \tag{3.72}$$

- Conversely, when each function $h_j(\mathbf{x})$ is affine, each function $g_i(\mathbf{x})$ is convex, and $(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda})$ satisfies (3.71) and (3.72), we have $\mathbf{x} \in \mathrm{SOL}(\mathscr{A}, \mathscr{F})$.

The existence and uniqueness of solutions obtained via $\mathscr{A}$ in (3.69) can be investigated under different conditions pertaining to the VI. Corollary 2.2.5 in [24] establishes that the solution set of $VI(\mathscr{A}, \mathscr{F})$ is nonempty, that is, a solution for $VI(\mathscr{A}, \mathscr{F})$ exists when

1. $\mathscr{A}$ is a nonempty, convex, and compact subset of a finite-dimensional Euclidean space; and
2. $\mathscr{F}$ is a continuous mapping. The solution to $VI(\mathscr{A}, \mathscr{F})$ is unique when $\mathscr{F}$ is continuous and strongly monotone on the convex and closed set $\mathscr{A}$ (Theorem 2.3.3(b) in [24]). Moreover, the strong monotonicity of $\mathscr{F}$ is sufficient for the existence of a solution. Theorem 3.7 and the P-matrix property of matrix $\boldsymbol{\Upsilon}$ in (3.9) can also be used to investigate the existence and uniqueness of solutions, as per Section 12 in [2].

The preceding reformulation of games via the KKT conditions and NCP can also be applied to games with pricing [2]. Consider the nominal game in the underlay CRN in Section 1.4.2.1 of Chapter 1, and the case where PUs in $\mathscr{Q}$ impose pricing on SUs in $\mathscr{S}$ through an exogenous price vector $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_Q\}$ as a way to ensure that the SUs' interference on PUs is below a predefined threshold. In this case, the modified utility function of SU s is

$$v_s^{\text{pricing}} = v_s(\mathbf{p}_s, \mathbf{p}_{-s}) + \sum_{q \in \mathscr{Q}} \lambda_q \Big(\sum_{k \in \mathscr{K}} p_s^k g_{sq}^k \Big), \quad \forall s \in \mathscr{S}, \tag{3.73}$$

and the optimization problem of SU s is changed to

$$\max_{\mathbf{p}_s \in \mathscr{A}_s(\mathbf{p})} v_s^{\text{pricing}}, \quad \forall s \in \mathscr{S}, \tag{3.74}$$

where $\mathscr{A}_s(\mathbf{p})$ is the strategy set of SU s, which is dependent on other SUs' strategies via λ_q as

$$\mathbf{0} \leq \lambda_q \perp \sum_{S\in\mathscr{S}} \sum_{k\in\mathscr{K}} p_s^k g_{sq}^k \geq \mathbf{0}, \quad \forall q \in \mathscr{Q}. \tag{3.75}$$

The equilibrium point of this game is the NCP's solution for both fixed and variable prices in (3.75) [2, 45]. The conditions for the NE's existence and uniqueness in this type of game are presented in [2, 45, 92, 93]. Specifically, it is shown in Section 12.3 in [2] that such games can be reformulated via NCP, and the conditions for the NE's existence (Theorem 12.2) and uniqueness (Proposition 12.9) are derived. A distributed algorithm to obtain equilibrium is also developed in [2] (Proposition 12.18).

Appendix 2: Proof of $\Psi_n(\mathbf{p}_n, \mathbf{p}_{-n})$ Convexity

Since $\mathbf{f}_n$ is a linear function of other users' strategies and system parameters, and the norm function is convex and bounded to $\varepsilon_{\mathbf{f}_n}$ (Section 2.2.2 in [94]), as per Section 3.2 in [94], $\widetilde{\mathscr{A}}_n(\mathbf{p}_{-n})$ is a convex, bounded, and closed set. To prove that (3.26) is concave with respect to $\mathbf{p}_n$, we follow Section 3.1 in [94]. For $\mathbf{p}_n = \mu\mathbf{p}_n' + (1-\mu)\mathbf{p}_n''$ when $\mu \in [0, 1]$, we have

$$\begin{aligned}\Psi_n(\mathbf{p}_n, \mathbf{p}_{-n}) &= \min_{\mathbf{f}_n\in\mathscr{R}_{\mathbf{f}_n}(\mathbf{p}_{-n})} u_n\left(\mu\mathbf{p}_n' + (1-\mu)\mathbf{p}_n'', \mathbf{f}_n\right)\\ &\geq \min_{\mathbf{f}_n\in\mathscr{R}_{\mathbf{f}_n}(\mathbf{p}_{-n})} \mu u_n(\mathbf{p}_n', \mathbf{f}_n) + (1-\mu)u_n(\mathbf{p}_n'', \mathbf{f}_n)\\ &= \mu\Psi_n(\mathbf{p}_n', \mathbf{p}_{-n}') + (1-\mu)\Psi_n(\mathbf{p}_n'', \mathbf{p}_{-n}''). \end{aligned} \tag{3.76}$$

Since (3.76) assumes the convexity of $u_n(\mathbf{p}_n, \mathbf{f}_n)$ with respect to $\mathbf{p}_n$, the function $\Psi_n(\mathbf{p}_n, \mathbf{p}_{-n})$ is concave in $\mathbf{p}_n$. The same is true of the convexity of $\Psi_n(\mathbf{p}_n, \mathbf{p}_{-n})$ with respect to $\mathbf{f}_n$. Since $\Psi_n(\mathbf{p}_n, \mathbf{p}_{-n})$ is concave, the Lagrange dual function for (3.26) in the uncertainty region is

$$L(\mathbf{p}_n, \mathbf{f}_n, \lambda_n) = \sum_{k=1}^{K} u_n^k(p_n^k, f_n^k) - \lambda_n \left(\varepsilon_{\mathbf{f}_n}^2 - \sum_{k=1}^{K}(f_n^k - \bar{f}_n^k)^2\right),$$

where λ_n is the nonnegative Lagrange multiplier that satisfies

$$\lambda_n^* \times \left(\varepsilon_{\mathbf{f}_n}^2 - \sum_{k=1}^{K}(f_n^{k*} - \bar{f}_n^k)^2\right) = 0, \tag{3.77}$$

where λ_n^* and f_n^{k*} are the optimal solutions to (3.77). The solution to (3.77) for f_n^k is obtained from $\frac{\partial L(\mathbf{p}_n,\mathbf{f}_n,\lambda_n)}{\partial f_n^k} = 0$ [94, 95], which yields

$$\frac{\partial u_n^k(p_n^k, f_n^{k*})}{\partial f_n^k} = -2\lambda_n^* \times (f_n^{k*} - \bar{f}_n^k), \qquad \forall k \in \mathscr{K},$$

which can be rewritten as $(f_n^{k*} - \bar{f}_n^k) = \frac{1}{-2\lambda_n^*} \times \frac{\partial u_n^k(p_n^k, f_n^{k*})}{\partial f_n^k}$. Using this in (3.77), λ_n^* is

$$\lambda_n^* = \frac{1}{2\varepsilon_{\mathbf{f}_n}} \times \sqrt{\sum_{n=1}^{N} \left(\frac{\partial u_n^k(p_n^k, f_n^{k*})}{\partial f_n^k} \right)^2}.$$

Hence, the uncertain parameter is $\mathbf{f}_n^* = \bar{\mathbf{f}}_n - \varepsilon_{\mathbf{f}_n} \boldsymbol{\vartheta}_n$, where $\mathbf{f}_n^* = [f_n^{1*}, \ldots, f_n^{K*}]$, $\boldsymbol{\vartheta}_n = [\vartheta_n^1, \ldots, \vartheta_n^K]$, and ϑ_n^k is (3.27). By replacing ϑ_n^k in u_n, we have $\Psi_n(\mathbf{p}_n, \mathbf{p}_{-n}) = u_n(\mathbf{p}_n, \mathbf{f}_n)|_{\mathbf{f}_n^* = \bar{\mathbf{f}}_n - \varepsilon_{\mathbf{f}_n} \boldsymbol{\vartheta}_n}$. The difference between $\Psi_n(\mathbf{p}_n, \mathbf{p}_{-n})$ and the utility function of the nominal game $\upsilon_n(\mathbf{p}_n, \mathbf{f}_n)$ is the extra term $\varepsilon_{\mathbf{f}_n} \boldsymbol{\vartheta}_n$. From Assumptions A1–A3 in Section 3.2.1.3 in this chapter, $\varepsilon_{\mathbf{f}_n} \boldsymbol{\vartheta}_n$ is continuous. Thus, $\Psi_n(\mathbf{p}_n, \mathbf{p}_{-n})$ is continuous in $\mathbf{p} = (\mathbf{p}_n, \mathbf{p}_{-n})$. Now, the derivative of Ψ_n with respect to $\mathbf{p}_n$ is

$$\begin{aligned} \nabla_{\mathbf{p}_n} \Psi_n(\mathbf{p}_n, \mathbf{p}_{-n}) &= \nabla_{\mathbf{p}_n} u_n(\mathbf{p}_n, \bar{\mathbf{f}}_n - \varepsilon_{\mathbf{f}_n} \boldsymbol{\vartheta}_n) + \nabla_{\mathbf{f}_n} u_n(\mathbf{p}_n, \bar{\mathbf{f}}_n - \varepsilon_{\mathbf{f}_n} \boldsymbol{\vartheta}_n) \times \mathbf{1}_K \times \nabla_{\mathbf{p}_n} \mathbf{f}_n \times \mathbf{1}_K^{\mathrm{T}} \\ &= \nabla_{\mathbf{p}_n} u_n(\mathbf{p}_n, \bar{\mathbf{f}}_n - \varepsilon_{\mathbf{f}_n} \boldsymbol{\vartheta}_n) - \varepsilon_{\mathbf{f}_n} \nabla_{\mathbf{f}_n} u_n(\mathbf{p}_n, \bar{\mathbf{f}}_n - \varepsilon_{\mathbf{f}_n} \boldsymbol{\vartheta}_n) \times \mathbf{1}_K \times \nabla_{\mathbf{p}_n} \boldsymbol{\vartheta}_n \times \mathbf{1}_K^{\mathrm{T}}, \end{aligned} \tag{3.78}$$

where $\mathbf{1}_K$ is a $1 \times K$ vector whose elements are equal to one. The last term in (3.78) contains $\frac{\partial^2 u_n^k}{\partial p_n^k \partial f_n^k}$. From Assumption A3 in Section 3.2.1.3 in this chapter, $\Psi_n(\mathbf{p}_n, \mathbf{p}_{-n})$ is differentiable with respect to $\mathbf{p}_n$, and

$$\nabla_{\mathbf{p}_{-n}} \Psi_n(\mathbf{p}_n, \mathbf{p}_{-n}) = \nabla_{\mathbf{f}_n} u_n(\mathbf{p}_n, \bar{\mathbf{f}}_n - \varepsilon_{\mathbf{f}_n} \boldsymbol{\vartheta}_n) \times \mathbf{1}_K \times \nabla_{\mathbf{p}_{-n}} \mathbf{f}_n \times \mathbf{1}_K^{\mathrm{T}},$$

which is continuous from Assumptions A1–A3 in Section 3.2.1.3 in this chapter, the definition of $\mathbf{f}_n$, and the definition of $\boldsymbol{\vartheta}_n$ in (3.27) in this chapter.

Appendix 3: Proof of Lemma 3.1

For the robust game we have $GVI(\mathscr{A}, \widetilde{\mathscr{F}})$, and $\widetilde{\mathscr{F}}(\mathbf{p}) = (\widetilde{\mathscr{F}}_n(\mathbf{p}))_{n=1}^N$, where $\widetilde{\mathscr{F}}_n(\mathbf{p})$ for user n is obtained from (3.78). Variations in CSI between transmitters and receivers and in users' strategies cause variations in $\mathbf{f}_n$ for user n. When these variations are negligible, we have $\|\widehat{\mathbf{e}}_n\| = \varepsilon_{\mathbf{f}_n} = 0$. The Taylor series expansion of $\widetilde{\mathscr{F}}_n(\mathbf{p})$ around $\varepsilon_{\mathbf{f}_n}$ is $\widetilde{\mathscr{F}}_n(\mathbf{p}) = \left[\widetilde{\mathscr{F}}_n(\mathbf{p})\right]_{\varepsilon_{\mathbf{f}_n}=0} + \left[\sum_{i=1}^{\infty} \frac{1}{i!} (\varepsilon_{\mathbf{f}_n})^i (\nabla_{\mathbf{f}_n}^i \widetilde{\mathscr{F}}_n)\right]_{\varepsilon_{\mathbf{f}_n}=0}$. From (3.78), the Taylor series expansion is

$$\widetilde{\mathscr{F}}_n(\mathbf{p}) = -\Big[\nabla_{\mathbf{p}_n} u_n(\mathbf{p}_n, \bar{\mathbf{f}}_n - \varepsilon_{\mathbf{f}_n}\boldsymbol{\vartheta}_n)\Big]_{(\varepsilon_{\mathbf{f}_n}=0)}$$
$$-\varepsilon_{\mathbf{f}_n}\Big[\nabla^2_{\mathbf{p}_n\mathbf{f}_n} u_n(\mathbf{p}_n, \bar{\mathbf{f}}_n - \varepsilon_{\mathbf{f}_n}\boldsymbol{\vartheta}_n) \times (\mathbf{1}_K^{\mathrm{T}} - \varepsilon_{\mathbf{f}_n}\nabla_{\mathbf{f}_n}\boldsymbol{\vartheta}_n \times \mathbf{1}_K^{\mathrm{T}})\Big]_{(\varepsilon_{\mathbf{f}_n}=0)}$$
$$-\frac{\varepsilon_n^2}{2}\Big[\nabla^3_{\mathbf{p}_n\mathbf{f}_n^2} u_n(\mathbf{p}_n, \bar{\mathbf{f}}_n - \varepsilon_{\mathbf{f}_n}\boldsymbol{\vartheta}_n) \times (\mathbf{1}_K^{\mathrm{T}} - \varepsilon_{\mathbf{f}_n}\nabla_{\mathbf{f}_n}\boldsymbol{\vartheta}_n \times \mathbf{1}_K^{\mathrm{T}}) \times (\mathbf{1}_K^{\mathrm{T}} - \varepsilon_{\mathbf{f}_n}\nabla_{\mathbf{f}_n}\boldsymbol{\vartheta}_n \times \mathbf{1}_K^{\mathrm{T}})$$
$$+\nabla^2_{\mathbf{p}_n\mathbf{f}_n} u_n(\mathbf{p}_n, \bar{\mathbf{f}}_n - \varepsilon_{\mathbf{f}_n}\boldsymbol{\vartheta}_n) \times (\varepsilon_{\mathbf{f}_n}\nabla^2_{\mathbf{f}_n\mathbf{f}_n}\boldsymbol{\vartheta}_n \times \mathbf{1}_K^{\mathrm{T}})\Big]_{(\varepsilon_{\mathbf{f}_n}=0)} + o(\cdot). \tag{3.79}$$

From (3.5), and since $\mathbf{f}_n = \bar{\mathbf{f}}_n$ for $\varepsilon_{\mathbf{f}_n} = 0$, we rewrite (3.79) as

$$\widetilde{\mathscr{F}}_n(\mathbf{p}) = \mathscr{F}_n(\mathbf{p}) - \varepsilon_{\mathbf{f}_n}\left[\nabla^2_{\mathbf{p}_n\mathbf{f}_n} v_n(\mathbf{p}_n, \mathbf{f}_n) \times \mathbf{1}_K^{\mathrm{T}}\right] - \frac{\varepsilon_n^2}{2}\left[\nabla^3_{\mathbf{p}_n\mathbf{f}_n^2} v_n(\mathbf{p}_n, \mathbf{f}_n) \times \mathbf{1}_K^{\mathrm{T}}\right] + o(\cdot). \tag{3.80}$$

For $0 < \varepsilon_{\mathbf{f}_n} \ll 1$, the value of $o(\cdot)$ is small and can be ignored.[4] From Assumption A1 in Section 3.2.1.3 in this chapter, all derivatives of $v_n(\mathbf{p}_n, \mathbf{f}_n)$ are bounded. Hence, the last three terms on the right-hand side of (3.80) are bounded, and $\widetilde{\mathscr{F}}_n(\mathbf{p})$ is the bounded perturbed version of $\mathscr{F}_n(\mathbf{p})$. Note that (3.80) also shows that $\widetilde{\mathscr{F}}_n(\mathbf{p})$ is continuous over $\mathbf{p}$ for small values of $\varepsilon_{\mathbf{f}_n}$.

Appendix 4: Proof of Theorem 3.9

- As was shown in Lemma 3.1, when uncertainty is small, the RNE is a perturbed solution to $VI(\mathscr{A}, \mathscr{F})$ and can be obtained from $VI(\mathscr{A}, \mathscr{F} + \mathbf{q})$. Recall that when (3.9) is a P-matrix, $\mathscr{F}(\mathbf{p})$ is strongly monotone, and the utility is strongly convex (see Proof of Theorem 3.7 and Lemma 3.2). Since $\mathscr{A}$ is convex in $\mathbb{R}^K$, and $\mathscr{F}(\mathbf{p}) : K \to \mathbb{R}^K$ is a continuous mapping on $\mathscr{A}$, the solution to $VI(\mathscr{A}, \mathscr{F} + \mathbf{q})$ is always a compact and convex set (Corollary 2.6.4 in [24]). This solution set contains $\mathbf{p}^*$ and $\widetilde{\mathbf{p}}^*$, and we have

$$(\widetilde{\mathbf{p}}^* - \mathbf{p}^*)\mathscr{F}(\mathbf{p}^*) \geq 0, \qquad \text{and} \qquad (\mathbf{p}^* - \widetilde{\mathbf{p}}^*)\widetilde{\mathscr{F}}(\widetilde{\mathbf{p}}^*) \geq 0.$$

  From the preceding expressions we get

$$(\widetilde{\mathbf{p}}^* - \mathbf{p}^*)\Big(\mathscr{F}(\mathbf{p}^*) - \widetilde{\mathscr{F}}(\widetilde{\mathbf{p}}^*)\Big) \geq 0. \tag{3.81}$$

[4] In general, when higher-order terms in the Taylor series expansion are ignored, the comparison results hold in the neighborhood of the equilibrium at which Taylor series expansion is applied. Our approximation is for small values of $\varepsilon_{\mathbf{f}_n}$, meaning that the robust equilibrium, which is a bounded perturbed version of the nominal equilibrium, is in fact in its neighborhood, and higher-order terms in the Taylor series can be ignored as they are multiplied by a higher power of $\varepsilon_{\mathbf{f}_n}$.

When the utility of user n at the RNE is greater than that at the nominal NE (i.e., when $\Psi_n(\widetilde{\mathbf{p}}_n^*, \widetilde{\mathbf{p}}_{-n}^*) > v_n\left(\mathbf{p}_n^*, \bar{\mathbf{f}}_n(\mathbf{p}_{-n}^*, \mathbf{s}_n)\right))$, we have $\mathscr{F}_n(\mathbf{p}^*) > \widetilde{\mathscr{F}}_n(\widetilde{\mathbf{p}}^*)$. When $\mathscr{F}(\mathbf{p}^*) > \widetilde{\mathscr{F}}(\widetilde{\mathbf{p}}^*)$, we have $\widetilde{\mathbf{p}}^* \geq \mathbf{p}^*$, and when ε_n is small, $\widetilde{\mathscr{F}}(\widetilde{\mathbf{p}}^*) \approx \mathscr{F}(\widetilde{\mathbf{p}}^*)$. Hence, from (3.81) we have

$$(\widetilde{\mathbf{p}}^* - \mathbf{p}^*)\left(\mathscr{F}(\widetilde{\mathbf{p}}^*) - \mathscr{F}(\mathbf{p}^*)\right) \leq 0. \tag{3.82}$$

Note that (3.82) contradicts the strong monotonicity of $\mathscr{F}(\mathbf{p})$, and this contradiction implies that each user's utility at the RNE is less than that at the nominal NE. Consequently, the social utility (the sum rate of all users) at the RNE is less than that at the nominal NE.

- Since $\mathscr{A}$ is a closed convex set and $\widetilde{\mathscr{F}}(\mathbf{p})$ is continuous and strongly monotone, there is a unique solution to $VI(\mathscr{A}, \widetilde{\mathscr{F}}(\mathbf{p}))$ [24], denoted by $\widetilde{\mathbf{p}}^* = \Phi^*(\mathbf{q})$, that can be considered the worst-case robust solution to $\widetilde{\mathscr{G}}$ for $\|\mathbf{q}\|_2 \leq \|\boldsymbol{\varepsilon}\|_2$. Now, both $\mathbf{p}_n^*$ and $\widetilde{\mathbf{p}}_n^*$ must satisfy

$$0 \leq \left(\Phi^*(\mathbf{q}) - \Phi^*(\mathbf{0})\right)\left(\mathscr{F}\left(\Phi^*(\mathbf{0})\right)\right) \tag{3.83}$$

and

$$0 \leq \left(\Phi^*(\mathbf{0}) - \Phi^*(\mathbf{q})\right)\left(\mathscr{F}\left(\Phi^*(\mathbf{q})\right) + \mathbf{q}\right), \tag{3.84}$$

where $\mathbf{0} = (\mathbf{0}_K)_1^N$, and $\mathbf{0}_K$ is the $K \times 1$ all zero vector. From (3.83) and (3.84) we get

$$\left(\Phi^*(\mathbf{0}) - \Phi^*(\mathbf{q})\right)\left(\mathscr{F}\left(\Phi^*(\mathbf{0})\right) - \mathscr{F}\left(\Phi^*(\mathbf{q})\right)\right) \leq \left(\Phi^*(\mathbf{0}) - \Phi^*(\mathbf{q})\right)\mathbf{q}. \tag{3.85}$$

Using the Schwartz inequality for the right-hand side, we have

$$\|(\Phi^*(\mathbf{0}) - \Phi^*(\mathbf{q}))\mathbf{q}\|_2 \leq \|\Phi^*(\mathbf{0}) - \Phi^*(\mathbf{q})\|_2\|\mathbf{q}\|_2.$$

Since $\Phi^*(\mathbf{q})$ is the co-coercive function of $\mathbf{q}$ (Proposition 2.3.11 in [24]), the left-hand side of (3.85) is always greater than $c_{\text{sm}}\|\Phi^*(\mathbf{0}) - \Phi^*(\mathbf{q})\|_2^2$. Therefore, (3.85) can be rewritten as

$$c_{\text{sm}}\|\Phi^*(\mathbf{0}) - \Phi^*(\mathbf{q})\|_2^2 \leq \|\Phi^*(\mathbf{0}) - \Phi^*(\mathbf{q})\|_2\|\mathbf{q}\|_2,$$

which is simplified to

$$c_{\text{sm}}\|\Phi^*(\mathbf{0}) - \Phi^*(\mathbf{q})\|_2 \leq \|\mathbf{q}\|_2.$$

Recall that $\Phi^*(\mathbf{0})$ and $\Phi^*(\mathbf{q})$ correspond to $\mathbf{p}_n^*$ and $\widetilde{\mathbf{p}}_n^*$, respectively, and the upper bound on $\mathbf{q}$ is $\boldsymbol{\varepsilon}$. Hence, $c_{\text{sm}}\|\mathbf{p}_n^* - \widetilde{\mathbf{p}}_n^*\|_2 \leq \|\boldsymbol{\varepsilon}\|_2$, which is the same as (3.41).

- **Proof of** (3.39)**:** Since $\widetilde{\mathscr{F}}(\mathbf{p})$ is a bounded perturbed version of $\mathscr{F}(\mathbf{p})$, the difference between the utility values of user n at the RNE and at the nominal NE can be approximated by the first term of the Taylor series expansion of $u_n^k(\mathbf{p}_n, \mathbf{f}_n)$, which is equal to $\varepsilon_{\mathbf{f}_n}(\frac{\partial u_n^k(p_n^k, f_n^k)}{\partial \varepsilon_{\mathbf{f}_n}})_{\varepsilon_{\mathbf{f}_n}=\mathbf{0}}$. By some mathematical manipulations, we get

$$u_n^k(p_n^k, f_n^k) - v_n^k(p_n^k, f_n^k) \approx \varepsilon_{\mathbf{f}_n}\left(\frac{\partial v_n^k(p_n^k, f_n^k)}{\partial p_n^k} \times \frac{\partial p_n^k}{\partial \varepsilon_{\mathbf{f}_n}} + \frac{\partial v_n^k(p_n^k, f_n^k)}{\partial f_n^k} \times \frac{\partial f_n^k}{\partial \varepsilon_{\mathbf{f}_n}}\right),$$

for all $n \in \mathscr{N}$ and $k \in \mathscr{K}$, which is

$$u_n^k(p_n^k, f_n^k) - v_n^k(p_n^k, f_n^k) \approx \varepsilon_{\mathbf{f}_n}\left(\frac{\partial v_n^k(p_n^k, f_n^k)}{\partial p_n^k} \times \frac{\partial p_n^k}{\partial \varepsilon_{\mathbf{f}_n}} + \frac{\partial v_n^k(p_n^k, f_n^k)}{\partial f_n^k} \times \sum_{m \neq n} h_{mn}^k \frac{\partial p_m^k}{\partial \varepsilon_{\mathbf{f}_n}}\right). \tag{3.86}$$

When $\varepsilon_{\mathbf{f}_n}$ is sufficiently small, $\frac{\partial p_n^k}{\partial \varepsilon_{\mathbf{f}_n}} = \lim_{\varepsilon_{\mathbf{f}_n} \to 0} \frac{\tilde{p}_n^{*k} - p_n^{*k}}{\varepsilon_{\mathbf{f}_n}}$. Now, (3.86) for all users is

$$\|v(\mathbf{p}^*) - u(\tilde{\mathbf{p}}^*)\|_2 \approx \|\mathbf{w}(\mathbf{p}^*)\|_2 \times \|\mathbf{p}^* - \widetilde{\mathbf{p}}^*\|. \tag{3.87}$$

By substituting (3.41) into (3.87), the approximation (3.39) is obtained. In (3.87), we use the first term in the Taylor series for v_n to approximate $\|v(\mathbf{p}^*) - u(\tilde{\mathbf{p}}^*)\|_2$. The remainder of these differences for user n is always less than or equal to $\frac{\varepsilon_{\mathbf{f}_n}^2}{2!}\frac{\partial^2 v_n}{\partial^2 \mathbf{p}_n}$ [96]. For all users, this remainder is upper bounded to $\frac{\|\mathbf{J}(\mathscr{F})\|_2\|\boldsymbol{\varepsilon}\|_2^2}{2}$.

Appendix 5: Proof of Theorem 3.10

To derive the convergence condition for the proximal response approach, one needs to show that the solution to (3.43) is a contraction mapping. For any vector $\mathbf{z} \in \mathscr{A}$ in (3.43), we have

$$\left(\mathbf{z} - \mathbf{p}^{\text{PRM}}(\mathbf{b})\right)\left[\widetilde{\mathscr{F}}\left(\mathbf{p}^{\text{PRM}}(\mathbf{b}), \mathbf{b}\right) + \left(\mathbf{p}^{\text{PRM}}(\mathbf{b}) - \mathbf{b}\right)^{\text{T}}\right] \geq 0, \tag{3.88}$$

$$\left(\mathbf{z} - \mathbf{p}^{\text{PRM}}(\mathbf{b}')\right)\left[\widetilde{\mathscr{F}}\left(\mathbf{p}^{\text{PRM}}(\mathbf{b}'), \mathbf{b}'\right) + \left(\mathbf{p}^{\text{PRM}}(\mathbf{b}') - \mathbf{b}'\right)^{\text{T}}\right] \geq 0. \tag{3.89}$$

When $\mathbf{z} = \mathbf{p}^{\mathrm{PRM}}(\mathbf{b}')$ in (3.88), and $\check{\mathbf{z}} = \mathbf{p}^{\mathrm{PRM}}(\mathbf{b})$ in (3.89), we have

$$\begin{aligned} 0 &\le \Big(\mathbf{p}^{\mathrm{PRM}}(\mathbf{b}') - \mathbf{p}^{\mathrm{PRM}}(\mathbf{b})\Big)\Big[\widetilde{\mathscr{F}}\big(\mathbf{p}^{\mathrm{PRM}}(\mathbf{b}), \mathbf{b}\big) + \big(\mathbf{p}^{\mathrm{PRM}}(\mathbf{b}) - \mathbf{b}\big)^{\mathrm{T}}\Big] \\ &\quad + \big(\mathbf{p}^{\mathrm{PRM}}(\mathbf{b}) - \mathbf{p}^{\mathrm{PRM}}(\mathbf{b}')\big)\Big[\widetilde{\mathscr{F}}\big(\mathbf{p}^{\mathrm{PRM}}(\mathbf{b}'), \mathbf{b}'\big) + \big(\mathbf{p}^{\mathrm{PRM}}(\mathbf{b}') - \mathbf{b}'\big)^{\mathrm{T}}\Big] \\ &= \big(\mathbf{p}^{\mathrm{PRM}}(\mathbf{b}') - \mathbf{p}^{\mathrm{PRM}}(\mathbf{b})\big)\Big[\widetilde{\mathscr{F}}\big(\mathbf{p}^{\mathrm{PRM}}(\mathbf{b}), \mathbf{b}\big) - \widetilde{\mathscr{F}}\big(\mathbf{p}^{\mathrm{PRM}}(\mathbf{b}'), \mathbf{b}'\big)\Big] \\ &\quad - \|\mathbf{p}^{\mathrm{PRM}}(\mathbf{b}') - \mathbf{p}^{\mathrm{PRM}}(\mathbf{b})\| + \Big(\mathbf{p}^{\mathrm{PRM}}(\mathbf{b}') - \mathbf{p}^{\mathrm{PRM}}(\mathbf{b})\Big)(\mathbf{b} - \mathbf{b}')^{\mathrm{T}}. \end{aligned} \tag{3.90}$$

When $\frac{\partial^3 v_n}{\partial \mathbf{p}_n \partial^2 \mathbf{f}_n} = \frac{\partial^3 v_n}{\partial^2 \mathbf{p}_n \partial \mathbf{f}_n} = 0$, we have

$$\widetilde{\mathscr{F}}_n = -\nabla_{\mathbf{p}_n} u_n(\mathbf{p}_n, \bar{\mathbf{f}}_n + \varepsilon_{\mathbf{f}_n} \boldsymbol{\vartheta}_n) - \varepsilon_{\mathbf{f}_n} \nabla_{\mathbf{f}_n} u_n(\mathbf{p}_n, \bar{\mathbf{f}}_n + \varepsilon_{\mathbf{f}_n} \boldsymbol{\vartheta}_n) \times \mathbf{1}_K \times \nabla_{\mathbf{p}_n} \boldsymbol{\vartheta}_n \times \mathbf{1}_k^{\mathrm{T}},$$

$$\nabla_{\mathbf{p}_n} \widetilde{\mathscr{F}}_n = -\nabla^2_{\mathbf{p}_n, \mathbf{p}_n} u_n + \varepsilon_{\mathbf{f}_n} \times \nabla^3_{\mathbf{p}_n \mathbf{p}_n \mathbf{f}_n} u_n,$$

$$\nabla_{\mathbf{p}_m} \widetilde{\mathscr{F}}_n = -\nabla^2_{\mathbf{p}_n \mathbf{p}_m} u_n + \varepsilon_{\mathbf{f}_n} \times \nabla^3_{\partial \mathbf{p}_n \partial^2 \mathbf{f}_n} u_n \times \mathbf{1}_K^{\mathrm{T}} \times \mathbf{h}_{mn}.$$

Consequently, (3.90) can be rewritten as

$$\begin{aligned} &\big(\mathbf{p}^{\mathrm{PRM}}(\mathbf{b}') - \mathbf{p}^{\mathrm{PRM}}(\mathbf{b})\big)\Big[\sum_{n \in \mathcal{N}} -\nabla^2_{\mathbf{p}_n \mathbf{p}_n} u_n\Big]\Big(\mathbf{p}^{\mathrm{PRM}}(\mathbf{b}') - \mathbf{p}^{\mathrm{PRM}}(\mathbf{b})\Big)^{\mathrm{T}} \\ &\quad + \big(\mathbf{p}^{\mathrm{PRM}}(\mathbf{b}) - \mathbf{p}^{\mathrm{PRM}}(\mathbf{b}')\big)\Big[\sum_{m \in \mathcal{N}, m \neq n} -\nabla^2_{\mathbf{p}_n \mathbf{p}_m} u_n\Big](\mathbf{b} - \mathbf{b}')^{\mathrm{T}} \\ &\quad - \|\mathbf{p}^{\mathrm{PRM}}(\mathbf{b}') - \mathbf{p}^{\mathrm{PRM}}(\mathbf{b})\| + \big(\mathbf{p}(\mathbf{b}') - \mathbf{p}(\mathbf{b})\big)\big(\mathbf{b} - \mathbf{b}'\big)^{\mathrm{T}} \ge 0. \end{aligned} \tag{3.91}$$

For $\widetilde{\alpha}_n(\mathbf{p}) \triangleq$ smallest eigenvalue of $-\nabla^2_{\mathbf{p}_n} u_n(\mathbf{p}_n, \mathbf{f}_n)$, $\widetilde{\beta}_{nm}(\mathbf{p}) \triangleq \|-\nabla_{\mathbf{p}_n \mathbf{p}_m} u_n(\mathbf{p}_n, \mathbf{f}_n)\|$, $\forall n \neq m$, and $\mathbf{z} \triangleq \tau\big(\mathbf{p}^{\mathrm{PRM}}(\mathbf{b}), \mathbf{b}\big) + (1-\tau)\big(\mathbf{p}^{\mathrm{PRM}}(\mathbf{b}'), \mathbf{b}'\big)$, from (3.91), we have

$$\big(1 + \widetilde{\alpha}_n(\mathbf{z})\big)\|\mathbf{p}^{\mathrm{PRM}}(\mathbf{b}') - \mathbf{p}^{\mathrm{PRM}}(\mathbf{b})\| \le \sum_{n=1}^{N} \widetilde{\beta}_{nm}(\mathbf{z}_n)\|\mathbf{b}_{-n} - \mathbf{b}'_{-n}\|. \tag{3.92}$$

On the other hand,

$$-\nabla^2_{\mathbf{p}_n \mathbf{p}_n} u_n = -\nabla_{\mathbf{p}_n \mathbf{p}_n} u_n(\mathbf{p}_n, \bar{\mathbf{f}}_n - \varepsilon_{\mathbf{f}_n} \boldsymbol{\vartheta}_n) + \varepsilon_{\mathbf{f}_n} \nabla_{\mathbf{f}_n} u_n(\mathbf{p}_n, \bar{\mathbf{f}}_n - \varepsilon_{\mathbf{f}_n} \boldsymbol{\vartheta}_n) \times \mathbf{1}_K \times \nabla_{\mathbf{p}_n} \boldsymbol{\vartheta}_n.$$

Since the utility is convex with respect to $\mathbf{p}_n$, we have

$$-\nabla_{\mathbf{p}_n \mathbf{p}_n} u_n(\mathbf{p}_n, \bar{\mathbf{f}}_n - \varepsilon_{\mathbf{f}_n} \boldsymbol{\vartheta}_n) > 0 \quad \text{and} \quad \|\nabla^2_{\mathbf{p}_n \mathbf{p}_n} u_n\| \ge \|\nabla^2_{\mathbf{p}_n \mathbf{p}_n} v_n\|.$$

In addition, $\nabla^2_{\mathbf{p}_n\mathbf{p}_m} u_n = \nabla_{\mathbf{p}_n\mathbf{p}_m} u_n(\mathbf{p}_n, \bar{\mathbf{f}}_n + \varepsilon_{\mathbf{f}_n}\boldsymbol{\vartheta}_n) - \varepsilon_{\mathbf{f}_n} \nabla_{\mathbf{f}_n} u_n(\mathbf{p}_n, \bar{\mathbf{f}}_n - \varepsilon_{\mathbf{f}_n}\boldsymbol{\vartheta}_n) \times \mathbf{1}_K \times \nabla_{\mathbf{p}_n}\boldsymbol{\vartheta}_n$, which leads to

$$\|\nabla^2_{\mathbf{p}_n\mathbf{p}_m} u_n\| \leq \|\nabla^2_{\mathbf{p}_n\mathbf{p}_m} v_n\|.$$

From the preceding expression, (3.92) is transformed into

$$\big(1 + \alpha_n(\mathbf{z})\big)\|\mathbf{p}^{\mathrm{PRM}}(\mathbf{b}') - \mathbf{p}^{\mathrm{PRM}}(\mathbf{b})\| \leq \sum_{n=1}^{N} \beta_{nm}(\mathbf{z}^n)\|\mathbf{b}_{-n} - \mathbf{b}'_{-n}\|,$$

which means that when (3.9) is a P-matrix, (3.44) is a contraction mapping (Proposition 12.17 in Section 12 in [2]). Therefore, the distributed algorithm in Table 3.4 converges to a unique RNE.

Appendix 6: Proof of (3.42)

Recall that $\mathscr{M}$ is strongly monotone on $\mathscr{A}$ if there exists a c_{sm} such that for all $\mathbf{p} = (\mathbf{p}_n)_{n\in\mathscr{N}}$ and $\mathbf{p}' = (\mathbf{p}'_n)_{n\in\mathscr{N}}$ we have

$$(\mathbf{p} - \mathbf{p}')\Big(\mathscr{M}(\mathbf{p}) - \mathscr{M}(\mathbf{p}')\Big) \geq c_{\mathrm{sm}} \parallel \mathbf{p} - \mathbf{p}' \parallel .$$

For $d_n^k = \Big(p_n^k - (p_n^k)'\Big)$, we write

$$\begin{aligned}
&(\mathbf{p}_n - \mathbf{p}'_n)\Big(\mathscr{M}_n(\mathbf{p}) - \mathscr{M}_n(\mathbf{p}')\Big)\\
&= (\mathbf{p}_n - \mathbf{p}'_n)\Big(\sum_{m=1}^{N}\big(\mathbf{M}_{nm}(\mathbf{p}_m)^{\mathrm{T}} - \mathbf{M}_{nm}(\mathbf{p}'_m)^{\mathrm{T}}\big)\Big)\\
&= \sum_{k=1}^{K}\Big(p_n^k - (p_n^k)'\Big)\Big(\sum_{m=1}^{N} M_{nm}^{kk}(p_n^k - (p_n^k)')\Big)\\
&\geq \sum_{k=1}^{K}(d_n^k)^2 - \sum_{m=1,\, m\neq n}^{N} |\sum_{k=1}^{K} d_m^k \frac{\bar{h}_{mn}^k}{\bar{h}_{mm}^k} d_n^k|\\
&\geq \sum_{k=1}^{K}(d_n^k)^2 - \sum_{m=1,\, m\neq n}^{N} (\sum_{k=1}^{K} d_n^k)^2 \max_{k\in K}\frac{\bar{h}_{mn}^k}{\bar{h}_{mm}^k}\Big(\sum_{k=1}^{K}(d_m^k)^2\Big)^2\\
&\geq \|\mathbf{d}_n\|_2 \sum_{m=1}^{N}[\mathbf{M}^{\max}]_{nm}\|\mathbf{d}_m\|_2,
\end{aligned} \tag{3.93}$$

where $\mathbf{d}_n = [d_n^1 \cdots d_n^k]^{\mathrm{T}}$. Hence, $(\mathbf{p} - \mathbf{p}')\Big(\mathscr{M}(\mathbf{p}) - \mathscr{M}(\mathbf{p}')\Big) \geq \mathbf{d}^{\mathrm{T}}\mathbf{M}^{\max}\mathbf{d}$ for all $n \in \mathscr{N}$ and $\mathbf{d}^{\mathrm{T}} = [\mathbf{d}_1^{\mathrm{T}}, \ldots, \mathbf{d}_N^{\mathrm{T}}]$. Since $\mathbf{d}^{\mathrm{T}}\mathbf{M}^{\max}\mathbf{d} \geq \lambda_{\min}(\mathbf{M}^{\max})\|\mathbf{d}\|_2$, the matrix $\mathbf{M}^{\max}$ is positive semidefinite. Consequently, $c_{\mathrm{sm}}(\mathscr{F}) = \lambda_{\min}(\mathbf{M}^{\max})$, and from (3.41) Eq. (3.42) is derived.

Appendix 7: Performance at RSE1 and RSE2 in Single-Leader/Single-Follower Scenario (Table 3.6)

Proof of Validity of Statement in Column 1, Row 2 in Table 3.6

At RSE1, we have $\nabla_{\mathbf{p}_1^{*\mathrm{RSE1}}} u_1(\mathbf{p}_1^{*\mathrm{RSE1}}, \mathbf{f}_1^{*\mathrm{RSE1}}) \geq \mathbf{0}$, where $\mathbf{0}$ is the all zero $K \times 1$ vector, and

$$\left[\mathbf{J}_{\mathbf{p}_1\mathbf{p}_1} \nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_1^{*\mathrm{RSE1}} + \mathbf{J}_{\mathbf{f}_1\mathbf{p}_1} \nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{f}_1^{*\mathrm{RSE1}}\right]_{\varepsilon_{\mathbf{f}_1}=0} = \mathbf{0}. \tag{3.94}$$

From (3.56), the value of $[\nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{f}_1^{*\mathrm{RSE1}}]_{\varepsilon_{\mathbf{f}_1}=0}$ in (3.94) is equal to $-\vartheta_1^{\mathrm{T}}$. By rearranging (3.94), we get

$$\nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_1^{*\mathrm{RSE1}} = (\mathbf{J}_{\mathbf{p}_1\mathbf{p}_1})^{-1} \mathbf{J}_{\mathbf{f}_1\mathbf{p}_1} \vartheta_1^{\mathrm{T}}. \tag{3.95}$$

From Assumptions A1–A3 in Section 3.2.1.3 in this chapter, the right-hand side of (3.95) is negative. Hence,

$$\nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_1^{*\mathrm{RSE1}} < \mathbf{0},$$

meaning that the follower's strategy is a decreasing function of $\varepsilon_{\mathbf{f}_1}$.

Proof of Validity of Statement in Column 1, Row 3 in Table 3.6

At RSE1, we have

$$\nabla_{\mathbf{p}_0^{*\mathrm{RSE1}}} v_0(\mathbf{p}_0^{*\mathrm{RSE1}}, \mathbf{f}_0^{*\mathrm{RSE1}}) \geq \mathbf{0},$$

$$\left[\mathbf{J}_{\mathbf{p}_0\mathbf{p}_0} \nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_0^{*\mathrm{RSE1}} + \mathbf{J}_{\mathbf{f}_0\mathbf{p}_0} \overline{\mathbf{H}}_{10} \nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_1^{*\mathrm{RSE1}}\right]_{\varepsilon_{\mathbf{f}_1}=0} = \mathbf{0}, \tag{3.96}$$

which is equivalent to

$$\nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_0^{*\mathrm{RSE1}} = -(\mathbf{J}_{\mathbf{p}_0\mathbf{p}_0})^{-1} \mathbf{J}_{\mathbf{f}_0\mathbf{p}_0} \overline{\mathbf{H}}_{10} \nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_1^{*\mathrm{RSE1}}. \tag{3.97}$$

From Assumptions A1–A3 in Section 3.2.1.3 in this chapter, the right-hand side of (3.97) is positive. Hence,

$$\nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_0^{*\mathrm{RSE1}} > \mathbf{0},$$

meaning that the leader's strategy is an increasing function of $\varepsilon_{\mathbf{f}_1}$.

Proof of Validity of Statement in Column 1, Row 4 in Table 3.6

The Taylor series expansion of the leader's utility around the uncertain parameter is

$$v_0(\mathbf{p}_0^{*\mathrm{RSE1}}, \mathbf{f}_0^{*\mathrm{RSE1}}) = v_0(\mathbf{p}_0^{*\mathrm{NSE}}, \mathbf{f}_0^{*\mathrm{NSE}}) + \varepsilon_{\mathbf{f}_1} \left[(\overline{\mathbf{H}}_{10}\mathbf{J}_{\mathbf{f}_0})^{\mathrm{T}} \nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_1^{*\mathrm{RSE1}} + (\mathbf{J}_{\mathbf{p}_0})^{\mathrm{T}} \nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_0^{*\mathrm{RSE1}} \right]_{\varepsilon_{\mathbf{f}_1}=0} + o. \tag{3.98}$$

In what follows, only the first term of the Taylor series expansion is considered, and higher terms for small values of $\varepsilon_{\mathbf{f}_1}$ are ignored. From Assumption A2 in Section 3.2.1.3 in this chapter, and since $\nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_1^{*\mathrm{RSE1}} < \mathbf{0}$, the value of $(\overline{\mathbf{H}}_{10}\mathbf{J}_{\mathbf{f}_0})^{\mathrm{T}} \nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_1^{*\mathrm{RSE1}}$ in (3.98) is always positive. Also, $(\mathbf{J}_{\mathbf{p}_0})^{\mathrm{T}} \nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_0^{*\mathrm{RSE1}}$ in (3.98) has positive elements only. Hence, the leader's utility at RSE1 is always greater than that at the NSE, and we have

$$\omega_0^{*\mathrm{RSE1}} - \omega_0^{*\mathrm{NSE}} \approx \varepsilon_{\mathbf{f}_1} \left[(\mathbf{J}_{\mathbf{p}_0})^{\mathrm{T}} \nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_0^{*\mathrm{RSE1}} + (\overline{\mathbf{H}}_{10}\mathbf{J}_{\mathbf{f}_0})^{\mathrm{T}} \nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_1^{*\mathrm{RSE1}} \right]. \tag{3.99}$$

The Taylor series expansion of the follower's utility around $\varepsilon_{\mathbf{f}_1}$ is

$$u_1(\mathbf{p}_1^{*\mathrm{RSE1}}, \mathbf{f}_1^{*\mathrm{RSE1}}) = v_1(\mathbf{p}_1^{*\mathrm{NSE}}, \mathbf{f}_1^{*\mathrm{NSE}}) + \varepsilon_{\mathbf{f}_1} \left[(\overline{\mathbf{H}}_{01}\mathbf{J}_{\mathbf{f}_1})^{\mathrm{T}} \nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_0^{*\mathrm{RSE1}} + (\mathbf{J}_{\mathbf{p}_1}^1)^{\mathrm{T}} \nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_1^{*\mathrm{RSE1}} \right]_{\varepsilon_{\mathbf{f}_1}=0} + o. \tag{3.100}$$

Since $\mathbf{J}_{\mathbf{f}_1} < \mathbf{0}$ and $\nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_0^{*\mathrm{RSE1}} > \mathbf{0}$, the value of $(\overline{\mathbf{H}}_{01}\mathbf{J}_{\mathbf{f}_1})^{\mathrm{T}} \nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_0^{*\mathrm{RSE1}}$ in (3.100) is always negative. Also, since $\nabla_{\mathbf{p}_1} v_1(\mathbf{p}_1, \mathbf{f}_1) > \mathbf{0}$ and $\nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_1^{*\mathrm{RSE1}} < \mathbf{0}$, the value of $(\mathbf{J}_{\mathbf{p}_1}^1)^{\mathrm{T}} \nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_1^{*\mathrm{RSE1}}$ in (3.100) is negative. Hence, the follower's utility at RSE1 is always less than that at the NSE, and we have

$$\omega_1^{*\mathrm{RSE1}} - \omega_1^{*\mathrm{NSE}} \approx \varepsilon_{\mathbf{f}_1} \times \left[(\overline{\mathbf{H}}_{01}\mathbf{J}_{\mathbf{f}_1})^{\mathrm{T}} \nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_0^{*\mathrm{RSE1}} + (\mathbf{J}_{\mathbf{p}_1})^{\mathrm{T}} \nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_1^{*\mathrm{RSE1}} \right]. \tag{3.101}$$

The social utility at RSE1 is increased when

$$\omega_0^{*\mathrm{RSE1}} - \omega_0^{*\mathrm{NSE}} + \omega_1^{*\mathrm{RSE1}} - \omega_1^{*\mathrm{NSE}} > 0,$$

which is equivalent to the sum of (3.99) and (3.101). To satisfy this, note that since $\nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_1^{*\mathrm{RSE1}} < \mathbf{0}$ and $\nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_0^{*\mathrm{RSE1}} > \mathbf{0}$, the sum of the terms multiplied by $\nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_1^{*\mathrm{RSE1}}$ should be negative and the sum of the terms multiplied by $\nabla_{\varepsilon_{\mathbf{f}_1}} \mathbf{p}_0^{*\mathrm{RSE1}}$ should be positive. Hence, we have $|\mathbf{J}_{\mathbf{p}_0}| - |\overline{\mathbf{H}}_{01}||\mathbf{J}_{\mathbf{f}_1}| > \mathbf{0}$ and $|\mathbf{J}_{\mathbf{p}_1}| - |\overline{\mathbf{H}}_{10}||\mathbf{J}_{\mathbf{f}_0}| < \mathbf{0}$, which are the same as C1 and C2 in Table 3.6.

Proof of Validity of Statement in Column 2, Row 2 in Table 3.6

For RSE2, the proof is similar to that for the statement in Column 1, Row 2 in Table 3.6, which was presented at the beginning of this appendix, except that (3.95) is changed to

$$\nabla_{\delta_{10}} \mathbf{p}_1^{*\mathrm{RSE2}} = -(\mathbf{J}_{\mathbf{p}_1\mathbf{p}_1})^{-1} \mathbf{J}_{\mathbf{f}_1\mathbf{p}_1} \nabla_{\delta_{10}} \mathbf{f}_1^{*\mathrm{RSE2}}. \tag{3.102}$$

From Statement 1 in [44], the value of $\nabla_{\delta_{10}} \mathbf{f}_1^{*\mathrm{RSE2}}$ is negative, and from Assumptions A1–A3 in Section 3.2.1.3 in this chapter, the value of $-(\mathbf{J}_{\mathbf{p}_1\mathbf{p}_1})^{-1} \mathbf{J}_{\mathbf{f}_1\mathbf{p}_1}$ is always negative. Hence, $\nabla_{\delta_{10}} \mathbf{p}_1^{*\mathrm{RSE2}}$ is always positive, meaning that the follower's action is an increasing function of δ_{10}.

Proof of Validity of Statement in Column 2, Row 3 in Table 3.6

We have

$$\nabla_{\delta_{10}} \mathbf{p}_0^{*\mathrm{RSE2}} = -(\mathbf{J}_{\mathbf{p}_0\mathbf{p}_0})^{-1} \mathbf{J}_{\mathbf{f}_0\mathbf{p}_0} \overline{\mathbf{H}}_{10} \nabla_{\delta_{10}} \mathbf{p}_1^{*\mathrm{RSE2}}. \tag{3.103}$$

From Assumptions A1–A3 in Section 3.2.1.3 in this chapter, and (3.102), we note that $\nabla_{\delta_{10}} \mathbf{p}_0^{*\mathrm{RSE2}}$ is negative.

Proof of Validity of Statement in Column 2, Row 4 in Table 3.6

We have

$$\begin{aligned} &\omega_1^{*\mathrm{RSE2}} - \omega_1^{*\mathrm{NSE}} \\ &\quad\approx \delta_{10} \times \left[(\mathbf{J}_{\mathbf{p}_1})^{\mathrm{T}} (\mathbf{J}_{\mathbf{p}_1\mathbf{p}_1})^{-1} \mathbf{J}_{\mathbf{f}_1\mathbf{p}_1} \nabla_{\delta_{10}} \mathbf{f}_1^{*\mathrm{RSE2}} - (\mathbf{J}_{\mathbf{f}_1})^{\mathrm{T}} \nabla_{\delta_{10}} \mathbf{f}_1^{*\mathrm{RSE2}} \right]. \end{aligned} \tag{3.104}$$

From Assumptions A1–A3 in Section 3.2.1.3 in this chapter and Statement 1 in [44], we note that the right-hand side of (3.104) is positive. Hence, the follower's utility at RSE2 is higher than that at the NSE. For the leader, we have similar steps, except that (3.104) is changed to

$$\omega_0^{*\mathrm{RSE2}} - \omega_0^{*\mathrm{NSE}} \approx \delta_{10} \times \left[-(\mathbf{J}_{\mathbf{p}_0})^{\mathrm{T}}(\mathbf{J}_{\mathbf{p}_0\mathbf{p}_0})^{-1}\overline{\mathbf{H}}_{10}\mathbf{J}_{\mathbf{f}_0\mathbf{p}_0} + (\mathbf{J}_{\mathbf{f}_0})^{\mathrm{T}}\overline{\mathbf{H}}_{10}\right] \nabla_{\delta_{10}}\mathbf{p}_1, \tag{3.105}$$

which, from Assumptions A1–A3 in Section 3.2.1.3 in this chapter and Statement 1 in [44], is always negative. Hence, the leader's utility at RSE2 is less than that at the NSE. Now we derive the conditions under which the social utility is increased (i.e., when C3 and C4 hold). Since $\nabla_{\delta_{10}}\mathbf{p}_0^{*\mathrm{RSE2}} < \mathbf{0}$, the sum of the terms multiplied by $\nabla_{\delta_{10}}\mathbf{p}_0^{*\mathrm{RSE2}}$ should be negative. Hence, the sum of the second term on the right-hand side of (3.104), which is

$$(\mathbf{J}_{\mathbf{f}_1})^{\mathrm{T}}\nabla_{\delta_{10}}\mathbf{f}_1^{*\mathrm{RSE2}} = (\mathbf{J}_{\mathbf{f}_1})^{\mathrm{T}}\overline{\mathbf{H}}_{01}\nabla_{\delta_{10}}\mathbf{p}_0^{*\mathrm{RSE2}},$$

and the first term on the right-hand side of (3.105), which is

$$-(\mathbf{J}_{\mathbf{p}_0})^{\mathrm{T}}(\mathbf{J}_{\mathbf{p}_0\mathbf{p}_0})^{-1}\overline{\mathbf{H}}_{01}\mathbf{J}_{\mathbf{f}_0\mathbf{p}_0} = -(\mathbf{J}_{\mathbf{p}_0})^{\mathrm{T}}\nabla_{\delta_{10}}\mathbf{p}_0^{*\mathrm{RSE2}},$$

should be negative, that is, $|\mathbf{J}_{\mathbf{p}_0}| - |\mathbf{J}_{\mathbf{f}_1}||\overline{\mathbf{H}}_{10}| < \mathbf{0}$. Also, since $\nabla_{\delta_{10}}\mathbf{p}_1^{*\mathrm{RSE2}} > \mathbf{0}$, the sum of the terms multiplied by $\nabla_{\delta_{10}}\mathbf{p}_1^{*\mathrm{RSE2}}$ should be positive. Hence, the first term on the right-hand side of (3.104) simplifies to $\delta_{10}\mathbf{J}_{\mathbf{p}_1}\nabla_{\delta_{10}}\mathbf{p}_1^{*\mathrm{RSE2}}$ and the second term on the right-hand side of (3.105), that is, $\delta_{10}(\mathbf{J}_{\mathbf{f}_0})^{\mathrm{T}}\overline{\mathbf{H}}_{10}\nabla_{\delta_{10}}\mathbf{p}_1$, should be positive, that is, $|\mathbf{J}_{\mathbf{p}_1}| - |\mathbf{J}_{\mathbf{f}_0}||\overline{\mathbf{H}}_{10}| > \mathbf{0}$. Note that these two conditions are equivalent to C3 and C4.

Appendix 8: Performance at RSE1 and RSE2 in Single-Leader/Multi-Follower Scenario (Table 3.7)

Proof of Validity of Statement in Column 1, Row 2 in Table 3.7

Lemma 3.3. *When $\boldsymbol{\Upsilon}$ is a P-matrix, and uncertainty is small, the followers' strategies are decreasing functions of $\boldsymbol{\varepsilon} = [\varepsilon_{\mathbf{f}_1}, \ldots, \varepsilon_{\mathbf{f}_{N_F}}]$.*

Proof. Let $\mathbf{P}_{N_\mathrm{F}} \triangleq [\mathbf{p}_1, \ldots, \mathbf{p}_{N_\mathrm{F}}]$, and assume that $\mathbf{P}_{N_\mathrm{F}}$ is an increasing function of $\boldsymbol{\varepsilon}$, that is, $\mathbf{P}_{N_\mathrm{F}}^{*\mathrm{RSE1}} \geq \mathbf{P}_{N_\mathrm{F}}^{*\mathrm{NSE}}$. When $\boldsymbol{\Upsilon}$ is a P-matrix, $\mathscr{J}(\mathbf{P}_{N_\mathrm{F}}) \triangleq (\mathbf{J}_{\mathbf{p}_{n_\mathrm{F}}}(\mathbf{p}))_{n_\mathrm{F}=1}^{N_\mathrm{F}}$ is strongly monotone (Theorem 12.5 in [2]), and

$$\mathscr{J}(\mathbf{p}_{N_\mathrm{F}}^{*\mathrm{RSE1}}) \geq \mathscr{J}(\mathbf{p}_{N_\mathrm{F}}^{*\mathrm{NSE}}). \tag{3.106}$$

(continued)

Lemma 3.3 (continued)
On the other hand, using (3.27), we have

$$\frac{\partial u_{n_F}^k(p_{n_F}^k, f_{n_F}^k)}{\partial p_{n_F}^k} = \frac{\partial v_{n_F}^k(p_{n_F}^k, f_{n_F}^{k*})}{\partial p_{n_F}^k} + \frac{\partial v_{n_F}^k(p_n^k, f_{n_F}^{k*})}{\partial f_{n_F}^{k*}} \times \frac{\partial f_{n_F}^{k*}}{\partial p_{n_F}^k},$$

$$\frac{\partial f_{n_F}^{k*}}{\partial p_{n_F}^k} = \frac{\partial f_{n_F}^{k*}}{\partial \vartheta_{n_F}^k} \times \frac{\partial \vartheta_{n_F}^k}{\partial p_{n_F}^k} = -\varepsilon_{\mathbf{f}_{n_F}} \times \frac{\partial^2 v_n^k(\mathbf{p}_{n_F}, f_{n_F}^*)}{\partial p_{n_F}^k \partial f_{n_F}^k} \times \frac{1}{\sqrt{\sum_{k=1}^K (\frac{\partial u_{n_F}^k(\mathbf{p}_{n_F}, \mathbf{f}_{n_F}^*)}{\partial f_{n_F}^k})^2}}. \tag{3.107}$$

Note that

$$\widetilde{p}_{n_F}^k = -\varepsilon_{\mathbf{f}_{n_F}} \times \frac{\partial v_{n_F}^k(\mathbf{p}_{n_F}, \mathbf{f}_{n_F}^*)}{\partial f_{n_F}^{k*}} \times \frac{\partial^2 v_{n_F}^k(\mathbf{p}_{n_F}, \mathbf{f}_{n_F}^*)}{\partial p_{n_F}^k \partial f_{n_F}^{k*}} \times \frac{1}{\sqrt{\sum_{k=1}^K (\frac{\partial u_{n_F}^k(\mathbf{p}_{n_F}, \mathbf{f}_{n_F}^*)}{\partial f_{n_F}^{k*}})^2}}\Bigg|_{\mathbf{p}_{n_F}=\mathbf{p}_{n_F}^{\mathrm{NSE}}}$$

is negative from Assumptions A1–A3 in Section 3.2.1.3 in this chapter. We rewrite (3.107) as

$$\mathscr{J}(\mathbf{p}_{\mathrm{N_F}}^{*\mathrm{RSE1}}) - \mathscr{J}(\mathbf{p}_{\mathrm{N_F}}^{*\mathrm{NSE}}) = \widetilde{\mathbf{p}} < \mathbf{0},$$

where $\widetilde{\mathbf{p}} = (\widetilde{\mathbf{p}}_{n_F})_{n_F=1}^{N_F}$, $\widetilde{\mathbf{p}}_{n_F}^{\mathrm{T}} = [\widetilde{p}_{n_F}^1, \dots, \widetilde{p}_{n_F}^K]$, and $\mathbf{0}$ is the zero vector with the same size as $\widetilde{\mathbf{p}}$. Note that this contradicts (3.106) and implies that our assumption was wrong. Hence, the followers' strategies at RSE1 are decreasing functions of $\boldsymbol{\varepsilon}_{\mathbf{f}_{n_F}}$.

Proof of Validity of Statement in Column 1, Row 3 in Table 3.7

From the preceding Lemma 3.3, since the followers' strategies are decreasing functions of $\boldsymbol{\varepsilon}_{\mathbf{f}_{n_F}}$, their interference $\mathbf{f}_0$ on the leader is reduced when $\boldsymbol{\varepsilon}_{\mathbf{f}_{n_F}}$ is increased, which implies $\omega_0^{*\mathrm{RSE1}} \geq \omega_0^{*\mathrm{NSE}}$ from Assumption A2 in Section 3.2.1.3 in this chapter and the Taylor series expansion of ω_0^{RSE1} around $\boldsymbol{\varepsilon}_{\mathbf{f}_{n_F}}$, which is

$$\omega_0^{*\mathrm{RSE1}} = \omega_0^{*\mathrm{NSE}} + \varepsilon_{\mathbf{f}_{n_F}} \times \left[(\nabla_{\boldsymbol{\varepsilon}_{\mathbf{f}_{n_F}}} \mathbf{p}_0)^{\mathrm{T}} \mathbf{J}_{\mathbf{p}_0} + \sum_{n_F=1}^{N_F} \overline{\mathbf{H}}_{n_F 0} \mathbf{J}_{\mathbf{f}_0} (\nabla_{\varepsilon_{\mathbf{f}_{n_F}}} \mathbf{p}_{n_F})^{\mathrm{T}} \right] + o. \tag{3.108}$$

Proof of Validity of Statement in Column 1, Row 4 in Table 3.7

The game between the followers in the multi-follower RSG in Section 3.3.3 is the same as the robust game in Section 3.1. From Theorem 3.7 in this chapter, when $\boldsymbol{\Upsilon}$ is a P-matrix and uncertainty is small, the followers' social utility at RSE1, denoted by $\omega_{n_{\mathrm{F}}}^{*\mathrm{RSE1}}$, is less than that at the NSE, denoted by $\omega_{n_{\mathrm{F}}}^{*\mathrm{NSE}}$. The Taylor series expansion of the utility of follower n_{F} around $\boldsymbol{\varepsilon}_{\mathbf{f}_{n_{\mathrm{F}}}}$ is

$$\omega_{n_{\mathrm{F}}}^{*\mathrm{RSE1}} = \omega_{n_{\mathrm{F}}}^{*\mathrm{NSE}} + \varepsilon_{\mathbf{f}_{n_{\mathrm{F}}}} \times \Bigg[(\mathbf{J}_{\mathbf{f}_{n_{\mathrm{F}}}})^{\mathrm{T}} \overline{\mathbf{H}}_{n_{\mathrm{F}}0} \nabla_{\boldsymbol{\varepsilon}_{\mathbf{f}_{n_{\mathrm{F}}}}} \mathbf{p}_0 + (\mathbf{J}_{\mathbf{f}_{n_{\mathrm{F}}}})^{\mathrm{T}} \left(\sum_{n_{\mathrm{F}}=1, m_{\mathrm{F}} \neq n_{\mathrm{F}}}^{N_{\mathrm{F}}} \overline{\mathbf{H}}_{m_{\mathrm{F}} n_{\mathrm{F}}} \nabla_{\varepsilon_{\mathbf{f}_{n_{\mathrm{F}}}}} \mathbf{p}_{m_{\mathrm{F}}} \right) + (\mathbf{J}_{\mathbf{p}_n}^{n})^{\mathrm{T}} \nabla_{\varepsilon_{\mathbf{f}_{n_{\mathrm{F}}}}} \mathbf{p}_{n_{\mathrm{F}}} \Bigg] + o, \tag{3.109}$$

where $\nabla_{\boldsymbol{\varepsilon}_{\mathbf{f}_{n_{\mathrm{F}}}}} \mathbf{p}_0$ is a $K \times 1$ vector whose kth element is $\sum_{n \in \mathcal{N}_{\mathrm{F}}} \frac{\partial p_0^k}{\partial \varepsilon_{\mathbf{f}_{n_{\mathrm{F}}}}}$.

Proof of Validity of Statement in Column 1, Row 5 in Table 3.7

When the sum of the second terms in (3.108) and (3.109) for all the followers is positive, the social utility at RSE1 is higher than that at the NSE. Note that the terms multiplied by $\nabla_{\boldsymbol{\varepsilon}_{\mathbf{f}_{n_{\mathrm{F}}}}} \mathbf{p}_0$ should be positive because $\nabla_{\boldsymbol{\varepsilon}_{\mathbf{f}_{n_{\mathrm{F}}}}} \mathbf{p}_0 > \mathbf{0}$. Also, since $\nabla_{\varepsilon_{\mathbf{f}_{n_{\mathrm{F}}}}} \mathbf{p}_{n_{\mathrm{F}}} < \mathbf{0}$, the terms multiplied by $\nabla_{\varepsilon_{\mathbf{f}_{n_{\mathrm{F}}}}} \mathbf{p}_{n_{\mathrm{F}}}$ should be negative. By some rearrangements, it can easily be shown that the terms multiplied by $\nabla_{\boldsymbol{\varepsilon}_{\mathbf{f}_{n_{\mathrm{F}}}}} \mathbf{p}_0$ are positive when C5 holds, and the terms multiplied by $\nabla_{\varepsilon_{\mathbf{f}_{n_{\mathrm{F}}}}} \mathbf{p}_{n_{\mathrm{F}}}$ are negative when C6 holds.

Proof of Validity of Statement in Column 2, Row 2 in Table 3.7

Lemma 3.4. *When $\boldsymbol{\Upsilon}$ is a P-matrix and uncertainty is small, the followers' strategies are increasing functions of $\boldsymbol{\delta}_0 \triangleq [\delta_{10}, \ldots, \delta_{N_F 0}]$.*

Proof. The proof is similar to that of Lemma 3.3, except that here we assume that the followers' strategies are decreasing functions of $\boldsymbol{\delta}_0$ and demonstrate that this assumption contradicts Assumptions A1–A3 in Section 3.2.1.3 in this chapter and Statement 1 in [44].

Proof of Validity of Statement in Column 2, Row 3 in Table 3.7

For this case, we have

$$\nabla_{\delta_{n_{\mathrm{F}}0}} \mathbf{p}_0^{*\mathrm{RSE2}} = -(\mathbf{J}_{\mathbf{p}_0\mathbf{p}_0})^{-1}\mathbf{J}_{\mathbf{f}_0\mathbf{p}_0}\overline{\mathbf{H}}_{n_{\mathrm{F}}0}\nabla_{\delta_{n_{\mathrm{F}}0}}\mathbf{p}_{n_{\mathrm{F}}}^{*\mathrm{RSE2}}. \tag{3.110}$$

From Assumptions A1–A3 in Section 3.2.1.3 in this chapter and Lemma 3.4, we note that $\nabla_{\delta_{n_{\mathrm{F}}0}} \mathbf{p}_0^{*\mathrm{RSE2}}$ is negative, that is, the leader's strategy is a decreasing function of $\delta_{n_{\mathrm{F}}0}$ for all $n_{\mathrm{F}} \in \mathcal{N}_{\mathrm{F}}$. From (3.110) and Lemma 3.4, the leader's utility in RSE2 is less than that at NSE.

Proof of Validity of Statement in Column 2, Row 4 in Table 3.7

From (3.110) and Lemma 3.4, the followers' social utility in RSE2 is greater than that at NSE.

Proof of Validity of Statement in Column 2, Row 5 in Table 3.7

The Taylor series expansion of the utility of the leader, that is, $\omega_0^{*\mathrm{RSE2}}$, around $\delta_{n_{\mathrm{F}}0}$ is

$$\omega_0^{*\mathrm{RSE2}} = \omega_0^{*\mathrm{NSE}} + \delta_{n_{\mathrm{F}}0} \times \left[\sum_{n_{\mathrm{F}}\in\mathcal{N}_{\mathrm{F}}} (\mathbf{J}_{\mathbf{f}_0})^{\mathrm{T}}\overline{\mathbf{H}}_{n_{\mathrm{F}}0}\nabla_{\delta_{n_{\mathrm{F}}0}}\mathbf{p}_{n_{\mathrm{F}}} + (\mathbf{J}_{\mathbf{p}_0})^{\mathrm{T}}\nabla_{\delta_{n_{\mathrm{F}}0}}\mathbf{p}_0\right] + o. \tag{3.111}$$

Also, the Taylor series expansion of $\omega_{n_{\mathrm{F}}}^{*\mathrm{RSE2}}$ around $\delta_{n_{\mathrm{F}}0}$ is

$$\begin{aligned}\omega_{n_{\mathrm{F}}}^{*\mathrm{RSE2}} = \omega_{n_{\mathrm{F}}}^{*\mathrm{NSE}} + \delta_{n_{\mathrm{F}}0} \times \Bigg[&(\mathbf{J}_{\mathbf{f}_{n_{\mathrm{F}}}})^{\mathrm{T}}\overline{\mathbf{H}}_{0n_{\mathrm{F}}}\nabla_{\delta_{n_{\mathrm{F}}0}}\mathbf{p}_0 \\ &+(\mathbf{J}_{\mathbf{f}_{n_{\mathrm{F}}}})^{\mathrm{T}}\left(\sum_{m_{\mathrm{F}}\neq n_{\mathrm{F}},\, m_{\mathrm{F}}\in\mathcal{N}_{\mathrm{F}}} \overline{\mathbf{H}}_{m_{\mathrm{F}}n_{\mathrm{F}}}\nabla_{\delta_{n_{\mathrm{F}}0}}\mathbf{p}_{m_{\mathrm{F}}}\right) + (\mathbf{J}_{\mathbf{p}_{n_{\mathrm{F}}}})^{\mathrm{T}}\nabla_{\delta_{n_{\mathrm{F}}0}}\mathbf{p}_{n_{\mathrm{F}}}\Bigg] + o.\end{aligned} \tag{3.112}$$

When the sum of the second terms in (3.111) and (3.112) for all the followers is positive, the social utility at RSE2 is higher than that at the NSE. Since $\nabla_{\delta_{n_{\mathrm{F}}0}}\mathbf{p}_0 < \mathbf{0}$ and $\nabla_{\delta_{n_{\mathrm{F}}0}}\mathbf{p}_{n_{\mathrm{F}}} > \mathbf{0}$, the terms multiplied by $\nabla_{\boldsymbol{\delta}_0}\mathbf{p}_0$ should be negative, and the terms multiplied by $\nabla_{\delta_{n_{\mathrm{F}}0}}\mathbf{p}_{n_{\mathrm{F}}}$ should be positive. By some rearrangements, it can easily be shown that the corresponding statement in Table 3.7 is valid when C7 and C8 in Table 3.7 hold.

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Chapter 4
Nonconvex Robust Problems

This chapter gives an overview of relaxation methods for solving nonconvex and intractable robust optimization problems for allocating resources in wireless networks. We begin by presenting a taxonomy of relaxation methods that have been widely used in this context and continue by giving several examples to demonstrate how such methods are utilized in practice.

4.1 Introduction

The nominal optimization problems in Chapters 2 and 3 either are convex or can be converted into convex problems. However, their robust counterparts may be nonconvex and nondeterministic polynomial-time (NP)–hard due to uncertainty in parameter values and new constraints. In Chapters 2 and 3 we showed that when uncertainty is confined to a small region, robust problems may become tractable and their computational complexity can be manageable. However, many emerging resource allocation problems in future wireless networks are inherently nonconvex and NP-hard, and their robust counterparts are significantly more complicated. This means that there is a need to present efficient techniques for converting such problems into more tractable formulations.

In this chapter, our focus is on relaxation methods that have been widely applied for such reformulations. Relaxation methods provide approximations of the original nonconvex and NP-hard optimization problem, where such approximations are easier to solve. Specifically, such methods entail replacing certain constraints with more conservative (i.e., safe[1]) ones with reduced complexity, where the solution to the modified problem is near the solution to the original nonconvex and NP-hard problem.

[1]In a safe approximation for a constraint, a feasible solution to the problem with approximated constraints is also a feasible solution to the original problem.

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We begin this chapter by presenting a taxonomy of relaxation methods that have been widely used for solving nonconvex and intractable robust optimization problems for allocating resources in wireless networks and proceed to give several examples demonstrating how such methods are utilized in practice. The examples are so chosen to include hot topics in future wireless networks, namely, beamforming, cooperative relaying, cognitive radio networks (CRNs), and physical-layer secure communications. The methods in this chapter may not lead to globally optimal points but in general include very efficient and tractable algorithms for reaching locally optimal or near globally optimal solutions.

Using relaxation methods, NP-hard problems can be efficiently solved in polynomial time, and bounds on the optimal value can be obtained. Relaxation methods can be categorized into the following two general classes:

- Direct relaxation, where each nonconvex objective or constraint is replaced by a looser or equivalent tractable convex counterpart
- Lagrangian relaxation, in which the Lagrangian dual of a nonconvex problem is solved to obtain a lower bound on the optimal solution of the nonconvex problem

4.2 Taxonomy of Relaxation Methods

Consider the following optimization problem whose constraints are in the form of both equalities and inequalities [1]:

$$\min_{\mathbf{x}} \quad f_0(\mathbf{x}), \tag{4.1a}$$

$$\text{subject to} \begin{cases} f_y(\mathbf{x}) = 0, \quad \forall y = 1, \ldots, Y, & (4.1b) \\ f_z(\mathbf{x}) \leq 0, \quad \forall z = 1, \ldots, Z. & (4.1c) \end{cases}$$

The objective is to obtain the vector $\mathbf{x}$ that minimizes $f_0(\mathbf{x})$ while satisfying (4.1b) and (4.1c). In the preceding problem:[2]

- $\mathbf{x} \in \mathbb{R}^n$ is the optimization variable, for example, transmit power, channel number, relay number, beamforming vector, or antenna number;
- $f_0 : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the objective function, for example, energy efficiency, fairness criteria, signal-to-interference-plus-noise ratio (SINR) outage probability, rate outage probability, transmit power, or interference power;
- $f_y(\mathbf{x}) = 0$, $y = 1, \ldots, Y$ are the equality constraints, for example, channel allocation in orthogonal frequency division multiple access (OFDMA)–based systems that guarantees each channel is exclusively allocated to at most one link or a zero forcing constraint for nulling the interference at the receiver, where $f_y : \mathbb{R}^n \longrightarrow \mathbb{R}$ are the equality constraint functions; and $f_z(\mathbf{x}) \leq 0$, $z = 1, \ldots, Z$

[2]In this chapter, a vector $\mathbf{x}$ is assumed to be in the column space and its transpose $\mathbf{x}^{\mathrm{T}}$ in the row space.

are inequality constraints, for example, transmit power, rate or SINR outage probability, or interference power, where $f_z : \mathbb{R}^n \longrightarrow \mathbb{R}$ are the inequality constraint functions.

The optimization problem is convex when all of the following conditions hold:

- Objective function (4.1a) and inequality constraint functions (4.1b) are convex.
- Equality constraint functions (4.1c) are affine.
- The set of points for which the objective and constraint functions are defined, that is, the domain of the optimization problem is convex.

When problem (4.1) is nonconvex and intractable or involves a mix of integer and continuous variables, the following relaxation methods may be deployed to convert it into a tractable optimization problem.

1. Direct relaxation methods:
    - Epigraph form (EF)
    - Charnes–Cooper transformation (CCT)
    - Schur complement (SC)
    - S-procedure (SP)
    - Bounding techniques (BTs):
        - Norm bounding:
            - Triangle inequality (TI)
            - Cauchy–Schwarz inequality
            - Norm approximation (NA)
        - Probabilistic bounding:
            - Markov's inequality (MI)
            - Bernstein approximation (BA)
            - Vysochanskii–Petunin inequality (VPI)
            - Conditional-value at risk (CVaR)
            - Difference of two convex approximations (DCA)
            - Large deviation inequality for complex Gaussian quadratic forms (LDI-CGQF)
            - Bernstein-type inequality for complex Gaussian quadratic forms (BTI-CGQF)
        - Trace bounding:
            - Determinant inequality (DI)
            - Von Neumann trace inequality (VNTI)
            - Trace of two matrices inequality (TTMI)
        - Taylor bounding (TB)
    - Semidefinite relaxation (SDR)
    - Nonlinear fractional programming (NLFP)

- DC programming (DCP) and successive convex approximation (SCA)
- Sequential parametric convex approximation (SPCA)

2. Lagrangian relaxation (LR):

- Duality
- Time sharing

In the sequel, we briefly discuss these methods.

4.2.1 Direct Relaxation

4.2.1.1 Epigraph Form

In some cases, the intractability of optimization problem (4.1) comes from its objective function, as in the following example:

$$\min_{\mathbf{x}} \quad \max_{k=1,\ldots,K} g_k(\mathbf{x}), \tag{4.2a}$$

$$\text{subject to} \begin{cases} f_y(\mathbf{x}) = 0, \quad \forall y = 1, \ldots, Y, & (4.2b) \\ f_z(\mathbf{x}) \leq 0, \quad \forall z = 1, \ldots, Z, & (4.2c) \end{cases}$$

where $g_k : \mathbb{R}^n \longrightarrow \mathbb{R}, \quad \forall k = 1, \ldots, K$, is convex.

In this case, the optimization problem's EF can be used to linearize the objective function. When the objective function is linear, simple algorithms can be developed to solve the reformulated optimization problem. The EF of problem (4.1) is [1]

$$\min_{\mathbf{x}} \quad t, \tag{4.3a}$$

$$\text{subject to} \begin{cases} f_0(\mathbf{x}) \leq t, & (4.3b) \\ f_y(\mathbf{x}) = 0, \quad \forall y = 1, \ldots, Y, & (4.3c) \\ f_z(\mathbf{x}) \leq 0, \quad \forall z = 1, \ldots, Z. & (4.3d) \end{cases}$$

Using the EF, problem (4.2) can be reformulated into a convex and tractable problem as

$$\min_{\mathbf{x}} \quad t, \tag{4.4a}$$

$$\text{subject to} \begin{cases} g_k(\mathbf{x}) \leq t, \quad \forall k = 1, \ldots, K, & (4.4b) \\ f_y(\mathbf{x}) = 0, \quad \forall y = 1, \ldots, Y, & (4.4c) \\ f_z(\mathbf{x}) \leq 0, \quad \forall z = 1, \ldots, Z. & (4.4d) \end{cases}$$

The EF can be used in many resource allocation problems in communication systems, for example, (1) to guarantee fairness among users via the maxi-min objective function or (2) to maximize the secrecy rate in a network with multiple eavesdroppers.

4.2.1.2 Charnes–Cooper Transformation

When the nonconvex optimization problem is a fractional linear programming problem, that is,

$$\max_{\mathbf{x} \succeq \mathbf{0}} \quad \frac{\mathbf{c}_1^{\mathrm{T}}\mathbf{x} + r_1}{\mathbf{c}_2^{\mathrm{T}}\mathbf{x} + r_2}, \tag{4.5a}$$

$$\text{subject to} \begin{cases} \mathbf{A}\mathbf{x} \preceq \mathbf{b}, & (4.5\text{b}) \\ \mathbf{G}\mathbf{x} = \mathbf{h}, & (4.5\text{c}) \end{cases}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{G} \in \mathbb{R}^{u \times n}$, $\mathbf{b} \in \mathbb{R}^{m}$, $\mathbf{h} \in \mathbb{R}^{u}$, $\mathbf{x}, \mathbf{c}_i \in \mathbb{R}^{n}$, and $r_i \in \mathbb{R}, \forall i = 1, 2$, problem (4.5) can be converted into a linear programming optimization problem using the CCT [2], in which

$$\mathbf{x} = \frac{\mathbf{y}}{t}. \tag{4.6}$$

When $\mathbf{y} \succeq \mathbf{0}$ and $t > 0$, problem (4.5) can be reformulated into the following linear programming optimization problem:

$$\max_{\mathbf{y} \succeq \mathbf{0},\ t > 0} \quad \mathbf{c}_1^{\mathrm{T}}\mathbf{y} + r_1 t, \tag{4.7a}$$

$$\text{subject to} \begin{cases} \mathbf{A}\mathbf{y} - \mathbf{b}t \preceq \mathbf{0}, & (4.7\text{b}) \\ \mathbf{c}_2^{\mathrm{T}}\mathbf{y} + r_2 t = 1, & (4.7\text{c}) \\ \mathbf{G}\mathbf{y} - \mathbf{h}t = \mathbf{0}. & (4.7\text{d}) \end{cases}$$

4.2.1.3 Schur Complement

When the objective function or the constraints in the optimization problem are quadratic, one can utilize the SC to convert them into linear ones and use semi definite programming (SDP) to solve the modified problem. For example, consider the following nonconvex optimization problem

$$\max_{\mathbf{x} \succeq \mathbf{0}} \quad -\mathbf{z}^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{z} + r, \tag{4.8a}$$

$$\text{subject to} \begin{cases} \mathbf{B} \succeq \mathbf{0}, & (4.8\text{b}) \\ \mathbf{x} \succeq \mathbf{0}, & (4.8\text{c}) \end{cases}$$

where $r \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y}_i \in \mathbb{R}^n$, $\mathbf{z} = \sum_{i=1}^{m} x_i \mathbf{y}_i$, $\mathbf{B} = \sum_{i=1}^{m} x_i \mathbf{F}_i$, and $\mathbf{F}_i$, $\forall i = 1, \ldots, m$, is an $n \times n$ symmetric and invertible matrix. Next, consider the matrix $\mathbf{X} \in \mathbb{S}^n$ partitioned as

$$\mathbf{X} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^{\mathrm{T}} & \mathbf{C} \end{bmatrix}, \tag{4.9}$$

where $\mathbf{A} \in \mathbb{R}^{u \times u}$, $\mathbf{B} \in \mathbb{R}^{u \times w}$, $\mathbf{C} \in \mathbb{R}^{w \times w}$, and $n = u + w$. The SCs in what follows have been defined depending on whether $\mathbf{A}$, $\mathbf{C}$, and $\mathbf{X}$ are positive definite or positive semidefinite.

Schur complement of A in X: Let $\mathbf{S}$ be the SC of $\mathbf{A}$ in $\mathbf{X}$ in (4.9), defined as

$$\mathbf{S} = \mathbf{C} - \mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{B}, \tag{4.10}$$

and assume $\det(\mathbf{A}) \neq 0$. We have [1, 3]:

- $\mathbf{X} \succ \mathbf{0}$ if and only if $\mathbf{A} \succ \mathbf{0}$ and $\mathbf{S} \succ \mathbf{0}$.
- If $\mathbf{A} \succ \mathbf{0}$, then $\mathbf{X} \succeq \mathbf{0}$ if and only if $\mathbf{S} \succeq \mathbf{0}$.

Schur complement of C in X: Let $\mathbf{D}$ be the SC of $\mathbf{C}$ in $\mathbf{X}$ in (4.9), defined as

$$\mathbf{D} = \mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^{\mathrm{T}}, \tag{4.11}$$

and assume $\det(\mathbf{C}) \neq 0$. We have [1, 3]:

- $\mathbf{X} \succ \mathbf{0}$ if and only if $\mathbf{C} \succ \mathbf{0}$ and $\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^{\mathrm{T}} \succ \mathbf{0}$.
- If $\mathbf{C} \succ \mathbf{0}$, then $\mathbf{X} \succeq \mathbf{0}$ if and only if $\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^{\mathrm{T}} \succeq \mathbf{0}$.

Generalized SC: When the generalized inverses of $\mathbf{A}$ and $\mathbf{C}$ are available (any matrix $\mathbf{A}^g$ that satisfies $\mathbf{A}\mathbf{A}^g\mathbf{A} = \mathbf{A}$ is called a generalized inverse of $\mathbf{A}$), the following conditions, called the generalized SC, are sufficient and necessary for $\mathbf{X}$ to be positive semidefinite: [4]

- $\mathbf{X} \succeq \mathbf{0}$ if and only if $\mathbf{A} \succeq \mathbf{0}$, $\mathbf{C} - \mathbf{B}^{\mathrm{T}}\mathbf{A}^g\mathbf{B} \succeq \mathbf{0}$ and $(\mathbf{I} - \mathbf{A}\mathbf{A}^g)\mathbf{B} = 0$.
- $\mathbf{X} \succeq \mathbf{0}$ if and only if $\mathbf{C} \succeq \mathbf{0}$; $\mathbf{A} - \mathbf{B}\mathbf{C}^g\mathbf{B}^{\mathrm{T}} \succeq \mathbf{0}$ and $(\mathbf{I} - \mathbf{C}\mathbf{C}^g)\mathbf{B}^{\mathrm{T}} = 0$.

Using the preceding expressions and the EF in Section 4.2.1.1 in this chapter, optimization problem (4.8) is reformulated as

$$\max_{\mathbf{x}\succeq\mathbf{0}} \quad t, \tag{4.12a}$$

$$\text{subject to} \begin{cases} \mathbf{B} \succeq \mathbf{0}, & (4.12\text{b}) \\ \mathbf{x} \succeq \mathbf{0}, & (4.12\text{c}) \\ -\mathbf{z}^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{z} + r \geq t. & (4.12\text{d}) \end{cases}$$

Using the SC, problem (4.12) is rewritten as the following SDP problem:

$$\max_{\mathbf{x}\succeq\mathbf{0}} \quad t, \tag{4.13a}$$

$$\text{subject to} \begin{cases} \begin{bmatrix} \mathbf{B} & \mathbf{z} \\ \mathbf{z}^{\mathrm{T}} & r-t \end{bmatrix} \succeq \mathbf{0}, & (4.13\text{b}) \\ \mathbf{x} \succeq \mathbf{0}. & (4.13\text{c}) \end{cases}$$

4.2.1.4 S-Procedure

When some constraints in the robust optimization of a quadratic programming problem are semi-infinite and intractable, the SP can be used to replace such constraints with linear matrix inequalities (LMIs) [1, 5–8]. This method can also be used to verify the nonnegativity of a quadratic function under quadratic constraints. When a constraint in an optimization problem has a single quadratic function, it is referred to as an S-lemma, and when there are at least two quadratic inequalities in the constraint set, the term *S-procedure* is used. Note that SP is the generalization of S-lemma [9, 10]. As an example, consider the following quadratically constrained quadratic program (QCQP) problem with a single constraint:

$$\min_{\mathbf{x}} \quad f_2(\mathbf{x}), \tag{4.14a}$$

$$\text{subject to} \quad f_1(\mathbf{x}) \geq 0, \tag{4.14b}$$

where $f_i : \mathbb{R}^n \longrightarrow \mathbb{R},$ for $i = 1, 2$, are quadratic functions defined as

$$f_i(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}_i\mathbf{x} + 2\mathbf{q}_i^{\mathrm{T}}\mathbf{x} + c_i, \quad \text{for} \quad i = 1, 2. \tag{4.15}$$

Note that the preceding problem is convex when $\mathbf{A}_1 \preceq \mathbf{0}$ and $\mathbf{A}_2 \succeq \mathbf{0}$. When the problem is nonconvex, we use Theorem 4.1 in what follows (also known as S-lemma) to rewrite it as an SDP.

Theorem 4.1 (S-Lemma). *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be symmetric* $n \times n$ *matrices, and assume* $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$. *We have*

$$\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0 \quad \Longrightarrow \quad \mathbf{x}^T\mathbf{B}\mathbf{x} \geq 0$$

if and only if

$$\exists \mu > 0, \quad \mathbf{B} \succeq \mu\mathbf{A}.$$

Theorem 4.2 below is another form of S-lemma when the quadratic function is complex.

Theorem 4.2. *Let* $f_i(\mathbf{x}) = \mathbf{x}^H\mathbf{A}_i\mathbf{x} + 2Re\{\mathbf{q}_i^H\mathbf{x}\} + c_i$, *for* $i = 1, 2$, *where* $\mathbf{A}_i \in \mathbb{H}^n$, $\mathbf{q}_i \in \mathbb{C}^n$, $c_i \in \mathbb{R}$, *and there exists a point* $\hat{\mathbf{x}}$ *such that* $f_1(\hat{\mathbf{x}}) < 0$. *We have*

$$f_1(\mathbf{x}) \leq 0 \quad \Longrightarrow \quad f_2(\mathbf{x}) \leq 0$$

if and only if

$$\exists \mu > 0, \quad \mu \begin{pmatrix} \mathbf{A}_1 & \mathbf{q}_1 \\ \mathbf{q}_1^H & c_1 \end{pmatrix} - \begin{pmatrix} \mathbf{A}_2 & \mathbf{q}_2 \\ \mathbf{q}_2^H & c_2 \end{pmatrix} \succeq \mathbf{0}.$$

When the optimization problem has at least two quadratic inequalities in its constraint set, the following theorem can be used.

Theorem 4.3 (S-Procedure). *In a linear vector space* $\mathscr{V}$ *for* $\mathbf{x}$, *let* $f_i(\mathbf{x}) = \mathbf{x}^T\mathbf{A}_i\mathbf{x} + 2\mathbf{q}_i^T\mathbf{x} + c_i$, $\forall i = 0, 1, \ldots, N$, *where* $f_i : \mathscr{V} \Longrightarrow \mathbb{R}$, $\forall i = 1, \ldots, N$. *We have*

$$C1: \quad \forall \mathbf{x} \in \mathscr{V}, \quad f_i(\mathbf{x}) \geq 0, \quad \forall i = 1, \ldots, N \quad \Longrightarrow \quad f_0(\mathbf{x}) \geq 0,$$

(continued)

Theorem 4.3 (continued)
if and only if

$$C2: \quad \exists \mu_i > 0, \quad \forall i = 1, \ldots, N, \quad f_0(\mathbf{x}) - \sum_{i=1}^{N} \mu_i f_i(\mathbf{x}) \geq 0, \quad \forall \mathbf{x} \in \mathscr{V}.$$

The preceding condition can also be written in matrix form:

$$\exists \mu_i > 0, \quad \begin{pmatrix} \mathbf{A}_0 & \mathbf{q}_0 \\ \mathbf{q}_0^H & c_0 \end{pmatrix} + \sum_{i=1}^{N} \mu_i \begin{pmatrix} \mathbf{A}_i & \mathbf{q}_i \\ \mathbf{q}_i^H & c_i \end{pmatrix} \succeq \mathbf{0}.$$

In brief, C1 holds if and only if C2 holds.

With the introduction of a new variable t, problem (4.14) can be rewritten as

$$\max_{\mathbf{x}} \quad t, \tag{4.16a}$$

$$\text{subject to} \quad f_1(\mathbf{x}) \geq 0 \quad \Longrightarrow \quad f_2(\mathbf{x}) \geq t. \tag{4.16b}$$

Using S-lemma, the optimization problem (4.16) can be reformulated as

$$\max_{\mu,\, \mathbf{x}} \quad t, \tag{4.17a}$$

$$\text{subject to} \begin{cases} f_2(\mathbf{x}) - t \geq \mu f_1(\mathbf{x}), \quad \forall \mathbf{x}, & (4.17b) \\ \mu \geq 0. & (4.17c) \end{cases}$$

By using $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ in Theorem 4.3, problem (4.17) is rewritten as

$$\max_{\mu,\, \mathbf{x}} \quad t, \tag{4.18a}$$

$$\text{subject to} \begin{cases} \mathbf{x}^{\mathrm{T}}\mathbf{A}_2\mathbf{x} + 2\mathbf{q}_2^{\mathrm{T}}\mathbf{x} + c_2 - t \geq \mu\big(\mathbf{x}^{\mathrm{T}}\mathbf{A}_1\mathbf{x} + 2\mathbf{q}_1^{\mathrm{T}}\mathbf{x} + c_1\big), \quad \forall \mathbf{x}, & (4.18b) \\ \mu \geq 0. & (4.18c) \end{cases}$$

The preceding problem can be written in matrix form,

$$\max_{\mu \geq 0,\, \mathbf{x}} \quad t, \tag{4.19a}$$

$$\text{subject to} \quad \mathbf{y}^{\mathrm{T}} \begin{bmatrix} \mathbf{A}_2 - \mu\mathbf{A}_1 & \frac{1}{2}(\mathbf{q}_2 - \mu\mathbf{q}_1) \\ \frac{1}{2}(\mathbf{q}_2^{\mathrm{T}} - \mu\mathbf{q}_1^{\mathrm{T}}) & c_2 - t - \mu c_1 \end{bmatrix} \mathbf{y} \geq 0, \quad \forall \mathbf{x}, \tag{4.19b}$$

where $\mathbf{y} = [\mathbf{x} \quad 1]^{\mathrm{T}}$. A symmetric $n \times n$ real-valued matrix $\mathbf{B}$ is positive semidefinite if the scalar $\mathbf{x}^{\mathrm{T}}\mathbf{B}\mathbf{x}$ is nonnegative for every nonzero column vector $\mathbf{x}$ of n real numbers. In other words, when $\mathbf{B} \succeq \mathbf{0}$, we have $\mathbf{x}^{\mathrm{T}}\mathbf{B}\mathbf{x} \geq 0$. Using the preceding explanation, problem (4.19) is equivalent to

$$\max_{\mu \geq 0} \quad t, \tag{4.20a}$$

$$\text{subject to} \quad \begin{bmatrix} \mathbf{A}_2 - \mu \mathbf{A}_1 & \frac{1}{2}(\mathbf{q}_2 - \mu \mathbf{q}_1) \\ \frac{1}{2}(\mathbf{q}_2^{\mathrm{T}} - \mu \mathbf{q}_1^{\mathrm{T}}) & c_2 - t - \mu c_1 \end{bmatrix} \succeq 0. \tag{4.20b}$$

4.2.1.5 Semidefinite Relaxation

In some cases, the robust resource allocation problem can be stated by a QCQP as

$$\min_{\mathbf{x}} \quad \mathbf{x}^{\mathrm{T}}\mathbf{A}_0\mathbf{x} + \mathbf{q}_0^{\mathrm{T}}\mathbf{x} + r_0, \tag{4.21a}$$

$$\text{subject to} \quad \mathbf{x}^{\mathrm{T}}\mathbf{A}_i\mathbf{x} + \mathbf{q}_i^{\mathrm{T}}\mathbf{x} + r_i \leq 0, \quad \forall i = 1, \ldots, m, \tag{4.21b}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{A}_i \in \mathbb{S}^n$, $\mathbf{q}_i \in \mathbb{R}^n$, and $r_i \in \mathbb{R}$. Problem (4.21) is nonconvex when at least one $\mathbf{A}_i$ is not positive semidefinite. A direct relaxation technique called semidefinite programming relaxation (SDR) [1, 11, 12] is used to relax the NP-hard problem (4.21) into a problem that can be solved in polynomial time.

SDP is an optimization problem where a linear function is minimized subject to a constraint that involves an affine combination of symmetric matrices, stated by [1, 13, 14]

$$\min_{\mathbf{x}} \quad \mathbf{c}^{\mathrm{T}}\mathbf{x}, \tag{4.22a}$$

$$\text{subject to} \begin{cases} x_1\mathbf{F}_1 + \cdots + x_n\mathbf{F}_n + \mathbf{G} \preceq \mathbf{0}, & (4.22\text{b}) \\ \mathbf{A}\mathbf{x} = \mathbf{b}, & (4.22\text{c}) \end{cases}$$

where $\mathbf{G}, \mathbf{F}_1, \ldots, \mathbf{F}_n \in \mathbb{S}^k$, $\mathbf{x}, \mathbf{c} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^w$, and $\mathbf{A} \in \mathbb{R}^{w \times n}$. Inequality constraint (4.22b) is an LMI. An SDP can be solved by the efficient interior-point method via some available tools such as CVX [15] and SeDuMi [16]. Prior to the introduction of SDR, QCQPs were solved by exhaustive search, which has a non-polynomial-time computational complexity.

Procedure for Converting QCQP into SDP

Step 1: Use $\mathbf{X} = \mathbf{x}\mathbf{x}^{\mathrm{T}}$ to linearize problem (4.21). This implies that rank $(\mathbf{X}) = 1$ and $\mathbf{x}^{\mathrm{T}}\mathbf{A}_i\mathbf{x} = \mathrm{tr}(\mathbf{x}^{\mathrm{T}}\mathbf{A}_i\mathbf{x}) = \mathrm{tr}(\mathbf{A}_i\mathbf{x}\mathbf{x}^{\mathrm{T}}) = \mathrm{tr}(\mathbf{A}_i\mathbf{X})$. Now, problem (4.21) can be rewritten as

$$\min_{\mathbf{x},\mathbf{X}} \quad \mathrm{tr}(\mathbf{A}_0\mathbf{X}) + \mathbf{q}_0^{\mathrm{T}}\mathbf{x} + r_0, \tag{4.23a}$$

$$\text{subject to} \begin{cases} \mathrm{tr}(\mathbf{A}_i\mathbf{X}) + \mathbf{q}_i^{\mathrm{T}}\mathbf{x} + r_i \le 0, \quad \forall i = 1,\ldots,m, & \text{(4.23b)} \\ \mathbf{X} = \mathbf{x}\mathbf{x}^{\mathrm{T}}. & \text{(4.23c)} \end{cases}$$

Step 2: Constraint (4.23c) is nonconvex but can be relaxed by replacing it with a looser positive semidefinite constraint $\mathbf{X} - \mathbf{x}\mathbf{x}^{\mathrm{T}} \succeq \mathbf{0}$. In this way, the relaxed problem is the following SDP:

$$\min_{\mathbf{x},\mathbf{X}} \quad \mathrm{tr}(\mathbf{A}_0\mathbf{X}) + \mathbf{q}_0^{\mathrm{T}}\mathbf{x} + r_0, \tag{4.24a}$$

$$\text{subject to} \begin{cases} \mathrm{tr}(\mathbf{A}_i\mathbf{X}) + \mathbf{q}_i^{\mathrm{T}}\mathbf{x} + r_i \le 0, \quad \forall i = 1,\ldots,m, & \text{(4.24b)} \\ \mathbf{X} - \mathbf{x}\mathbf{x}^{\mathrm{T}} \succeq \mathbf{0}. & \text{(4.24c)} \end{cases}$$

Step 3: Use the SC [1, 3], explained in Section 4.2.1.3 in this chapter, to convert (4.24c) into its convex approximation. Now, (4.24) can be rewritten as

$$\min_{\mathbf{x},\mathbf{X}} \quad \mathrm{tr}(\mathbf{A}_0\mathbf{X}) + \mathbf{q}_0^{\mathrm{T}}\mathbf{x} + r_0, \tag{4.25a}$$

$$\text{subject to} \begin{cases} \mathrm{tr}(\mathbf{A}_i\mathbf{X}) + \mathbf{q}_i^{\mathrm{T}}\mathbf{x} + r_i \le 0, \quad \forall i = 1,\ldots,m, & \text{(4.25b)} \\ \begin{bmatrix} \mathbf{X} & \mathbf{x} \\ \mathbf{x}^{\mathrm{T}} & 1 \end{bmatrix} \succeq \mathbf{0}. & \text{(4.25c)} \end{cases}$$

Problem (4.25) is the SDP relaxation of the original nonconvex problem (4.21). The optimal solution of the relaxed problem is the lower bound on the optimal solution of the original nonconvex QCQP.

A simpler relaxed problem can be obtained when the objective function and the constraints of the original problem (4.21) are homogeneous, that is, there are no linear terms $\mathbf{q}_i^{\mathrm{T}}\mathbf{x}$. We use $\mathbf{X} = \mathbf{x}\mathbf{x}^{\mathrm{T}}$ to linearize the homogeneous QCQP

$$\min_{\mathbf{x},\mathbf{X}} \quad \text{tr}(\mathbf{A}_0\mathbf{X}) + r_0, \tag{4.26a}$$

$$\text{subject to} \begin{cases} \text{tr}(\mathbf{A}_i\mathbf{X}) + r_i \leq 0, \quad \forall i = 1, \ldots, m, & (4.26\text{b}) \\ \mathbf{X} = \mathbf{x}\mathbf{x}^{\text{T}}. & (4.26\text{c}) \end{cases}$$

Constraint (4.26c) implies $\mathbf{X} \succeq \mathbf{0}$ and rank $(\mathbf{X}) = 1$. Hence, this constraint is equivalent to $\mathbf{X} \succeq \mathbf{0}$ and rank $(\mathbf{X}) = 1$. Now, problem (4.26) is rewritten

$$\min_{\mathbf{X}} \quad \text{tr}(\mathbf{A}_0\mathbf{X}) + r_0, \tag{4.27a}$$

$$\text{subject to} \begin{cases} \text{tr}(\mathbf{A}_i\mathbf{X}) + r_i \leq 0, \quad \forall i = 1, \ldots, m, & (4.27\text{b}) \\ \mathbf{X} \succeq \mathbf{0}, & (4.27\text{c}) \\ \text{rank}(\mathbf{X}) = 1. & (4.27\text{d}) \end{cases}$$

Note that nonhomogeneous quadratic functions with linear terms as in (4.21) can be "homogenized" by introducing an additional variable t and an additional constraint $t^2 = 1$. The nonhomogeneous QCQP (4.21) can be rewritten as the following homogeneous QCQP:

$$\min_{\mathbf{x},t} \quad \frac{1}{2}[\mathbf{x}^{\text{T}} \; t] \begin{bmatrix} 2\mathbf{A}_0 & \mathbf{q}_0 \\ \mathbf{q}_0^{\text{T}} & 0 \end{bmatrix} [\mathbf{x} \; t]^{\text{T}} + r_0, \tag{4.28a}$$

$$\text{subject to} \begin{cases} \frac{1}{2}[\mathbf{x}^{\text{T}} \; t] \begin{bmatrix} 2\mathbf{A}_i & \mathbf{q}_i \\ \mathbf{q}_i^{\text{T}} & 0 \end{bmatrix} [\mathbf{x} \; t]^{\text{T}} + r_i \leq 0, \quad \forall i = 1, \ldots, m, & (4.28\text{b}) \\ t^2 = 1. & (4.28\text{c}) \end{cases}$$

Note that the rank constraint (4.27d) is the only nonconvex constraint in problem (4.27), which can be relaxed. The approximate problem in SDP form is

$$\min_{\mathbf{X}} \quad \text{tr}(\mathbf{A}_0\mathbf{X}) + r_0, \tag{4.29a}$$

$$\text{subject to} \begin{cases} \text{tr}(\mathbf{A}_i\mathbf{X}) + r_i \leq 0, \quad \forall i = 1, \ldots, m, & (4.29\text{b}) \\ \mathbf{X} \succeq \mathbf{0}. & (4.29\text{c}) \end{cases}$$

A positive semidefinite matrix $\mathbf{X}^*$ is obtained using the SDR method, which can be used to obtain a feasible and near-optimal solution for the nonconvex QCQP. Note that the rank of the solution to problem (4.29), denoted by $\mathbf{X}^*$, is not necessarily equal to 1, and the following Gaussian sampling method is generally used to extract a feasible point. The interested reader is referred to [11, 12] for more details.

Gaussian Randomization Procedure:

Step 1: Choose a feasible solution $\mathbf{X}^*$ for the relaxed homogeneous QCQP, and the number of samples L.
Step 2: For $l = 1, \ldots, L$, generate $\mathbf{z}_l \sim N(\mathbf{0}; \mathbf{X}^*)$ for the homogeneous QCQP.
Step 3: Determine l^* in $\arg\min_l \{\mathbf{z}_l^{\mathrm{T}} \mathbf{A}_0 \mathbf{z}_l + r_0\}$ for the homogeneous QCQP.
Step 4: Output $\mathbf{x} = \mathbf{z}_{l^*}$ for the homogeneous QCQP.

The obtained solutions may not be feasible for the original optimization problem. In such cases, they can be turned into feasible solutions by rescaling [11, 12]. For example, in the homogeneous optimization problem when r_0 is -1, the following rescaling can be used:

$$\mathbf{a}_l = \frac{\mathbf{z}_l}{\max_{i=1,\ldots,m} \sqrt{\mathbf{z}_l^{\mathrm{T}} \mathbf{A}_i \mathbf{z}_l}}. \tag{4.30}$$

The aforementioned procedure for converting real-valued QCQP into SDP form are also applicable for both complex-valued QCQP and separable QCQP. Consider the following complex-valued homogeneous QCQP:

$$\min_{\mathbf{x}} \quad \mathbf{x}^{\mathrm{T}} \mathbf{A}_0 \mathbf{x} + r_0, \tag{4.31a}$$

$$\text{subject to} \quad \mathbf{x}^{\mathrm{T}} \mathbf{A}_i \mathbf{x} + r_i \le 0, \quad \forall i = 1, \ldots, m, \tag{4.31b}$$

where $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{A}_i \in \mathbb{H}^n$, and $r_i \in \mathbb{C}$, $\forall i = 0, 1, \ldots, m$. Similar to the real-valued case, the corresponding relaxed problem is

$$\min_{\mathbf{X} \in \mathbb{H}^n} \quad \mathrm{tr}(\mathbf{A}_0 \mathbf{X}) + r_0, \tag{4.32a}$$

$$\text{subject to} \begin{cases} \mathrm{tr}(\mathbf{A}_i \mathbf{X}) + r_i \le 0, \quad \forall i = 1, \ldots, m, & (4.32\text{b}) \\ \mathbf{X} \succeq \mathbf{0}, & (4.32\text{c}) \end{cases}$$

where $\mathbf{X} = \mathbf{x}\mathbf{x}^{\mathrm{H}}$.

The separable homogeneous QCQP is

$$\min_{\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{C}^n} \quad \sum_{i=1}^{k} \mathbf{x}_i^{\mathrm{T}} \mathbf{A}_{0i} \mathbf{x}_i + r_0, \tag{4.33a}$$

$$\text{subject to} \quad \sum_{l=1}^{k} \mathbf{x}^{\mathrm{T}} \mathbf{A}_{il} \mathbf{x} + r_i \le 0, \quad \forall i = 1, \ldots, m. \tag{4.33b}$$

Following the same line of argument as the real-valued case, the corresponding complex-valued relaxed problem is

$$\min_{\mathbf{X}_1,\ldots,\mathbf{X}_k \in \mathbb{H}^n} \quad \sum_{i=1}^{k} \operatorname{tr}(\mathbf{A}_{0i}\mathbf{X}_i) + r_0, \tag{4.34a}$$

$$\text{subject to} \begin{cases} \sum_{l=1}^{k} \operatorname{tr}(\mathbf{A}_{il}\mathbf{X}_l) + r_i \leq 0, & \forall i = 1, \ldots, m, \quad (4.34b) \\ \mathbf{X}_i \succeq \mathbf{0}, & \forall i = 1, \ldots, k, \quad (4.34c) \end{cases}$$

where $\mathbf{X}_i = \mathbf{x}_i\mathbf{x}_i^{\mathrm{H}}, \ \ \forall i = 1, \ldots, k$.

The tightness of the SDR solution is an important issue and is evaluated by the rank of the solution matrix. If the algorithm for solving the relaxed SDP form satisfies the rank constraint, the solution of the relaxed problem is also optimal for the original problem. In [17–19], the authors prove that the rank of the matrix solution of an SDP with an $n \times n$ matrix variable and m linear constraints is less than or equal to $\lfloor \frac{\sqrt{8m+1}-1}{2} \rfloor$. For example, when $m \leq 2$, the rank of the solution is 1, which implies that the relaxed SDP is equivalent to the original real-valued homogeneous QCQP. Hence, for a real-valued homogeneous QCQP with two constraints, the solution of the SDR is tight[3] and equivalent. In addition, in [11], the authors show that the rank of the solution of the complex-valued QCQP is less than or equal to $\sqrt{m}$. For complex-valued separable QCQP, the condition on the rank is

$$\sum_{i=1}^{k} \operatorname{rank}(\mathbf{X}_i^*)^2 \leq m, \tag{4.35}$$

and for real-valued separable QCQP problems, the condition on the rank is

$$\sum_{i=1}^{k} \frac{\operatorname{rank}(\mathbf{X}_i^*)\left(\operatorname{rank}(\mathbf{X}_i^*) + 1\right)}{2} \leq m. \tag{4.36}$$

To verify the tightness of solutions of the SDR for various QCQPs, direct methods that rely on rank 1 decomposition methods are used [20, 21]. However, such methods can be used when the number of constraints in the QCQP is not too high [12]. For example, the solution to the SDR form is optimal for the complex-valued homogeneous QCQP when the number of constraints is at most three, and the solution is tight when the number of constraints does not exceed four [21].

[3]When the approximate function or problem is close to its original counterpart, it is called a tight approximation.

4.2.1.6 Bounding Techniques

In robust optimization problems, the existence of error terms (uncertainty) in the objective function and in the constraints makes the problem intractable. A widely used technique for approximating intractable nonconvex optimization problems is to replace nonconvex functions with their tight and safe convex approximations. Depending on the problem, the lower or upper bounds can be used in such replacements, and the approach is called bounding technique (BT) [22–24]. When the bounds in the BT are not tight enough, this relaxation approach is a conservative one, that is, in utility maximization problems, the objective function will be guaranteed to be no less than the worst-case optimum [5, 22, 23]. In the sequel, we briefly review some of the more well-known BTs.

Norm Bounding In some intractable and NP-hard nonconvex optimization problems, the objective function or the constraints include norm functions, which makes the problem difficult to solve. As an example, consider the following intractable nonconvex optimization problem:

$$\max_{\mathbf{x}} \quad \|\mathbf{x}\|^2, \tag{4.37a}$$

$$\text{subject to} \begin{cases} \min\limits_{\|\mathbf{y}_i\|<\varepsilon_1} |\mathbf{x}^{\mathrm{H}}(\mathbf{a}_i + \mathbf{y}_i)|^2 \geq c_1, & \forall i = 1, \ldots, N, & (4.37\text{b}) \\ \max\limits_{\|\mathbf{z}_k\|<\varepsilon_2} |\mathbf{x}^{\mathrm{H}}(\mathbf{b}_k + \mathbf{z}_k)|^2 \leq c_2, & \forall k = 1, \ldots, K, & (4.37\text{c}) \end{cases}$$

where $\mathbf{x}, \mathbf{y}_i, \mathbf{a}_i$, $\mathbf{b}_k$, and $\mathbf{z}_k$ are the $M \times 1$ complex vectors, and $c_1, c_2, \varepsilon_1, \varepsilon_2$ are real nonnegative numbers. The Euclidean norm of $\mathbf{x}$, denoted by $\|\mathbf{x}\|$, is $\sqrt{\sum_{i=1}^{M} x_i^2}$. Fortunately, efficient approximate solutions to such problems can be obtained via the following three norm BTs.

(1) **Triangle Inequality:** When a nonconvex problem includes the norm of the sum of two independent variables $\|\mathbf{x} + \mathbf{y}\|$, it can be replaced by $\|\mathbf{x}\| + \|\mathbf{y}\|$, justified by the following inequality [called the triangle inequality (TI)] [25]:

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| . \tag{4.38}$$

(2) **Cauchy–Schwartz Inequality:** When a nonconvex problem includes $\|\mathbf{x} \odot \mathbf{y}\|$, it can be replaced by $\|\mathbf{x}\|\|\mathbf{y}\|$, justified by the following inequality [called the Cauchy–Schwarz inequality] [25]:

(continued)

$$\|\mathbf{x} \odot \mathbf{y}\| \leq \|\mathbf{x}\|\|\mathbf{y}\|. \tag{4.39}$$

(3) **Norm Approximation:** When a nonconvex problem includes $\|\mathbf{x}\|_u$, one can use the following two l_w-norm inequalities [26]:

- For $1 \leq w < \infty \implies \|\mathbf{x}\|_u \leq \|\mathbf{x}\|_w$;
- For $w = \infty \implies \|\mathbf{x}\|_u \leq \sqrt{N}\|\mathbf{x}\|_\infty = \max_i\{x_i\}$,

where $\mathbf{x} = [x_1, \ldots, x_N]$, $u, w \in \mathbb{R}_{++}$ and $u \geq w$. The l_w-norm of vector $\mathbf{x}$ for $w = 1$, $1 < w < \infty$, and $w = \infty$ are $|\mathbf{x}|$, $\left(\sum_{i=1}^{N} x_i^w\right)^{\frac{1}{w}}$, and $\max_i x_i$, respectively.

To simplify (4.37), constraints (4.37b) and (4.37c) are modified using the aforementioned norm BTs. From the TI it follows that

$$|\mathbf{x}^{\mathrm{H}}(\mathbf{b}_k + \mathbf{z}_k)| \leq |\mathbf{x}^{\mathrm{H}}\mathbf{b}_k| + |\mathbf{x}^{\mathrm{H}}\mathbf{z}_k|. \tag{4.40}$$

Applying $\|\mathbf{z}_k\| < \varepsilon_2$ and Cauchy–Schwarz inequality, we get

$$|\mathbf{x}^{\mathrm{H}}\mathbf{z}_k| \leq \|\mathbf{x}\|\|\mathbf{z}_k\| \leq \varepsilon_2\|\mathbf{x}\|. \tag{4.41}$$

Substituting (4.40) and (4.41) into the left-hand side of (4.37c), we get

$$\max_{\|\mathbf{z}_k\| < \varepsilon_2} |\mathbf{x}^{\mathrm{H}}(\mathbf{b}_k + \mathbf{z}_k)|^2 \leq (|\mathbf{x}^{\mathrm{H}}\mathbf{b}_k| + \varepsilon_2\|\mathbf{x}\|)^2. \tag{4.42}$$

Now, expanding the right-hand side of (4.42) and applying Cauchy–Schwarz inequality, we get

$$\begin{aligned}
(|\mathbf{x}^{\mathrm{H}}\mathbf{b}_k| + \varepsilon_2\|\mathbf{x}\|)^2 &= |\mathbf{x}^{\mathrm{H}}\mathbf{b}_k|^2 + \varepsilon_2^2\|\mathbf{x}\|^2 + 2\varepsilon_2\|\mathbf{x}\||\mathbf{x}^{\mathrm{H}}\mathbf{b}_k|, && (4.43\mathrm{a})\\
&\leq |\mathbf{x}^{\mathrm{H}}\mathbf{b}_k|^2 + \varepsilon_2^2\|\mathbf{x}\|^2 + 2\varepsilon_2\|\mathbf{x}\|^2\|\mathbf{b}_k\|, && (4.43\mathrm{b})\\
&= |\mathbf{x}^{\mathrm{H}}\mathbf{b}_k|^2 + \varepsilon_2(\varepsilon_2 + 2\|\mathbf{b}_k\|)\|\mathbf{x}\|^2, && (4.43\mathrm{c})\\
&= \mathbf{x}^{\mathrm{H}}\hat{\mathbf{B}}_k\mathbf{x}, && (4.43\mathrm{d})
\end{aligned}$$

where $\hat{\mathbf{B}}_k = \mathbf{b}_k\mathbf{b}_k^{\mathrm{H}} + \varepsilon_2\left(\varepsilon_2 + 2\sqrt{\mathbf{b}_k^{\mathrm{H}}\mathbf{b}_k}\right)\mathbf{I}$.

Following the same line of argument as with constraint (4.37c), the left-hand side of constraint (4.37b) is lower bounded,

$$\min_{\|\mathbf{y}_i\| < \varepsilon_1} |\mathbf{x}^{\mathrm{H}}(\mathbf{a}_i + \mathbf{y}_i)|^2 \geq \mathbf{x}^{\mathrm{H}}\hat{\mathbf{A}}_i\mathbf{x}, \tag{4.44a}$$

where $\hat{\mathbf{A}}_i = \mathbf{a}_k\mathbf{a}_k^{\mathrm{H}} + \varepsilon_1\left(\varepsilon_1 - 2\sqrt{\mathbf{a}_k^{\mathrm{H}}\mathbf{a}_k}\right)\mathbf{I}$. Note that it is assumed that $|\mathbf{x}^{\mathrm{H}}\mathbf{a}_i| \geq |\mathbf{x}^{\mathrm{H}}\mathbf{y}_i|$.

We use the preceding expressions to reformulate the original problem (4.37) as

$$\max_{\mathbf{x}} \quad \|\mathbf{x}\|^2, \tag{4.45a}$$

$$\text{subject to} \begin{cases} \mathbf{x}^{\mathrm{H}}\hat{\mathbf{A}}_i\mathbf{x} \geq c_1, & \forall i = 1, \ldots, N, & (4.45\text{b}) \\ \mathbf{x}^{\mathrm{H}}\hat{\mathbf{B}}_k\mathbf{x} \leq c_2, & \forall k = 1, \ldots, K. & (4.45\text{c}) \end{cases}$$

Note that the matrices $\hat{\mathbf{B}}_k$ are always positive definite, but the positive definiteness of the matrices $\hat{\mathbf{A}}_i$ depends on the value of ε_1. It can be easily seen that problem (4.45) is a nonconvex QCQP problem. In Section 4.2.1.5, we showed how to obtain an approximate solution for this problem using SDR.

Probabilistic Bounding The main challenge in robust resource allocation with outage-based quality-of-service (QoS) constraints is the lack of closed-form expressions for the corresponding probability constraints. In general, such problems may be intractable and nonconvex, and obtaining their optimal solutions may be difficult [27–29]. In other words, solving the probabilistically robust optimization problems consisting of outage constraints involving the data rate, SINR, and interference power levels is not straightforward. To reduce the computational complexity of solving such problems, probabilistic BTs can be utilized, where the original intractable probabilistic constraints are replaced by their conservative and tractable approximations that may be either deterministic or statistical.

As an example, consider the following intractable and NP-hard probabilistic nonconvex optimization problem:

$$\min_{\mathbf{X}_1,\ldots,\mathbf{X}_K \in \mathbb{H}^N} \quad \sum_{i=1}^{K} \mathrm{tr}(\mathbf{X}_i), \tag{4.46a}$$

$$\text{subject to} \begin{cases} \Pr\left\{ \dfrac{\mathbf{a}_i^{\mathrm{H}}\mathbf{X}_i\mathbf{a}_i}{r_i + \sum\limits_{k=1,k\neq i}^{K} \mathbf{a}_i^{\mathrm{H}}\mathbf{X}_k\mathbf{a}_i} \geq c_i \right\} \geq \rho_i, \quad \forall i = 1, \ldots, K, & (4.46\text{b}) \\ \mathbf{X}_1, \ldots, \mathbf{X}_K \succeq \mathbf{0}, & (4.46\text{c}) \end{cases}$$

where $\mathbf{a}_i \in \mathbf{C}^N$ is a random variable and $c_i \in \mathbb{R}_+$ and $\rho_i \in [0 \quad 1]$. As can be seen, the main challenge for solving this problem comes from the rate outage probability constraints in (4.46b), which do not admit simple closed-form expressions.

(1) **Markov's Inequality:** The probabilistic constraints or objective function that make the optimization problem intractable can be transformed into a statistically averaged optimization problem using MI. Also, when the distribution of

random variables in constraints or objective function are unknown and only the first two moments are available, this BT can be used to solve such problems [30, 31].

Markov's Inequality: For a nonnegative random variable $X : \Omega \longrightarrow \mathbb{R}$, where $X(s), \forall s \in \Omega$, and for any positive real number $a > 0$, we have

$$\Pr\{X \geq a\} \leq \frac{\mathbb{E}\{X\}}{a}. \tag{4.47}$$

As an example, consider the following nonconvex and intractable probabilistic optimization problem:

$$\min_{\mathbf{X}} \quad \mathrm{tr}(\mathbf{X}\mathbf{X}^{\mathrm{H}}), \tag{4.48a}$$

$$\text{subject to} \quad \sup_{\mathrm{vec}(\mathbf{Y})\sim A(\mathbf{0},\Sigma)} \Pr\{\|\mathbf{A}(\mathbf{B}+\mathbf{Y})\mathbf{X}-\mathbf{D}\|_F^2 \geq \varepsilon\} \leq \zeta, \tag{4.48b}$$

where $\mathbf{A} \in \mathbb{C}^{L\times M}$, $\mathbf{B} \in \mathbb{C}^{M\times N}$, and $\mathbf{X} \in \mathbb{C}^{N\times L}$ are deterministic matrices and $\mathbf{Y} \in \mathbb{C}^{M\times N}$ is a matrix of random variables with arbitrary distribution whose first two moments are known, that is, $\mathrm{vec}(\mathbf{Y}) \sim A(\mathbf{0}, \Sigma)$, and $\varepsilon \in \mathbb{R}_+$ and $\zeta \in [0 \quad 1]$ are deterministic variables. To eliminate the supremum and get a tractable upper bound on constraint (4.48b), we use MI:

$$\Pr\{\|\mathbf{A}(\mathbf{B}+\mathbf{Y})\mathbf{X}-\mathbf{D}\|_F^2 \geq \varepsilon\} \leq \frac{\mathbb{E}\{\|\mathbf{A}(\mathbf{B}+\mathbf{Y})\mathbf{X}-\mathbf{D}\|_F^2\}}{\varepsilon}, \tag{4.49}$$

where the right-hand side of (4.49) can be simplified to

$$\mathbb{E}\{\|\mathbf{A}(\mathbf{B}+\mathbf{Y})\mathbf{X}-\mathbf{D}\|_F^2\} = \mathbb{E}\{\|\mathbf{AYX}\|_F^2\} + \|\mathbf{ABX}-\mathbf{D}\|_F^2, \tag{4.50a}$$

$$= \mathbb{E}\{\|(\mathbf{X}^{\mathrm{T}} \otimes \mathbf{A})\mathrm{vec}(\mathbf{Y})\|^2\} + \|\mathbf{ABX}-\mathbf{D}\|_F^2 \tag{4.50b}$$

$$= \|\mathrm{vec}((\mathbf{X}^{\mathrm{T}} \otimes \mathbf{A})\Sigma^{\frac{1}{2}})^{\mathrm{T}}\mathrm{vec}(\mathbf{ABX}-\mathbf{D})^{\mathrm{T}}\|^2. \tag{4.50c}$$

Accordingly, optimization problem (4.48) is approximated by

$$\min_{\mathbf{X}} \quad \mathrm{tr}(\mathbf{X}\mathbf{X}^{\mathrm{H}}), \tag{4.51a}$$

$$\text{subject to} \quad \|\mathrm{vec}((\mathbf{X}^{\mathrm{T}} \otimes \mathbf{A})\Sigma^{\frac{1}{2}})^{\mathrm{T}}\mathrm{vec}(\mathbf{ABX}-\mathbf{D})^{\mathrm{T}}\|^2 \leq \varepsilon\zeta. \tag{4.51b}$$

This problem is convex, and using the EF, explained in Section 4.2.1.1 in this chapter, it can be converted to the following second-order cone programming (SOCP) problem:

$$\min_{\mathbf{X}} \quad t \tag{4.52a}$$

$$\text{subject to} \begin{cases} \|\text{vec}((\mathbf{X}^{\mathrm{T}} \otimes \mathbf{A})\Sigma^{\frac{1}{2}})^{\mathrm{T}}\text{vec}(\mathbf{ABX} - \mathbf{D})^{\mathrm{T}}\| \le \sqrt{\varepsilon\zeta}, & (4.52\text{b}) \\ \text{tr}(\mathbf{X}) \le t. & (4.52\text{c}) \end{cases}$$

(2) **Bernstein Approximation:** Consider the following intractable and probabilistic optimization problem:

$$\min_{\mathbf{x}} \quad g(\mathbf{x}), \tag{4.53a}$$

$$\text{subject to} \quad \Pr\left\{f_0(\mathbf{x}) + \sum_{n=1}^{N} y_n f_n(\mathbf{x}) \le 0\right\} \ge 1 - \zeta, \tag{4.53b}$$

where $\mathbf{x}$ is a deterministic parameter vector and y_n, $\forall n = 1, \ldots, N$ is a random variable with marginal distribution π_n, $g, f_n : \mathbb{R}^N \longrightarrow \mathbb{R} \ \ \forall n = 0, 1, \ldots, N$. Assume

- $f_n(\mathbf{x})$, $\forall n = 0, 1, \ldots, N$, is an affine function in $\mathbf{x}$;
- x_n, $\forall n = 1, \ldots, N$, is an independent random variable;
- π_n, $\forall n = 1, \ldots, N$, has a bounded support of $[-1, 1]$.

The feasible set of probabilistic constraint (4.53b) can be either convex or non-convex, depending on the distribution of random variables. However, even if the probabilistic constraint is convex, it may not be straightforward to write it in a closed form, which makes the optimization problem computationally intractable [32–34]. To overcome this difficulty, one can use a BA to replace the probabilistic constraint by a tractable and convex approximation. In this way, the feasible set of the approximate optimization problem is a subset of the original problem, and an optimal solution to the approximate optimization problem is a feasible suboptimal solution to the original optimization problem.

Using this BT, a conservative approximation of (4.53b) is

$$\inf_{\rho>0}\left\{f_0(\mathbf{x}) + \rho\sum_{n=1}^{N} \Omega_n\left(\rho^{-1}f_n(\mathbf{x})\right) + \rho\log\left(\frac{1}{\zeta}\right)\right\} \le 0, \tag{4.54}$$

where $\Omega_n(z) \triangleq \max_{\pi_n} \log\left(\int e^{wz} d\pi_n(w)\right)$. In addition, to make (4.54) computationally efficient, the following tractable upper bound for $\Omega_n(z)$ can be used:

$$\Omega_n(z) \le \max\left\{\mu_n^- z, \mu_n^+ z\right\} + \frac{\sigma_n^2 z^2}{2}, \tag{4.55}$$

where μ_n^-, μ_n^+, with $-1 \leq \mu_n^- \leq \mu_n^+ \leq +1$ and $\sigma_n \geq 0$, are constants that depend on their respective probability distributions. Some examples are given in Table 1.2 in Chapter 1, where the useful prior knowledge includes the support, unimodality (with respect to the center of the support), and symmetry of the distribution, as well as the range of the first- and second-order moments. Note that more prior knowledge leads to a tighter approximation [34]. Using (4.55), the upper bound of (4.53b) is

$$f_0(\mathbf{x}) + \sum_{n=1}^{N} \max\left\{\mu_n^- f_n(\mathbf{x}), \mu_n^+ f_n(\mathbf{x})\right\} + \sqrt{2\log\frac{1}{\zeta}}\sqrt{\sum_{n=1}^{N} \sigma_n^2 f_n^2(\mathbf{x})} \leq 0. \tag{4.56}$$

Since the last term in (4.56) involves the l_2-norm of the vector, one can further approximate it via an l_∞-norm approximation, explained in Section 4.2.1.6 in this chapter as the norm BT, as

$$f_0(\mathbf{x}) + \sum_{n=1}^{N} \max\left\{\mu_n^- f_n(\mathbf{x}), \mu_n^+ f_n(\mathbf{x})\right\} + \sqrt{2N\log\frac{1}{\zeta}}\max_n\{\sigma_n f_n(\mathbf{x})\} \leq 0. \tag{4.57}$$

Alternatively, one can use the l_1-norm approximation to obtain another substitute:

$$f_0(\mathbf{x}) + \sum_{n=1}^{N} \max\left\{\mu_n^- f_n(\mathbf{x}), \mu_n^+ f_n(\mathbf{x})\right\} + \sqrt{2\log\frac{1}{\zeta}}\sum_{n=1}^{N} |\sigma_n f_n(\mathbf{x})| \leq 0. \tag{4.58}$$

Substituting (4.58) into (4.53b), a tractable approximation of (4.53) is

$$\min_{\mathbf{x}} \quad g(\mathbf{x}), \tag{4.59a}$$

$$\text{subject to} \;\; f_0(\mathbf{x}) + \sum_{n=1}^{N} \max\left\{\mu_n^- f_n(\mathbf{x}), \mu_n^+ f_n(\mathbf{x})\right\} + \sqrt{2\log\frac{1}{\zeta}}\sum_{n=1}^{N} |\sigma_n f_n(\mathbf{x})| \leq 0. \tag{4.59b}$$

This BT can also be used for those probabilistic optimization problems in which the distribution of random variables is not fully known (complete information on the distribution is not available) [32, 33].

(3) **Conditional Value at Risk (CVaR):** Consider the following intractable joint chance constraint optimization problem:

$$\min_{\mathbf{x}} \quad h(\mathbf{x}), \tag{4.60a}$$

$$\text{subject to} \quad \Pr\left\{f_1(\mathbf{u},\mathbf{x}) \leq 0, \ldots, f_M(\mathbf{u},\mathbf{x}) \leq 0\right\} \geq 1-\alpha, \tag{4.60b}$$

where $\mathbf{u}$ is a random vector supported on a set $\Xi \subset \mathbb{R}^d$, $\mathbf{x} \in \mathbb{R}^n$ is a deterministic vector, $f_i : \mathbb{R}^n \times \Xi \longrightarrow \mathbb{R}$, $\forall i = 1, \ldots, M$, $h : \mathbb{R}^n \longrightarrow \mathbb{R}$, and $\alpha \in (0, 1)$. When

$M = 1$, probabilistic constraint (4.60b) is called a single chance constraint because only a single constraint needs to be satisfied with probability $1 - \alpha$. In general, a joint chance constraint is more difficult than a single chance constraint. For simplicity, the preceding problem can be written as

$$\min_{\mathbf{x}} \quad h(\mathbf{x}), \tag{4.61a}$$

$$\text{subject to} \quad w(\mathbf{x}) \leq \alpha, \tag{4.61b}$$

where

$$w(\mathbf{x}) = 1 - \Pr\left\{f_1(\mathbf{u}, \mathbf{x}) \leq 0, \ldots, f_M(\mathbf{u}, \mathbf{x}) \leq 0\right\}. \tag{4.62}$$

Solving the preceding problem is not straightforward in either of the following two cases. First, even if $f_1(\mathbf{u}, \mathbf{x}), \ldots, f_M(\mathbf{u}, \mathbf{x})$ are all convex functions of $\mathbf{x}$, the probabilistic constraint may not be convex. Hence, the optimization problem (4.60) may be nonconvex, and consequently, finding a global optimal solution would be very difficult. Second, obtaining the value of $w(\mathbf{x})$ is typically not easy, and obtaining a closed form for $w(\mathbf{x})$ is generally very difficult. Instead, Monte Carlo simulations are often performed to calculate $w(\mathbf{x})$ when its closed form is not available. In doing so, $w(\mathbf{x})$ is approximated by a conservative function $\tilde{w}(\mathbf{x}) \geq w(\mathbf{x}),\ \forall \mathbf{x} \in \mathbb{R}^n$. When $\tilde{w}(\mathbf{x})$ is close to $w(\mathbf{x})$, it is a good approximation to the optimal solution of optimization problem (4.61), and the approximate function is tight. Moreover, when $\tilde{w}(\mathbf{x})$ is convex, the following approximated optimization problem may be easier to solve [33, 35]:

$$\min_{\mathbf{x}} \quad h(\mathbf{x}) \tag{4.63a}$$

$$\text{subject to} \quad \tilde{w}(\mathbf{x}) \leq \alpha. \tag{4.63b}$$

In [33], a convex conservative approximation called the CVaR is proposed. Using $g(\mathbf{u}, \mathbf{x}) = \max\{f_1(\mathbf{u}, \mathbf{x}), \ldots, f_M(\mathbf{u}, \mathbf{x})\}$, the joint chance constraint (4.61b) is converted into a single chance constraint as

$$w(\mathbf{x}) = 1 - \Pr\left\{\max\{f_1(\mathbf{u}, \mathbf{x}), \ldots, f_M(\mathbf{u}, \mathbf{x})\} \leq 0\right\} = \Pr\left\{g(\mathbf{u}, \mathbf{x}) > 0\right\}. \tag{4.64}$$

Probabilistic inequality (4.61b) can be written as

$$w(\mathbf{x}) = \Pr\left\{g(\mathbf{u}, \mathbf{x}) > 0\right\} = \mathbb{E}\{\mathbf{1}_{(0,+\infty)}(g(\mathbf{u}, \mathbf{x}))\} \leq \alpha, \tag{4.65}$$

where $\mathbf{1}_{(\mathbb{A})}(z)$ is the indicator function of the set $\mathbb{A}$, which is equal to 1 if $z \in \mathbb{A}$ and 0 if $z \notin A$. The indicator function makes $w(\mathbf{x})$ nonconvex. A conservative convex approximation of $w(\mathbf{x}) \leq \alpha$ is

$$\inf_{t>0} \left\{t\mathbb{E}\{\phi(t^{-1}g(\mathbf{u}, \mathbf{x})) - t\alpha\}\right\} \leq 0, \tag{4.66}$$

where $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a nonnegative, nondecreasing, and convex function, lower bounded by $\phi(0)$, and can be obtained by

$$\phi(z) = [1 + z]^+, \tag{4.67}$$

where $[a]^+ = \max\{a, 0\}$. In other words, $\mathbf{1}_{(0,+\infty)}(z) \leq \phi(z)$ and $\tilde{w}(\mathbf{x}) = \mathbb{E}\{\phi(t^{-1}g(\mathbf{u}, \mathbf{x}))\}$.

Using (4.67) and after some mathematical manipulations, (4.66) is equivalently rewritten as

$$\inf_{t \in \mathbb{R}} \left\{\mathbb{E}\{[g(\mathbf{u}, \mathbf{x}) + t]^+\} - t\alpha\right\} \leq 0. \tag{4.68}$$

(4) **DC Approximation:** In the CVaR BT, the difference between the indicator function and the CVaR approximation, that is, $\mathbf{1}_{(\mathbb{A})}(z) - \phi(z)$, grows unboundedly as $z \longrightarrow +\infty$. Thus, $\phi(z)$ is not a good approximation for the indicator function $\mathbf{1}_{(\mathbb{A})}(z)$. A better approximation for joint chance constraint (4.61b), called the difference of two convex (DC) functions, is proposed in [35], which is

$$\phi(z) = [1 + z]^+ - z^+. \tag{4.69}$$

Since $[1+z]^+$ and z^+ are convex functions of z, $\phi(z)$ is a DC function of z. Moreover, since $\mathbf{1}_{(0,+\infty)}(z) \leq \phi(z), \forall z \in \mathbb{R}$, (4.69) is also a conservative approximation for $\mathbf{1}_{(0,+\infty)}(z)$. In contrast to CVaR approximation that can find feasible but suboptimal solutions, DC approximation is equivalent to the highly intractable probability constraint (4.61b). Hence, the intractable and NP-hard optimization problem (4.61) can be equivalently reformulated as a DC programming problem, which can be efficiently solved by existing algorithms, such as an SCA method, that will be explained subsequently in Section 4.2.1.8 of this chapter.

(5) **Large Deviation Inequality for Complex Gaussian Quadratic Forms:** Consider the following intractable nonconvex optimization problem in which the random variables are complex Gaussian in quadratic form:

$$\min_{\mathbf{X}_1,\ldots,\mathbf{X}_K \in \mathbb{H}_+^N} \quad \sum_{i=1}^{K} \mathrm{tr}(\mathbf{X}_i), \tag{4.70a}$$

$$\text{subject to} \quad \Pr\left\{\mathbf{e}^{\mathrm{H}}\mathbf{Q}_i(\mathbf{X}_i)\mathbf{e} + 2\mathrm{Re}\{\mathbf{e}^{\mathrm{H}}\mathbf{r}_i(\mathbf{X}_i)\} + s_i(\mathbf{X}_i) < 0\right\} \leq \rho_i, \quad \forall i = 1, \ldots, K, \tag{4.70b}$$

where $\mathbf{e} \sim CN(\mathbf{0}, \mathbf{I})$ is a standard circularly symmetric complex Gaussian random vector, $(\mathbf{X}_i, \mathbf{r}_i, s_i) \in \mathbb{H}_+^N \times \mathbb{C}^N \times \mathbb{R}$ is an arbitrary 3-tuple of deterministic variables, and $\mathbf{Q}_i(\mathbf{X}_i) = \mathbf{A}_i\mathbf{X}_i\mathbf{A}_i$, $\mathbf{r}_i(\mathbf{X}_i) = \mathbf{B}_i\mathbf{X}_i\mathbf{c}_i$, $\mathbf{s}_i(\mathbf{X}_i) = \mathbf{d}_i^{\mathrm{H}}\mathbf{X}_i\mathbf{d}_i$, $\mathbf{A}_i, \mathbf{B}_i \in \mathbb{C}^{N\times N}$, and $\mathbf{c}_i, \mathbf{d}_i \in \mathbb{C}^N$, $\forall i = 1, \ldots, K$. The probabilistic constraint entails a significant analytical and computational challenge. To tackle this challenge, the BT-large deviation inequality (BT-LDI) can be used to analytically obtain a tractable upper bound on the probability of the complex Gaussian quadratic form, which would lead to a convex, safe, and tractable approximation for the original probabilistic constraint.

To find a computationally efficient convex function $f_i : \mathbb{H}^n \times \mathbb{C}^N \times \mathbb{R} \to \mathbb{R}$ such that

$$\Pr\left\{\mathbf{e}^{\mathrm{H}}\mathbf{Q}_i(\mathbf{X}_i)\mathbf{e} + 2\mathrm{Re}\{\mathbf{e}^{\mathrm{H}}\mathbf{r}_i(\mathbf{X}_i)\} + s_i(\mathbf{X}_i) < 0\right\} \le f(\mathbf{Q}_i, \mathbf{r}_i, s_i), \tag{4.71}$$

the LDI can be used [36, 37]. The following lemma derives a convex upper bound for (4.70b) [38, 39].

Lemma 4.1. *For any* $v_i > \frac{1}{\sqrt{2}}$ *and* $\eta_i > 0$*, we have*

$$\Pr\left\{\mathbf{e}^{\mathrm{H}}\mathbf{Q}_i(\mathbf{X}_i)\mathbf{e} + 2\mathrm{Re}\{\mathbf{e}^{\mathrm{H}}\mathbf{r}_i(\mathbf{X}_i)\} < \mathrm{tr}(\mathbf{Q}_i(\mathbf{X}_i)) - \eta_i\right\}$$
$$\le \begin{cases} \exp\left(-\frac{\eta_i^2}{4T_i^2}\right), & \text{for} \quad 0 < \eta_i \le 2\overline{\theta}_i v_i T_i, \\ \exp\left(-\frac{\overline{\theta}_i v_i \eta_i}{T_i} + (\overline{\theta}_i v_i)^2\right), & \text{for} \quad \eta_i > 2\overline{\theta}_i v_i T_i, \end{cases} \tag{4.72}$$

where $\overline{\theta}_i = 1 - \frac{1}{2v_i^2}$, $T_i = v_i\|\mathbf{Q}_i(\mathbf{X}_i)\|_F + \frac{1}{\sqrt{2}}\|\mathbf{r}_i(\mathbf{X}_i)\|$, $\eta_i = tr(\mathbf{Q}_i(\mathbf{X}_i)) + s_i(\mathbf{X}_i)$, *and* v_i *is the solution to the quadratic equation* $(1 - \frac{1}{2v_i^2})v_i = \sqrt{\ln(\frac{1}{\rho_i})}$ *that satisfies* $v_i > \frac{1}{\sqrt{2}}$.

The detailed proof of the preceding lemma can be found in [37] and references therein. By this lemma, when $2\sqrt{\ln(\frac{1}{\rho_i})}T_i \le \eta_i \le 2\overline{\theta}_i v_i T_i$ or, equivalently, when $\eta_i = 2\sqrt{\ln(\frac{1}{\rho_i})}T_i$, probabilistic constraint (4.70b) is satisfied. On the other hand, when $\eta_i > 2\overline{\theta}_i v_i T_i = 2\sqrt{\ln(\frac{1}{\rho_i})}T_i$, from Lemma 4.1 we get

$$\Pr\left\{\mathbf{e}^{\mathrm{H}}\mathbf{X}_i\mathbf{e} + 2\mathrm{Re}\{\mathbf{e}^{\mathrm{H}}\mathbf{r}_i\} + s_i < 0\right\} \le \exp\left(-\frac{\overline{\theta}_i v_i \eta_i}{T_i} + (\overline{\theta}_i v_i)^2\right) < \exp\left(-(\overline{\theta}_i v_i)^2\right), \tag{4.73}$$

which implies that probabilistic constraint (4.70b) is satisfied. Thus, we have

$$\Pr\left\{\mathbf{e}^{\mathrm{H}}\mathbf{Q}_i(\mathbf{X}_i)\mathbf{e} + 2\mathrm{Re}\{\mathbf{e}^{\mathrm{H}}\mathbf{r}_i(\mathbf{X}_i)\} + s_i(\mathbf{X}_i) < 0\right\} \le \exp\left(-\frac{(\mathrm{tr}(\mathbf{Q}_i(\mathbf{X}_i)) + s_i(\mathbf{X}_i))^2}{4T_i^2}\right). \tag{4.74}$$

From the preceding expression, a safe approximation for probabilistic constraint (4.70b) is

$$\mathrm{tr}(\mathbf{Q}_i(\mathbf{X}_i)) + s_i(\mathbf{X}_i) \geq 2T_i \sqrt{\ln\left(\frac{1}{\rho_i}\right)}. \tag{4.75}$$

Now, (4.75) can be stated as a second-order cone (SOC) constraint, and intractable probabilistic constraint (4.70b) is reformulated as the following feasibility problem:

$$\text{Find} \quad \mathbf{Q}_i(\mathbf{X}_i), \mathbf{r}_i(\mathbf{X}_i), s_i(\mathbf{X}_i), z_i, y_i, \tag{4.76a}$$

$$\text{subject to} \begin{cases} \mathrm{tr}(\mathbf{Q}_i(\mathbf{X}_i)) + s_i(\mathbf{X}_i) \geq 2(z_i + y_i)\sqrt{\ln\left(\frac{1}{\rho_i}\right)}, & (4.76\text{b}) \\ v_i \|\mathbf{Q}_i(\mathbf{X}_i)\|_F \leq y_i, & (4.76\text{c}) \\ \frac{1}{\sqrt{2}} \|\mathbf{r}_i(\mathbf{X}_i)\| \leq z_i. & (4.76\text{d}) \end{cases}$$

By applying the preceding concepts to the probabilistic optimization problem (4.70), the following conic program with LMI and SOC constraints is obtained:

$$\min_{\mathbf{X}_i,\, z_i,\, y_i,\, \forall i=1,\ldots,K} \quad \sum_{i=1}^{K} \mathrm{tr}(\mathbf{X}_i), \tag{4.77a}$$

$$\text{subject to} \begin{cases} \mathrm{tr}(\mathbf{Q}_i(\mathbf{X}_i)) + s_i(\mathbf{X}_i) \geq 2(z_i + y_i)\sqrt{\ln\left(\frac{1}{\rho_i}\right)}, & \forall i = 1, \ldots, K, \quad (4.77\text{b}) \\ v_i \|\mathrm{vec}(\mathbf{Q}_i(\mathbf{X}_i))\| \leq y_i, & \forall i = 1, \ldots, K, \quad (4.77\text{c}) \\ \frac{1}{\sqrt{2}} \|\mathbf{r}_i(\mathbf{X}_i)\| \leq z_i, & \forall i = 1, \ldots, K, \quad (4.77\text{d}) \\ \mathbf{X}_1, \ldots, \mathbf{X}_K \in \mathbb{H}_+^N. & (4.77\text{e}) \end{cases}$$

The preceding problem can be easily solved using existing convex optimization software such as CVX or SeDuMi.

(6) **Bernstein-type Inequality for Complex Gaussian Quadratic Forms:** An alternative way to obtain a safe approximation for (4.70b) is to use the BTI-CGQF [37, 40, 41], as explained in what follows.

Lemma 4.2. *For any $\eta_i > 0$, we have*

$$\Pr\left\{\mathbf{e}^{\mathrm{H}}\mathbf{Q}_i(\mathbf{X}_i)\mathbf{e} + 2\mathrm{Re}\{\mathbf{e}^{\mathrm{H}}\mathbf{r}_i(\mathbf{X}_i)\} \geq \Upsilon(\eta_i)\right\} \geq 1 - e^{-\eta_i}, \tag{4.78}$$

where $\boldsymbol{\Upsilon} : \mathbb{R}_{++} \to \mathbb{R}$ *is*

$$\boldsymbol{\Upsilon}(\eta) = \mathrm{tr}(\mathbf{Q}_i(\mathbf{X}_i)) - \sqrt{2\eta}\sqrt{\|\mathbf{Q}_i(\mathbf{X}_i)\|_F^2 + 2\|\mathbf{r}_i(\mathbf{X}_i)\|^2} - \eta\lambda^+(\mathbf{Q}_i(\mathbf{X}_i)). \tag{4.79}$$

Since $\boldsymbol{\Upsilon}$ is a monotonic and decreasing function, its inverse mapping can be well defined, and the Bernstein-type inequality (4.78) is rewritten as

$$\Pr\left\{\mathbf{e}^{\mathrm{H}}\mathbf{Q}_i(\mathbf{X}_i)\mathbf{e} + 2\mathrm{Re}\{\mathbf{e}^{\mathrm{H}}\mathbf{r}_i(\mathbf{X}_i)\} + s_i(\mathbf{X}_i) > 0\right\} \geq 1 - e^{-\Upsilon^{-1}(-s_i(\mathbf{X}_i))}. \tag{4.80}$$

A safe approximation for (4.70b), that is, $f(\mathbf{Q}_i, \mathbf{r}_i, s_i) \leq \rho_i$, is

$$\mathrm{tr}(\mathbf{Q}_i(\mathbf{X}_i)) - \sqrt{2\ln\left(\frac{1}{\rho_i}\right)}\sqrt{\|\mathbf{Q}_i(\mathbf{X}_i)\|_F^2 + 2\|\mathbf{r}_i(\mathbf{X}_i)\|^2} + \ln(\rho_i)\lambda^+(\mathbf{Q}_i(\mathbf{X}_i)) + s_i(\mathbf{X}_i) \geq 0, \tag{4.81}$$

where $\lambda^+(\mathbf{Q}_i(\mathbf{X}_i)) = \max\{0, \lambda_{\max}(-\mathbf{Q}_i(\mathbf{X}_i))\}$ and $\lambda_{\max}(-\mathbf{Q}_i(\mathbf{X}_i))$ is the maximum eigenvalue of $-\mathbf{Q}_i(\mathbf{X}_i)$.

Using slack variables, (4.81) is reformulated as a feasibility problem consisting of SOC and LMI constraints:

$$\text{Find} \quad \mathbf{Q}_i(\mathbf{X}_i), \mathbf{r}_i(\mathbf{X}_i), s_i(\mathbf{X}_i), z_i, y_i, \tag{4.82a}$$

$$\text{subject to} \begin{cases} \mathrm{tr}(\mathbf{Q}_i(\mathbf{X}_i)) - z_i\sqrt{2\ln\left(\frac{1}{\rho_i}\right)} + y_i\ln(\rho_i) + s_i(\mathbf{X}_i) \geq 0, & (4.82\text{b}) \\ \sqrt{\|\mathbf{Q}_i(\mathbf{X}_i)\|_F^2 + 2\|\mathbf{r}_i(\mathbf{X}_i)\|^2} \leq z_i, & (4.82\text{c}) \\ y_i\mathbf{I} + \mathbf{Q}_i(\mathbf{X}_i) \succeq \mathbf{0}, & (4.82\text{d}) \\ y_i \geq 0. & (4.82\text{e}) \end{cases}$$

Application of the preceding problem to probabilistic optimization problem (4.70b) makes it possible to convert the original optimization problem (4.70) into SDP form as

$$\min_{\mathbf{X}_i, z_i, y_i, \forall i=1,\ldots,K} \quad \sum_{i=1}^{K} \mathrm{tr}(\mathbf{X}_i), \tag{4.83a}$$

$$\text{subject to} \begin{cases} \operatorname{tr}(\mathbf{Q}_i(\mathbf{X}_i)) - z_i\sqrt{2\ln\left(\frac{1}{\rho_i}\right)} + y_i\ln(\rho_i) + s_i(\mathbf{X}_i) \geq 0, & \forall i = 1,\ldots,K, \quad (4.83\text{b}) \\ \sqrt{\|\operatorname{vec}(\mathbf{Q}_i(\mathbf{X}_i))\|^2 + 2\|\mathbf{r}_i(\mathbf{X}_i)\|^2} \leq z_i, & \forall i = 1,\ldots,K, \quad (4.83\text{c}) \\ y_i\mathbf{I}_N + \mathbf{Q}_i(\mathbf{X}_i) \succeq \mathbf{0}, & \forall i = 1,\ldots,K, \quad (4.83\text{d}) \\ y_i \geq 0, & \forall i = 1,\ldots,K, \quad (4.83\text{e}) \\ \mathbf{X}_1,\ldots,\mathbf{X}_K \in \mathbb{H}_+^N. & (4.83\text{f}) \end{cases}$$

The computational complexity of the BT-LDI is less than that of the BTI-CGQF, but the latter demonstrates better performance.

(7) **Vysochanskii–Petunin Inequality:** When the objective function or constraints are probabilistic with unimodal[4] and non-Gaussian distributions, the optimization problem is intractable. In such cases, the VPI can be used to transform the problem into a statistically averaged optimization problem. As an example, consider the following intractable chance constraint problem:

$$\min_{\mathbf{x}} \quad f_2(\mathbf{x}), \tag{4.84a}$$

$$\text{subject to} \quad \Pr\left\{\mathbf{y}^{\mathrm{H}}\mathbf{A}(\mathbf{x})\mathbf{y} \geq \epsilon\right\} \leq \zeta, \tag{4.84b}$$

where $\mathbf{x} \in \mathbb{C}^N$, $\mathbf{A}(\mathbf{x}) \in \mathbb{H}^N$, $\epsilon \in \mathbb{R}_+$, and $\zeta \in [0 \ \ 1]$ are deterministic variables, and $\mathbf{y} \in \mathbb{C}^N$ is a random variable. It is assumed that $\mathbf{y}^{\mathrm{H}}\mathbf{A}(\mathbf{x})\mathbf{y}$ is a unimodal random variable.

Vysochanskii–Petunin Inequality: Let x be a unimodal random variable and $a \in \mathbb{R}_{++}$ be a deterministic parameter. A tight upper bound on $\Pr\left\{|x| \geq a\right\}$ is [42–44]

$$\Pr\{|x| \geq a\} \leq \max\left\{\frac{4\mathbb{E}\{|x|^2\}}{9a^2}, \frac{4\mathbb{E}\{|x|^2\}}{3a^2} - \frac{1}{3}\right\}. \tag{4.85}$$

In [44], the authors generalized (4.85) to the rth moment as

$$\Pr\{|x| \geq a\} \leq \max\left\{\frac{r^r\mathbb{E}\{|x|^r\}}{(r+1)^r a^r}, \frac{s\mathbb{E}\{|x|^r\}}{a^r(s-1)} - \frac{1}{s-1}\right\}, \tag{4.86}$$

where s is a fixed and deterministic variable that satisfies $s > r + 1$ and $s(s-r-1)^r = r^r$.

[4] Unimodality means there is only a single highest value in the distribution function. For example, random variables with a normal distribution are unimodal.

Since $\mathbf{y}^{\mathrm{H}}\mathbf{A}(\mathbf{x})\mathbf{y}$ is a unimodal random variable, using the VPI, an upper bound on (4.84b) is

$$\Pr\left\{\mathbf{y}^{\mathrm{H}}\mathbf{A}(\mathbf{x})\mathbf{y} \geq \epsilon\right\} \geq \Pr\left\{|\mathbf{y}^{\mathrm{H}}\mathbf{A}(\mathbf{x})\mathbf{y}| \geq \epsilon\right\} \tag{4.87a}$$

$$\leq \max\left\{\frac{r^r\mathbb{E}\{|\mathbf{y}^{\mathrm{H}}\mathbf{A}(\mathbf{x})\mathbf{y}|^r\}}{(r+1)^r\epsilon^r}, \frac{s\mathbb{E}\{|\mathbf{y}^{\mathrm{H}}\mathbf{A}(\mathbf{x})\mathbf{y}|^r\}}{\epsilon^r(s-1)} - \frac{1}{s-1}\right\}. \tag{4.87b}$$

Replacing the outage probability in (4.84b) with the upper bound in (4.87b), a unified safe approximation for the probabilistic optimization problem (4.84) is

$$\min_{\mathbf{x}} \quad f_2(\mathbf{x}), \tag{4.88a}$$

$$\text{subject to} \quad \mathbb{E}\{|\mathbf{y}^{\mathrm{H}}\mathbf{A}(\mathbf{x})\mathbf{y}|^r\} \leq \phi(\zeta, r)\epsilon^r, \tag{4.88b}$$

where

$$\phi(\zeta, r) = \begin{cases} (1+\frac{1}{r})^r, & \zeta \in (0, b_1], \\ \frac{(s-1)\zeta+1}{s}, & \zeta \in [b_1, 1), \end{cases} \tag{4.89}$$

and $b_1 = \frac{1}{s((1+\frac{1}{r})^r-1)+1}$.

Trace Bounding When an optimization problem involves the trace of a matrix, the following inequalities can be used to make the problem tractable.

(1) **Determinant Inequality:** When an optimization problem includes a nonconvex determinant function $\det(\mathbf{I}+\mathbf{A})$, it can be replaced by $1+\mathrm{tr}(\mathbf{A})$, justified by the inequality [5]

$$\det(\mathbf{I}+\mathbf{A}) \geq 1+\mathrm{tr}(\mathbf{A}), \tag{4.90}$$

where $\mathbf{A} \succeq 0$.

Lemma 4.3. *The equality in* (4.90) *holds if and only if the rank of* $\mathbf{A}$ *is 1.*

Proof. Let $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_r > 0$ be the eigenvalues of $\mathbf{A}$ with rank r. We have

$$\det(\mathbf{I}+\mathbf{A}) = \prod_{i=1}^{r}(1+\lambda_i) = 1+\sum_{i=1}^{r}\lambda_i+\sum_{k\neq i}\lambda_k\lambda_i+\cdots \geq 1+\sum_{i=1}^{r}\lambda_i = 1+\mathrm{tr}(\mathbf{A}), \tag{4.91}$$

where equality (4.91) holds if and only if $r = 1$.

As an example, consider the following nonconvex optimization problem:

$$\min_{\mathbf{X}\succeq\mathbf{0}} \quad \mathrm{tr}(\mathbf{X}), \tag{4.92a}$$

$$\text{subject to} \quad \frac{\det(\mathbf{I}+\mathbf{A}^{\mathrm{H}}\mathbf{X}\mathbf{A})}{1+\mathbf{b}^{\mathrm{H}}\mathbf{X}\mathbf{b}} \le \zeta, \tag{4.92b}$$

where $\mathbf{X} \in \mathbb{H}^N$, $\mathbf{A} \in \mathbb{C}^{N\times N}$, $\mathbf{b} \in \mathbb{C}^N$, and $\zeta \in \mathbb{R}_{++}$. Since the nonconvexity of the preceding optimization problem comes from the determinant function in constraint (4.92b), the BT-DI is a good choice for transforming the problem into a convex one. Accordingly, a relaxation of (4.92) is

$$\min_{\mathbf{X}\succeq\mathbf{0}} \quad \mathrm{tr}(\mathbf{X}), \tag{4.93a}$$

$$\text{subject to} \quad \frac{1+\mathrm{tr}(\mathbf{A}^{\mathrm{H}}\mathbf{X}\mathbf{A})}{1+\mathbf{b}^{\mathrm{H}}\mathbf{X}\mathbf{b}} \le \zeta. \tag{4.93b}$$

Optimization problem (4.93) is convex and can be formulated in SDP form:

$$\min_{\mathbf{X}\succeq\mathbf{0}} \quad \mathrm{tr}(\mathbf{X}), \tag{4.94a}$$

$$\text{subject to} \quad 1+\mathrm{tr}(\mathbf{A}^{\mathrm{H}}\mathbf{X}\mathbf{A}) \le \zeta(1+\mathbf{b}^{\mathrm{H}}\mathbf{X}\mathbf{b}). \tag{4.94b}$$

The optimal solution of the preceding optimization problem can be obtained efficiently using any existing solver such as CVX. Note that from Lemma 4.3, this relaxation is tight when the rank of the optimal solution to problem (4.94) is 1.

(2) **Trace of Two Matrices Inequality:** When the optimization problem includes a trace of two matrices, it can be replaced by a tractable bound, as defined in what follows.

Lemma 4.4. *When* $\mathbf{X}$ *and* $\mathbf{Y}$ *are* $n \times n$ *positive semidefinite Hermitian matrices, we have [45, 46]*

$$\mathrm{tr}(\mathbf{X}\mathbf{Y}) \ge \sum_{i=1}^{n} \lambda_i(\mathbf{X})\lambda_{n-i+1}(\mathbf{Y}), \tag{4.95}$$

where $\lambda_i(\mathbf{X})$ *and* $\lambda_i(\mathbf{Y})$ *are the ith eigenvalues of* $\mathbf{X}$ *and* $\mathbf{Y}$*, respectively, in decreasing order.*

As an example, consider the following intractable and NP-hard problem:

$$\min_{\mathbf{x}} \quad \max_{\mathbf{z}_k,\ \forall k=1,\ldots,K} \quad \mathbf{x}^{\mathrm{H}}\mathbf{A}\mathbf{x}, \tag{4.96a}$$

$$\text{subject to} \quad \mathbf{z}_k \in \{\mathbf{z}_k^{\mathrm{H}}\mathbf{B}_k\mathbf{z}_k \le 1\}, \quad \forall k = 1,\ldots,K, \tag{4.96b}$$

where $\mathbf{x}, \mathbf{z}_k \in \mathbb{C}^N$, $\mathbf{B}_k \succ \mathbf{0}$, $\mathbf{A} = \text{diag}\{a_{11}, \ldots, a_{N,N}\}$, $a_{n,n} = \sum_{k=1}^{K} |y_{nk} + z_{nk}|^2$, and $\mathbf{y}_k = [y_{1,k}, \ldots, y_{N,k}] \in \mathbb{C}^N$. Using Cauchy–Schwarz inequality, an upper bound on the objective function is

$$\mathbf{x}^{\text{H}}\mathbf{A}\mathbf{x} = \mathbf{x}^{\text{H}}\tilde{\mathbf{A}}\mathbf{x} + \sum_{k=1}^{K} \|\mathbf{x} \odot \mathbf{z}_k\|^2 \leq \mathbf{x}^{\text{H}}\tilde{\mathbf{A}}\mathbf{x} + \sum_{k=1}^{K} \|\mathbf{x}\|^2 \|\mathbf{z}_k\|^2, \tag{4.97}$$

where $\tilde{\mathbf{A}} = \text{diag}\{\tilde{a}_{11}, \ldots, \tilde{a}_{N,N}\}$, and $\tilde{a}_{n,n} = \sum_{k=1}^{K} |y_{nk}|^2$. Now, from Lemma 4.4 and since the rank of $\mathbf{z}_k\mathbf{z}_k^{\text{H}}$ is 1, we have

$$\mathbf{z}_k^{\text{H}}\mathbf{B}_k\mathbf{z}_k = \text{tr}(\mathbf{B}_k\mathbf{z}_k\mathbf{z}_k^{\text{H}}) \geq \lambda_{\texttt{min}}(\mathbf{B}_k)\lambda_1(\mathbf{z}_k\mathbf{z}_k^{\text{H}}) = \lambda_{\texttt{min}}(\mathbf{B}_k)\|\mathbf{z}_k\|^2. \tag{4.98}$$

From constraint (4.96b) and (4.98), we write

$$\|\mathbf{z}_k\| \leq \frac{1}{\sqrt{\lambda_{\texttt{min}}(\mathbf{B}_k)}}. \tag{4.99}$$

Using the preceding inequality, (4.97) is approximated by

$$\mathbf{x}^{\text{H}}\tilde{\mathbf{A}}\mathbf{x} + \sum_{k=1}^{K} \|\mathbf{x}\|^2\|\mathbf{z}_k\|^2 \leq \mathbf{x}^{\text{H}}\tilde{\mathbf{A}}\mathbf{x} + \sum_{k=1}^{K} \mu_k^2\|\mathbf{z}_k\|^2 = \mathbf{x}^{\text{H}}(\tilde{\mathbf{A}} + \eta\mathbf{I})\mathbf{x}, \tag{4.100}$$

where $\eta = \sum_{k=1}^{K} \mu_k^2$ and $\mu_k = \frac{1}{\sqrt{\lambda_{\texttt{min}}(\mathbf{B}_k)}}$.

From the preceding expression, the optimization problem (4.97) can be solved by solving the following convex optimization problem:

$$\min_{\mathbf{x}} \quad \mathbf{x}^{\text{H}}(\tilde{\mathbf{A}} + \eta\mathbf{I})\mathbf{x}. \tag{4.101}$$

(3) **Von Neumann Trace Inequality:** For any two $n \times n$ complex matrices $\mathbf{A}$ and $\mathbf{B}$ with singular values $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$, respectively, we get [47]

$$|\text{tr}(\mathbf{AB})| \leq \sum_{m=1}^{n} \alpha_m\beta_m. \tag{4.102}$$

The equality is achieved when $\mathbf{A}$ and $\mathbf{B}$ are simultaneously unitarily diagonalizable.

Taylor Bounding Any differentiable nonconvex and intractable function $f(\mathbf{x}) : \mathbb{R}^N \to \mathbb{R}$ can be approximated at $\tilde{\mathbf{x}}$ by a linear or quadratic function using the first- or second-order terms in its Taylor series expansion, respectively. The first-order (linear) approximation function is [48]

$$g_1(\mathbf{x}) = f(\tilde{\mathbf{x}}) + (\mathbf{x} - \tilde{\mathbf{x}})^{\text{T}}\nabla f(\tilde{\mathbf{x}}), \tag{4.103}$$

where $\nabla f(\tilde{\mathbf{x}})$ is the first derivative of $f(\mathbf{x})$ at $\tilde{\mathbf{x}}$.

The second-order (quadratic) approximation is

$$g_2(\mathbf{x}) = f(\tilde{\mathbf{x}}) + (\mathbf{x} - \tilde{\mathbf{x}})^{\mathrm{T}} \nabla f(\tilde{\mathbf{x}}) + \frac{1}{2} (\mathbf{x} - \tilde{\mathbf{x}})^{\mathrm{T}} \nabla^2 f(\tilde{\mathbf{x}}) (\mathbf{x} - \tilde{\mathbf{x}}), \tag{4.104}$$

where $\nabla^2 f(\tilde{\mathbf{x}})$ is the second derivative of $f(\mathbf{x})$ at $\tilde{\mathbf{x}}$. Note that these approximations are tight if $\mathbf{x}$ is sufficiently close to $\tilde{\mathbf{x}}$. In addition, the quadratic approximation is tighter than the linear approximation when $\mathbf{x}$ is very close to $\tilde{\mathbf{x}}$, but the linear approximation is more tractable and simpler than the quadratic approximation.

4.2.1.7 Nonlinear Fractional Programming

Consider the following nonlinear fractional optimization problem [49]:

$$q^* = \max_{\mathbf{x} \in \mathbb{X}} \frac{f_1(\mathbf{x})}{f_2(\mathbf{x})}, \tag{4.105a}$$

$$\text{subject to} \quad f_3(\mathbf{x}) \le 0, \tag{4.105b}$$

where $\mathbb{X} \subset \mathbb{R}^n$ is a nonempty convex compact set, $f_i : \mathbb{R}^n \longrightarrow \mathbb{R},\ i = 2, 3$, are real-valued, continuously differentiable, and convex functions, and $f_1 : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a real-valued continuously differentiable and concave function.

Theorem 4.4. *The maximum value of the objective function, denoted by q^*, is achieved if and only if* $\max_{\mathbf{x}'} f_1(\mathbf{x}') - q^* f_2(\mathbf{x}') = f_1(\mathbf{x}^*) - q^* f_2(\mathbf{x}^*) = 0$ *for* $f_1(\mathbf{x}) \ge 0$ *and* $f_2(\mathbf{x}) > 0$*, where x^* is the global optimal solution.*

From Theorem 4.4, for any optimization problem whose objective function is in fractional form, there exists an equivalent objective function in subtractive form, for example, $f_1(\mathbf{x}) - q f_2(\mathbf{x})$. To solve problem (4.105), an iterative algorithm is used that converges to the optimal solution with an acceptable convergence speed [50].

Iterative Algorithm for Solving (4.105):

Step 0: Choose a predetermined error tolerance $\epsilon > 0$ and the maximum iteration number $L_{\mathtt{max}}$.
Step 1: Set $q = 0$ and iteration index $l = 0$.
Step 2: While (Convergence = **false** and $l \le L_{\mathtt{max}}$)
Step 3: Solve the convex problem (4.106) for a given q.
Step 4: If $f_1(\mathbf{x}') - q f_2(\mathbf{x}') < \epsilon$, **then**

(continued)

Step 4.1: Convergence =**true**
Step 4.2: **return** $\mathbf{x}^* = \mathbf{x}'$ and $q^* = \frac{f_1(\mathbf{x}')}{f_2(\mathbf{x}')}$
Step 5: else
Step 5.1: Set $q = \frac{f_1(\mathbf{x}')}{f_2(\mathbf{x}')}$ and $l = l + 1$.
Step 5.2: Convergence = **false**.
Step 6: end if
Step 7: end while
Step 8: return $\mathbf{x}^* = \mathbf{x}'$ and $q^* = \frac{f_1(\mathbf{x}')}{f_2(\mathbf{x}')}$.

The preceding algorithm at each iteration solves the following convex problem:

$$\max_{\mathbf{x}' \in \mathbb{X}} \quad f_1(\mathbf{x}') - q f_2(\mathbf{x}'), \tag{4.106a}$$

$$\text{subject to} \quad f_3(\mathbf{x}') \leq 0. \tag{4.106b}$$

4.2.1.8 Sequential Parametric Convex Approximation

The SPCA is a general iterative scheme for solving nonconvex optimization problems, where in each iteration the nonconvex optimization problem is replaced by its convex approximation. When this is done, the nonconvex constraints are replaced by their safe approximations. As described in what follows, when certain conditions are satisfied, a monotonic convergence to a Karush–Kuhn–Tucker (KKT) point is achieved [51]. Consider the following generic optimization problem:

$$\min_{\mathbf{x} \in \mathbb{R}^n} \quad f_0(\mathbf{x}), \tag{4.107a}$$

$$\text{subject to} \quad f_z(\mathbf{x}) \leq 0, \quad \forall z = 1, \ldots, Z, \tag{4.107b}$$

where f_0 and f_z, $\forall z = 1, \ldots, Z$, are all continuously differentiable functions over $\mathbb{R}^n$. Assume that the last $Z - m$ (for $m \leq Z$) constraints $f_{m+1}, \ldots, f_Z$ are convex over $\mathbb{R}^n$. In this case, the nonconvexity of the problem is due to the nonconvexity of the first m constraints $f_1, \ldots, f_m$. When $m = Z$, all the constraints are nonconvex. For the sake of simplicity, we only focus on (4.107) with inequality constraints since linear equality constraints do not significantly change the analysis.

Assume that for every f_z, $\forall z = 1, \ldots, m$ there exists a set $\mathbb{Y} \subset \mathbb{R}^n$ (for some positive integer n) and there is a convex upper estimate function $F_z : \mathbb{R}^n \times \mathbb{Y} \to \mathbb{R}$ such that

$$f_z(\mathbf{x}) \leq F_z(\mathbf{x}, \mathbf{y}_z^l), \quad \forall \mathbf{x} \in \mathbb{R}^n, \quad \forall \mathbf{y}_z^l \in \mathbb{Y}, \tag{4.108}$$

where, for a fixed $\mathbf{y}_z^l$, $\forall z = 1, \ldots, m$, $l = 1, 2, \ldots$, the function $F_z(\cdot, \mathbf{y}_z^l)$ is convex and continuously differentiable. The vector $\mathbf{y}_z^l$ is the parameter vector and $\mathbb{Y}$ is the admissible parameter set. The basic idea is that at each iteration, each $f_z(\mathbf{x})$ is replaced by its upper convex approximation $F_z(\mathbf{x}, \mathbf{y}_z^l)$, $\forall z = 1, \ldots, m$ for some $\mathbf{y}_z^l$ that satisfies inequality (4.108). Thus, in step $l \geq 1$, the following approximated convex problem is solved:

$$\min_{\mathbf{x}\in\mathbb{R}^n} \quad f_0(\mathbf{x}), \tag{4.109a}$$

$$\text{subject to} \begin{cases} F_z(\mathbf{x}, \mathbf{y}_z^l) \leq 0, & \forall i = 1, \ldots, m, \quad (4.109\text{b}) \\ f_z(\mathbf{x}) \leq 0, & \forall i = m+1, \ldots, Z. \quad (4.109\text{c}) \end{cases}$$

The vector $\mathbf{y}_z^l \in \mathbb{Y}$ is a fixed parameter vector that depends on the solution of the relaxed convex problem in step l. The convex upper estimate functions should ensure that for every $z = 1, \ldots, m$ there is a continuous function $\Psi_z : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that for any given $\mathbf{x} \in \mathbb{R}^n$, the vector $\mathbf{y}_z^l := \Psi_z(\mathbf{x}) \in \mathbb{Y}$ satisfies the following two equalities:

$$f_z(\mathbf{x}) = F_z(\mathbf{x}, \mathbf{y}_z^l), \tag{4.110a}$$

$$\nabla f_z(\mathbf{x}) = \nabla_{\mathbf{x}} F_z(\mathbf{x}, \mathbf{y}_z^l). \tag{4.110b}$$

SPCA Method

Step 0: Initialization. Choose an arbitrary and feasible solution $\mathbf{x}_0$ for problem (4.107), and set $\mathbf{y}_z^{l=1} = \Psi_z(\mathbf{x}_0)$.
For $l = 1, \ldots$ do
Step 1: Stop if either the KKT necessary optimality conditions are approximately satisfied or no improvement in the value of objective function $f_0(\cdot)$ is achieved.
Step 2: Obtain a solution $\mathbf{x}^l$ by solving convex problem (4.109).
Step 3: Set $\mathbf{y}_z^{l+1} = \Psi_z(\mathbf{x}^l), \forall z = 1, \ldots, m$ and $l = l + 1$, and go to Step 1.

In [51], it is shown that the SPCA method produces a sequence of feasible solutions for problem (4.107) whose objective function is monotonically nonincreasing and that the SPCA method is a descent scheme. Furthermore, the general SPCA method converges to a KKT point under certain conditions.

DC Programming and Sequence of Convex Approximation Consider the following nonlinear optimization problem [35]:

$$\min_{\mathbf{x}\in\mathbb{X}} \quad f_0(\mathbf{x}), \tag{4.111a}$$

$$\text{subject to} \quad f_1(\mathbf{x}) - f_2(\mathbf{x}) \leq 0, \tag{4.111b}$$

where $\mathbb{X} \subset \mathbb{R}^n$ is a nonempty convex compact set, and $f_i : \mathbb{R}^n \longrightarrow \mathbb{R},\ i = 0, 1, 2,$ are real-valued, continuously differentiable, and convex functions. Since $f_2(\mathbf{x})$ is a convex function, for any $\mathbf{y} \in \mathbb{X}$ we have

$$f_2(\mathbf{x}) \geq f_2(\mathbf{y}) + \nabla f_2(\mathbf{y})^{\mathrm{T}}(\mathbf{x} - \mathbf{y}), \tag{4.112}$$

which implies

$$f_1(\mathbf{x}) - f_2(\mathbf{x}) \leq f_1(\mathbf{x}) - \left[f_2(\mathbf{y}) + \nabla f_2(\mathbf{y})^{\mathrm{T}}(\mathbf{x} - \mathbf{y})\right]. \tag{4.113}$$

Using (4.113), a convex conservative approximation of (4.111) is obtained as

$$\min_{\mathbf{x} \in \mathbb{X}} \quad f_0(\mathbf{x}) \tag{4.114a}$$

$$\text{subject to} \quad f_1(\mathbf{x}) - \left[f_2(\mathbf{y}) + \nabla f_2(\mathbf{y})^{\mathrm{T}}(\mathbf{x} - \mathbf{y})\right] \leq 0. \tag{4.114b}$$

The following algorithm can be used to solve (4.111).

SCA Method

Step 0: Initialization. Choose an arbitrary and feasible solution $\mathbf{x}_0$ to problem (4.111), and set $\mathbf{y} = \mathbf{x}_0$ and $l = 0$.
Step 1: Stop if either the KKT necessary optimality conditions are approximately satisfied or no improvement in the objective value $f_0(\cdot)$ is achieved.
Step 2: Obtain a solution $\mathbf{x}^l$ by solving the convex problem (4.114).
Step 3: Set $l = l + 1$, $\mathbf{y} = \mathbf{x}^l$, and go to Step 1.

In [35], it is shown that the SCA method produces a sequence of feasible points for problem (4.111) whose objective function is monotonically nonincreasing. It is also shown that the SCA method is a descent scheme.

4.2.2 *Lagrangian Relaxation*

In the LR method, a lower bound on the optimal value of a nonconvex problem is obtained by solving the Lagrangian dual of the nonconvex problem [1, 14, 52].

4.2.2.1 Duality

The Lagrangian function of optimization problem (4.1) is

$$L(\mathbf{x}, \mathbf{w}, \mathbf{u}) = f_0(\mathbf{x}) + \sum_{z=1}^{Z} w_z f_z(\mathbf{x}) + \sum_{y=1}^{Y} u_y f_y(\mathbf{x}), \tag{4.115}$$

where $w_z \in \mathbb{R}_+$ is the Lagrange multiplier for inequality constraint (4.1b), and $u_y \in \mathbb{R}$ is the Lagrange multiplier for equality constraint (4.1c). The vectors $\mathbf{w}$ and $\mathbf{u}$ are also called the dual variables for problem (4.1). The dual function for problem (4.1) is

$$g(\mathbf{w}, \mathbf{u}) = \inf_{\mathbf{x}} L(\mathbf{x}, \mathbf{w}, \mathbf{u}). \tag{4.116}$$

The dual function gives the lower bound on the optimal value o^* for problem (4.1). In other words, for any $\mathbf{w} \succeq \mathbf{0}$ and any $\mathbf{u}$ we have

$$g(\mathbf{w}, \mathbf{u}) \leq o^*. \tag{4.117}$$

This property can be used to find the best lower bound by solving the following optimization problem:

$$\max_{\mathbf{w},\,\mathbf{u}} \quad g(\mathbf{w}, \mathbf{u}) \tag{4.118a}$$

$$\text{subject to} \quad \mathbf{w} \succeq \mathbf{0}. \tag{4.118b}$$

This problem is the dual of problem (4.1). The solution to the dual problem corresponds to the optimal values of dual variables $(\mathbf{w}^*, \mathbf{u}^*)$. The optimal solution of the dual problem is denoted by d^*, and from (4.117) we get

$$d^* \leq o^*, \tag{4.119}$$

which holds even if the original optimization problem is nonconvex. This feature of the LR method is also known as weak duality. When (4.119) holds with equality, that is, when $d^* = o^*$, strong duality holds. The difference $o^* - d^*$ is called the optimal duality gap of the original problem because it is the gap between the optimal value of the original optimization problem and the best lower bound obtained via the dual function [1]. Note that this gap is always nonnegative and can be zero when strong duality holds.

As an example, consider nonconvex problem (4.21) whose Lagrangian is

$$L(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}_0\mathbf{x} + \mathbf{q}_0^{\mathrm{T}}\mathbf{x} + r_0 + \sum_{i=1}^{m} \lambda_i\big(\mathbf{x}^{\mathrm{T}}\mathbf{A}_i\mathbf{x} + \mathbf{q}_i^{\mathrm{T}}\mathbf{x} + r_i\big) \tag{4.120a}$$

$$= \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} + \mathbf{q}\mathbf{x} + \mathbf{r}, \tag{4.120b}$$

where $\mathbf{A} = \mathbf{A}_0 + \sum_{i=1}^{m} \lambda_i\mathbf{A}_i$, $\mathbf{q} = \mathbf{q}_0^{\mathrm{T}} + \sum_{i=1}^{m} \lambda_i\mathbf{q}_i^{\mathrm{T}}$, and $\mathbf{r} = r_0 + \sum_{i=1}^{m} \lambda_i r_i$. The dual function of problem (4.21) is

$$g(\boldsymbol{\lambda}) = \inf_{\mathbf{x}} L(\mathbf{x}, \boldsymbol{\lambda}) \tag{4.121a}$$

$$= \begin{cases} \mathbf{r} - \frac{1}{4}\mathbf{q}^{\mathrm{T}}\mathbf{A}^{\mathrm{H}}\mathbf{q}, & \text{if } \mathbf{A} \succeq \mathbf{0} \text{ and } \mathbf{A} \in \mathscr{R}(\mathbf{A}), \\ -\infty, & \text{otherwise.} \end{cases} \tag{4.121b}$$

The dual problem is

$$\max_{\boldsymbol{\lambda}} \quad \mathbf{r} - \frac{1}{4}\mathbf{q}^{\mathrm{T}}\mathbf{A}^{\mathrm{H}}\mathbf{q}, \tag{4.122a}$$

$$\text{subject to} \begin{cases} \mathbf{A} \succeq \mathbf{0}, & (4.122b) \\ \boldsymbol{\lambda} \succeq \mathbf{0}. & (4.122c) \end{cases}$$

By introducing an auxiliary variable γ to rewrite optimization problem (4.122) in the EF, explained in Section 4.2.1.1 in this chapter, and utilizing the SC, explained in Section 4.2.1.3 in this chapter, we get the following SDP:

$$\max_{\boldsymbol{\lambda},\gamma} \quad \gamma, \tag{4.123a}$$

$$\text{subject to} \begin{cases} \begin{bmatrix} \mathbf{A} & \frac{1}{2}\mathbf{q} \\ \frac{1}{2}\mathbf{q}^{\mathrm{T}} & \mathbf{r} - \gamma \end{bmatrix} \succeq \mathbf{0}, & (4.123b) \\ \boldsymbol{\lambda} \succeq \mathbf{0}. & (4.123c) \end{cases}$$

Consequently, problem (4.123) can be solved using any existing solver, such as CVX.

4.2.2.2 Time-Sharing Condition

To reach a zero duality gap, the time-sharing condition introduced in [52] should be satisfied. If this condition holds, the duality gap of the optimization problem is always zero, regardless of the convexity of the optimization problem. In [52], it is shown that this condition holds for practical multiuser spectrum optimization problems in multichannel systems as the number of channels goes to infinity.

Theorem 4.5 (Zero Duality Gap [52]). *Consider the following nonconvex optimization problem:*

$$\max_{\mathbf{z}} \sum_{n=1}^{N} f_n(\mathbf{x}_n), \tag{4.124a}$$

$$\text{subject to} \sum_{n=1}^{N} h_n(\mathbf{x}_n) \leq \theta, \tag{4.124b}$$

(continued)

Theorem 4.5 (continued)
where $\mathbf{x}_n \in \mathbb{R}^U$, $\mathbf{z} = [\mathbf{x}_1, \ldots, \mathbf{x}_N]$, $f_n : \mathbb{R}^U \to \mathbb{R}$, *and* $h_n : \mathbb{R}^U \to \mathbb{R}$. *When optimization problem* (4.124) *is convex, that is, when* $f_n(\mathbf{x}_n), \forall n$, *is concave and* $h_n(\mathbf{x}_n), \forall n$, *is convex, the duality gap is zero. When the preceding problem is not convex, the dual method gives an upper bound and the duality gap is not zero. Let* $\mathbf{x}_n^*$ *and* $\mathbf{y}_n^*$ *denote the optimal solutions to* (4.124) *when* $\boldsymbol{\theta} = \boldsymbol{\theta}_x$ *and* $\boldsymbol{\theta} = \boldsymbol{\theta}_y$, *respectively. When there is a feasible solution* z_n *that satisfies the following two conditions (called the time-sharing conditions),*

- $\sum_{n=1}^{N} h_n(z_n) \leq \nu\boldsymbol{\theta}_x + (1-\nu)\boldsymbol{\theta}_y$,
- $\sum_{n=1}^{N} f_n(z_n) \geq \nu f_n(x_n^*) + (1-\nu) f_n(y_n^*)$,

the duality gap is zero, where $0 \leq \nu \leq 1$.

The time-sharing condition implies that the maximum value of the optimization problem is a concave function of θ. When problem (4.124) is convex, the time-sharing condition is always satisfied. However, the converse is not necessarily true. The time-sharing conditions can be held even when problem (4.124) is not convex.

4.3 Application of Relaxation Methods for Robust Resource Allocation

In this section, a number of applications for relaxation methods in robust optimization problems in communication are reviewed. We will mainly focus on the following three types of nonconvex problem: (1) partial CSI feedback with bounded uncertainty, (2) partial CSI feedback with stochastic uncertainty (not bounded), and (3) no CSI feedback. As was shown in [53], the solutions to many optimization problems suffer from sensitivity to uncertainty in side information, and even minor uncertainty can make the solutions suboptimal. In the literature, the relaxation methods applied to partial CSI feedback with bounded uncertainty are as follows:

- Time sharing
- Lagrangian relaxation
- Semidefinite relaxation
- Charnes–Cooper transformation
- S-procedure
- Schur complement
- Triangle inequality
- Cauchy–Schwarz inequality
- Epigraph form
- Determinant inequality
- Norm approximation
- Nonlinear fractional programming

The relaxation methods applied to partial CSI feedback with stochastic uncertainty are as follows:

- Semidefinite relaxation
- Bernstein-type inequality for complex Gaussian quadratic forms
- Large deviation inequality for complex Gaussian quadratic forms
- S-procedure
- Schur complement
- Lagrangian relaxation
- Vysochanskii–Petunin inequality
- Norm approximation
- Cauchy–Schwarz inequality
- Markov's inequality
- DC programming and sequence of convex approximation
- Epigraph form
- Conditional value at risk
- DC approximations
- Bernstein approximation

The relaxation methods applied for no CSI feedback are as follows:

- DC approximations
- Semidefinite relaxation
- DC programming and sequence of convex approximations
- Bernstein-type inequality for complex Gaussian quadratic forms

4.3.1 Partial CSI Feedback: Bounded Uncertainty

4.3.1.1 Robust Secure Transmission

As was mentioned in Chapter 1, physical-layer security is a challenging problem involving uncertain parameters and belongs to the set of intractable nonconvex optimization problems. In what follows, we will discuss several examples of such problems and demonstrate how the relaxation methods presented in this chapter can be applied to solve them.

Example 1. A multi-input/single-output (MISO) channel overheard by a set of $\mathscr{E} = \{1, \ldots, E\}$ multi-antenna eavesdroppers is considered in [5]. By assuming uncertainty in CSI of legitimate users and eavesdroppers, we will study the following two optimization problems:

Problem 1. Minimizing the transmit power of legitimate users subject to the minimum required secrecy rate and

Problem 2. Maximizing the secrecy rate subject to the maximum transmit power constraint via the worst case approach.

Let $\mathbf{h} \in \mathbb{C}^{U_t}$ be the channel gain between the legitimate transmitter and its receiver, where U_t is the number of antennas of the authorized transmitter, and $\mathbf{G}_e \in \mathbb{C}^{U_t \times U_e}$ be the channel gain between the legitimate transmitter and eavesdropper e, where U_e is the number of antennas of eavesdropper e. The uncertainty regions for CSI values are

$$\mathscr{R}_{\mathbf{h}} = \{\mathbf{h} \in \mathbb{C}^{U_t} | \|\hat{\mathbf{h}}\| \leq \epsilon_{\mathbf{h}}\}, \tag{4.125}$$

$$\mathscr{R}_{\mathbf{G}_e} = \{\mathbf{G}_e \in \mathbb{C}^{U_t \times U_e} | \|\hat{\mathbf{G}}_e\|_F \leq \epsilon_{\mathbf{G}_e}\}, \tag{4.126}$$

where $\hat{\mathbf{h}} = \mathbf{h} - \overline{\mathbf{h}}$, $\hat{\mathbf{G}}_e = \mathbf{G}_e - \overline{\mathbf{G}}_e$, and $\epsilon_{\mathbf{h}}, \epsilon_{\mathbf{G}_1}, \ldots, \epsilon_{\mathbf{G}_E}$ are known constants. The worst-case secrecy rate is

$$\phi(\mathbf{W}) = \min_{e=1,\ldots,E} \psi_e(\mathbf{W}), \tag{4.127}$$

where $\psi_k(\mathbf{W}) = \min_{\mathbf{h} \in \mathscr{R}_{\mathbf{h}}, \mathbf{G}_e \in \mathscr{R}_{\mathbf{G}_e}} \log(1 + \mathbf{h}^{\mathrm{H}}\mathbf{W}\mathbf{h}) - \log\det(\mathbf{I} + \mathbf{G}_e^{\mathrm{H}}\mathbf{W}\mathbf{G}_e)$, and $\mathbf{W}$ is the covariance matrix of the signal transmitted by the legitimate source $\mathbf{x}$, which is $\mathbf{W} = \mathbb{E}\{\mathbf{x}\mathbf{x}^{\mathrm{H}}\}$. Noise variance is assumed to be unity.

The robust optimization for the preceding **problem 1** is

$$\min_{\mathbf{W} \succeq \mathbf{0}} \quad \mathrm{tr}(\mathbf{W}) \tag{4.128a}$$

$$\text{subject to} \quad \phi(\mathbf{W}) \geq R, \tag{4.128b}$$

where R is the minimum required secrecy rate for the authorized user. By some mathematical manipulation of (4.128b), problem (4.128) is rewritten as

$$\min_{\mathbf{W} \succeq \mathbf{0}} \quad \mathrm{tr}(\mathbf{W}), \tag{4.129a}$$

$$\text{subject to} \quad \frac{\max_{\mathbf{G}_e \in \mathscr{R}_{\mathbf{G}_e}} \det(\mathbf{I} + \mathbf{G}_e^{\mathrm{H}}\mathbf{W}\mathbf{G}_e)}{\min_{\mathbf{h} \in \mathscr{R}_{\mathbf{h}}} \ 1 + \mathbf{h}^{\mathrm{H}}\mathbf{W}\mathbf{h}} \leq 2^{-R}, \quad \forall e = 1, \ldots, E. \tag{4.129b}$$

By applying the DI in the BT in Section 4.2.1.6 of this chapter, an approximation to (4.129) is obtained as

$$\min_{\mathbf{W} \succeq \mathbf{0}} \quad \mathrm{tr}(\mathbf{W}), \tag{4.130a}$$

$$\text{subject to} \quad \frac{\max_{\mathbf{G}_e \in \mathscr{R}_{\mathbf{G}_e}} \ 1 + \mathrm{tr}(\mathbf{G}_e^{\mathrm{H}}\mathbf{W}\mathbf{G}_e)}{\min_{\mathbf{h} \in \mathscr{R}_{\mathbf{h}}} \ 1 + \mathbf{h}^{\mathrm{H}}\mathbf{W}\mathbf{h}} \leq 2^{-R}, \quad \forall e = 1, \ldots, E. \tag{4.130b}$$

Problem (4.130) can be formulated as an SDP by decoupling the fractional constraint into two linear constraints as

$$\min_{\mathbf{W} \succeq \mathbf{0}, \theta} \quad \mathrm{tr}(\mathbf{W}), \tag{4.131a}$$

$$\text{subject to} \begin{cases} \min\limits_{\mathbf{h}\in\mathscr{R}_{\mathbf{h}}} \ 1+\mathbf{h}^{\mathrm{H}}\mathbf{W}\mathbf{h} \geq \theta, & (4.131\text{b}) \\ \max\limits_{\mathbf{G}_e\in\mathscr{R}_e} \ 1+\operatorname{tr}(\mathbf{G}_e^{\mathrm{H}}\mathbf{W}\mathbf{G}_e) \leq 2^{-R}\theta, \quad \forall e=1,\ldots,E, & (4.131\text{c}) \end{cases}$$

where $\theta > 0$ is the slack variable. With the transformation of (4.131b) and (4.131c) into LMIs, the SP in Section 4.2.1.4 can be used. When this is done for (4.131b), we let $\mathbf{h} = \bar{\mathbf{h}} + \hat{\mathbf{h}}$ and rewrite (4.131b) as

$$\hat{\mathbf{h}}^{\mathrm{H}}\hat{\mathbf{h}} \leq \epsilon_{\mathbf{h}}^2 \quad \Longrightarrow \quad \hat{\mathbf{h}}^{\mathrm{H}}\mathbf{W}\hat{\mathbf{h}} + 2\mathrm{Re}\{\bar{\mathbf{h}}^{\mathrm{H}}\mathbf{W}\hat{\mathbf{h}}\} + \bar{\mathbf{h}}^{\mathrm{H}}\mathbf{W}\bar{\mathbf{h}} + 1 - \theta \geq 0. \tag{4.132}$$

By the use of the SP, (4.132) is transformed into an LMI as

$$\mathbf{T}_{\mathbf{h}}(\mathbf{W},\mu_{\mathbf{h}},\theta) = \begin{bmatrix} \mu_{\mathbf{h}}\mathbf{I}_{U_t} + \mathbf{W} & \mathbf{W}\bar{\mathbf{h}} \\ \bar{\mathbf{h}}^{\mathrm{H}}\mathbf{W} & -\mu_{\mathbf{h}}\epsilon_{\mathbf{h}}^2 - \theta + \bar{\mathbf{h}}^{\mathrm{H}}\mathbf{W}\bar{\mathbf{h}} + 1 \end{bmatrix} \succeq \mathbf{0}, \tag{4.133}$$

where $\mu_{\mathbf{h}} \geq 0$. Similarly, (4.131c) is rewritten

$$\hat{\mathbf{g}}_e^{\mathrm{H}}\hat{\mathbf{g}}_e \leq \epsilon_{\mathbf{G}_e}^2 \quad \Longrightarrow \quad \hat{\mathbf{g}}_e^{\mathrm{H}}\mathbf{W}'_e\hat{\mathbf{g}}_e + 2\mathrm{Re}\{\bar{\mathbf{g}}_e^{\mathrm{H}}\mathbf{W}'_e\hat{\mathbf{g}}_e\} + \bar{\mathbf{g}}_e^{\mathrm{H}}\mathbf{W}'_e\bar{\mathbf{g}}_e + 1 - 2^{-R}\theta \leq 0, \tag{4.134}$$

where $\mathbf{W}'_e = \mathbf{I}_{U_e} \otimes \mathbf{W}$ and $\bar{\mathbf{g}}_e = \operatorname{vec}(\overline{\mathbf{G}}_e)$. Using the SP, (4.134) is converted into an LMI,

$$\mathbf{T}_e(\mathbf{W},\lambda_e,\theta) = \begin{bmatrix} \lambda_e\mathbf{I}_{U_eU_t} - \mathbf{W}'_e & -\mathbf{W}'_e\bar{\mathbf{g}}_e \\ -\bar{\mathbf{g}}_e^{\mathrm{H}}\mathbf{W}'_e & -\lambda_e\epsilon_{\mathbf{G}_e}^2 + \theta 2^{-R} - \bar{\mathbf{g}}_e^{\mathrm{H}}\mathbf{W}'_e\bar{\mathbf{g}}_e - 1 \end{bmatrix} \succeq \mathbf{0}, \tag{4.135}$$

where $\lambda_e \geq 0, \quad \forall e = 1,\ldots,E.$

Replacing (4.131b) and (4.131c) with (4.133) and (4.135), respectively, problem (4.131) is transformed into SDP form as

$$\min_{\mathbf{W}\succeq\mathbf{0},\,\theta>0,\,\mu_{\mathbf{h}}\geq 0,\,-_{\mathbf{e}}\geq 0} \quad \operatorname{tr}(\mathbf{W}), \tag{4.136a}$$

$$\text{subject to} \begin{cases} \mathbf{T}_{\mathbf{h}}(\mathbf{W},\mu_{\mathbf{h}},\theta) \succeq \mathbf{0}, & (4.136\text{b}) \\ \mathbf{T}_e(\mathbf{W},\lambda_e,\theta) \succeq \mathbf{0}, \qquad \forall e = 1,\ldots,E. & (4.136\text{c}) \end{cases}$$

The preceding problem can be solved via any existing SDP solver.

From Lemma 4.3, the relaxed convex problem (4.136) is tight when the rank of the optimal solution to problem (4.136) is 1.

Proposition 4.1. *When the worst-case secrecy rate is positive and problem (4.136) is feasible, the optimal solution to problem (4.136) must be of rank 1 and unique.*

Proof. The proof is based on an examination of the KKT conditions for problem (4.136).

Using Proposition 4.1 and Lemma 4.3, we get the following corollary.

Corollary 4.1. *If the worst-case secrecy rate is positive and problem (4.128) is feasible, the optimal solutions to problems (4.136) and (4.128) are equivalent to one another. In addition, the optimal solution to problem (4.128) is unique and of rank 1 (Appendix 1).*

Next, we consider **Problem 2**, which is

$$\max_{\mathbf{W}\succeq\mathbf{0}} \quad \phi(\mathbf{W}), \tag{4.137a}$$

$$\text{subject to} \quad \operatorname{tr}(\mathbf{W}) \le p^{\max}, \tag{4.137b}$$

where $p^{\max}$ is the maximum transmit power of the legitimate user. **Problem 2** can be rewritten as

$$\gamma^* = \min_{\mathbf{W}\succeq\mathbf{0}} \quad \max_{e=1,\ldots,E} \frac{\max_{\mathbf{G}_e\in\mathscr{R}_{\mathbf{G}_e}} \det(\mathbf{I} + \mathbf{G}_e^{\mathrm{H}}\mathbf{W}\mathbf{G}_e)}{\min_{\mathbf{h}\in\mathscr{R}_{\mathbf{h}}} \quad 1 + \mathbf{h}^{\mathrm{H}}\mathbf{W}\mathbf{h}}, \tag{4.138a}$$

$$\text{subject to} \quad \operatorname{tr}(\mathbf{W}) \le p^{\max}, \tag{4.138b}$$

where $0 < \gamma^* \le 1$. Using the BT-DI in Section 4.2.1.6 of this chapter, problem (4.138) is converted into

$$\gamma^*_{\text{relax}} = \min_{\mathbf{W}\succeq\mathbf{0}} \quad \max_{e=1,\ldots,E} \frac{\max_{\mathbf{G}_e\in\mathscr{R}_e} 1 + \operatorname{tr}(\mathbf{G}_e^{\mathrm{H}}\mathbf{W}\mathbf{G}_e)}{\min_{\mathbf{h}\in\mathscr{R}_{\mathbf{h}}} \quad 1 + \mathbf{h}^{\mathrm{H}}\mathbf{W}\mathbf{h}}, \tag{4.139a}$$

$$\text{subject to} \quad \operatorname{tr}(\mathbf{W}) \le p^{\max}. \tag{4.139b}$$

Using the epigraph method and CCT in Sections 4.2.1.1 and 4.2.1.2 in this chapter, respectively, the SDP transformation of (4.139) is

$$\min_{\mathbf{Z}\succeq\mathbf{0},\,\zeta>0} \quad \tau, \tag{4.140a}$$

$$\text{subject to}\begin{cases}\max\limits_{\mathbf{G}_e\in\mathscr{R}_{\mathbf{G}_e}} \zeta+\mathrm{tr}(\mathbf{G}_e\mathbf{G}_e^{\mathrm{H}}\mathbf{Z})\le\tau, \quad \forall e=1,\ldots,E, & (4.140\mathrm{b})\\ \mathrm{tr}(\mathbf{Z})\le\zeta P^{\max}, & (4.140\mathrm{c})\\ \min\limits_{\mathbf{h}\in\mathscr{R}_{\mathbf{h}}} \zeta+\mathbf{h}\mathbf{h}^{\mathrm{H}}\mathbf{Z}=1, & (4.140\mathrm{d})\end{cases}$$

where $\mathbf{W}=\frac{\mathbf{Z}}{\zeta}$. Using the SP, the semi-infinite constraints are transformed into LMI constraints, and we write

$$\min_{\mathbf{Z}\succeq\mathbf{0},\,\zeta>0,\,\mu_{\mathbf{h}}\ge0,\,\boldsymbol{\lambda}\succeq\mathbf{0}} \tau, \tag{4.141a}$$

$$\text{subject to}\begin{cases}\mathbf{M}_b(\mathbf{Z},\mu_{\mathbf{h}},\zeta)\succeq\mathbf{0}, & (4.141\mathrm{b})\\ \mathbf{M}_e(\mathbf{Z},\lambda_e,\zeta,\tau)\succeq\mathbf{0}, \quad \forall e=1,\ldots,E, & (4.141\mathrm{c})\\ \mathrm{tr}(\mathbf{Z})\le\zeta p^{\max}, & (4.141\mathrm{d})\end{cases}$$

where

$$\mathbf{M}_b(\mathbf{Z},\mu_{\mathbf{h}},\zeta)=\begin{bmatrix}\mu_{\mathbf{h}}\mathbf{I}_{U_t}+\mathbf{Z} & \mathbf{Z}\overline{\mathbf{h}}\\ \overline{\mathbf{h}}^{\mathrm{H}}\mathbf{Z} & -\mu_{\mathbf{h}}\epsilon_{\mathbf{h}}^2+\zeta+\overline{\mathbf{h}}^{\mathrm{H}}\mathbf{Z}\overline{\mathbf{h}}-1\end{bmatrix}, \tag{4.142}$$

$$\mathbf{M}_e(\mathbf{Z},\lambda_e,\zeta,\tau)=\begin{bmatrix}\lambda_e\mathbf{I}_{U_eU_t}-\mathbf{Z}_e' & -\mathbf{Z}_e'\overline{\mathbf{g}}_e\\ -\overline{\mathbf{g}}_e^{\mathrm{H}}\mathbf{Z}_e' & -\lambda_e\epsilon_{\mathbf{G}_e}^2-\zeta-\overline{\mathbf{g}}_k^{\mathrm{H}}\mathbf{Z}_e'\overline{\mathbf{g}}_e+\tau\end{bmatrix}, \tag{4.143}$$

and $\mathbf{Z}'=\mathbf{I}_{U_e}\bigotimes\mathbf{Z}$. Problem (4.141) can be solved using any existing problem solver. Since the DI relaxation method is used, the tightness of the relaxed problem can be proved when the solution of the relaxed problem is of rank 1 (Appendix 2).

Example 2 Another important nonconvex problem arises when friendly jammers are introduced. In [6], the system includes a legitimate transmitter, a friendly jammer, a legitimate receiver, and an eavesdropper. The legitimate transmitter and the friendly jammer have U_t and U_j antennas, respectively. Let $\mathbf{h}_{nn}$ and $\mathbf{g}_{ne}$ be the $1\times U_t$ channel vector gain between the legitimate transmitter and its receiver and that between the legitimate transmitter and the eavesdropper, respectively. Also, let $\mathbf{g}_{jn}$ and $\mathbf{g}_{je}$ be the $1\times U_j$ channel vector gain between the friendly jammer and the legitimate receiver and that between the friendly jammer and the eavesdropper, respectively. In this setup, noise at the legitimate receiver and the eavesdropper are assumed to be zero-mean circular complex Gaussian variables with variance σ_d^2 and σ_e^2, respectively, where $\sigma_d^2=\sigma_e^2=\sigma^2$ for notational simplicity.

We assume that only the uncertain values of channel gains between the eavesdropper and other nodes are available. In particular, the legitimate transmitter has an uncertain value of $\mathbf{g}_{ne}$ only, and the error is $\hat{\mathbf{g}}_{ne}=\mathbf{g}_{ne}-\overline{\mathbf{g}}_{ne}$. In addition, the jammer has an uncertain value of $\mathbf{g}_{je}$ only, and the error is $\hat{\mathbf{g}}_{je} = \mathbf{g}_{je} - \overline{\mathbf{g}}_{je}$. In the worst-case robust optimization, the uncertainty regions are

$$\mathscr{R}_{\mathbf{g}_{ne}} = \{\hat{\mathbf{g}}_{ne} : \|\hat{\mathbf{g}}_{ne}\|^2 \leq \varepsilon^2_{\mathbf{g}_{ne}}\},$$

$$\mathscr{R}_{\mathbf{g}_{je}} = \{\hat{\mathbf{g}}_{je} : \|\hat{\mathbf{g}}_{je}\|^2 \leq \varepsilon^2_{\mathbf{g}_{je}}\},$$

where $\varepsilon^2_{\mathbf{g}_{se}}$ and $\varepsilon^2_{\mathbf{g}_{je}}$ are the bounds on the respective uncertainty regions.

In this setup, the worst-case secrecy rate is

$$R_s = \log_2\left(1 + \frac{\mathbf{h}_{nn}\mathbf{Q}_s\mathbf{h}_{nn}^{\mathrm{H}}}{\sigma^2 + \mathbf{g}_{jn}\mathbf{Q}_j\mathbf{g}_{jn}^{\mathrm{H}}}\right) - \log_2\left(1 + \frac{\varXi(\mathbf{Q}_s, \hat{\mathbf{g}}_{ne})}{\sigma^2 + \varXi(\mathbf{Q}_j, \hat{\mathbf{g}}_{je})}\right), \tag{4.144}$$

where $\varXi(\mathbf{Q}_s, \hat{\mathbf{g}}_{ne}) = (\overline{\mathbf{g}}_{ne}+\hat{\mathbf{g}}_{ne})\mathbf{Q}_s(\overline{\mathbf{g}}_{ne}+\hat{\mathbf{g}}_{ne})^{\mathrm{H}}$, $\varXi(\mathbf{Q}_j, \hat{\mathbf{g}}_{je}) = (\overline{\mathbf{g}}_{je}+\hat{\mathbf{g}}_{je})\mathbf{Q}_j(\overline{\mathbf{g}}_{je}+\hat{\mathbf{g}}_{je})^{\mathrm{H}}$, $\mathbf{Q}_s$ is the covariance matrix of the signal transmitted by the legitimate transmitter $\mathbf{x}_s$, given by $\mathbf{Q}_s = \mathbb{E}\{\mathbf{x}_s\mathbf{x}_s^{\mathrm{H}}\}$, and the power constraint is such that $\mathbf{Q}_s \in \mathscr{Q}_s = \{\mathbf{Q}_s : \mathbf{Q}_s \succeq 0, \mathrm{tr}(\mathbf{Q}_s) \leq p_s^{\max}\}$, where $p_s^{\max}$ is the maximum allowable transmit power by the legitimate transmitter. The covariance matrix of the signal transmitted by jammer $\mathbf{x}_j$ is $\mathbf{Q}_j$, given by $\mathbf{Q}_j = \mathbb{E}\{\mathbf{x}_j\mathbf{x}_j^{\mathrm{H}}\}$, and the power constraint is such that $\mathbf{Q}_j \in \mathscr{Q}_j = \{\mathbf{Q}_j : \mathbf{Q}_j \succeq 0, \mathrm{tr}(\mathbf{Q}_j) \leq p_j^{\max}\}$, where $p_j^{\max}$ is the maximum transmit power of the jammer.

The objective is to maximize the secrecy rate via the worst-case optimization theory. The resource allocation problem is

$$\max_{\mathbf{Q}_s\in\mathscr{Q}_s,\, \mathbf{Q}_j\in\mathscr{Q}_j} \quad \min_{\hat{\mathbf{g}}_{ne}\in\mathscr{R}_{\mathbf{g}_{ne}},\, \hat{\mathbf{g}}_{je}\in\mathscr{R}_{\mathbf{g}_{je}}} \quad R_s, \tag{4.145}$$

which is nonconvex. To develop a tractable algorithm for solving (4.145), a zero-forcing (ZF) constraint on the jamming signal is considered,

$$\mathbf{g}_{jn}\mathbf{Q}_j\mathbf{g}_{jn}^{\mathrm{H}} = 0,$$

which converts the nonconvex problem into a convex one, for which the global optimum is guaranteed. With the ZF constraint, maximization of R_s over $\mathbf{Q}_j$ does not depend on $\mathbf{Q}_s$, although the optimal $\mathbf{Q}_j$ still depends on $\mathbf{Q}_s$. The optimization process is decoupled into two convex problems, in which $\mathbf{Q}_j$ is calculated first, followed by $\mathbf{Q}_s$ calculation.[5] The following steps are taken for solving (4.145) with ZF constraints:

[5] Quantifying the impact of the ZF constraint remains an open problem [6].

- **Step 1**: The optimal $\mathbf{Q}_j$ in (4.144) is obtained via

$$\max_{\mathbf{Q}_j \in \mathscr{Q}_j} \min_{\hat{\mathbf{g}}_{je} \in \mathscr{R}_{\mathbf{g}_{je}}} (\overline{\mathbf{g}}_{je} + \hat{\mathbf{g}}_{je})\mathbf{Q}_j(\overline{\mathbf{g}}_{je} + \hat{\mathbf{g}}_{je})^{\mathrm{H}}, \tag{4.146a}$$

$$\text{subject to} \quad \mathbf{g}_{jn}\mathbf{Q}_j\mathbf{g}_{jn}^{\mathrm{H}} = 0, \tag{4.146b}$$

where the maximin problem in (4.146a) can be transformed into

$$\max_{\mathbf{Q}_j \in \mathscr{Q}_j} \quad \nu, \tag{4.147a}$$

$$\text{subject to} \begin{cases} \|\hat{\mathbf{g}}_{je}\|^2 \le \varepsilon_{\mathbf{g}_{je}}^2, & (4.147\text{b}) \\ (\overline{\mathbf{g}}_{je} + \hat{\mathbf{g}}_{je})\mathbf{Q}_j(\overline{\mathbf{g}}_{je} + \hat{\mathbf{g}}_{je})^{\mathrm{H}} \ge \nu. & (4.147\text{c}) \end{cases}$$

Constraints (4.147b) and (4.147c) can be stated by

$$\hat{\mathbf{g}}_{je}\mathbf{Q}_j\hat{\mathbf{g}}_{je}^{\mathrm{H}} + 2\mathrm{Re}\{\overline{\mathbf{g}}_{je}\mathbf{Q}_j\hat{\mathbf{g}}_{je}^{\mathrm{H}}\} + \overline{\mathbf{g}}_{je}\mathbf{Q}_j\overline{\mathbf{g}}_{je}^{\mathrm{H}} - \nu \ge 0, \tag{4.148a}$$

$$\forall \hat{\mathbf{g}}_{je} : -\hat{\mathbf{g}}_{je}\hat{\mathbf{g}}_{je}^{\mathrm{H}} + \varepsilon_{\mathbf{g}_{je}}^2 \ge 0. \tag{4.148b}$$

From the SP, (4.148) holds if and only if there exists a $\mu \ge 0$ such that

$$\begin{bmatrix} \mu\mathbf{I}_{N_t} + \mathbf{Q}_j & \mathbf{Q}_j\hat{\mathbf{g}}_{je}^{\mathrm{H}} \\ \overline{\mathbf{g}}_{je}\mathbf{Q}_j & -\mu\epsilon_{\mathbf{g}_{je}}^2 - \nu + \overline{\mathbf{g}}_{je}\mathbf{Q}_j\overline{\mathbf{g}}_{je}^{\mathrm{H}} \end{bmatrix} \succeq \mathbf{0}. \tag{4.149}$$

Using $\psi = -\mu\epsilon_{\mathbf{g}_{je}}^2 - \nu + \overline{\mathbf{g}}_{je}\mathbf{Q}_j\overline{\mathbf{g}}_{je}^{\mathrm{H}}$, where $\psi \ge 0$, we transform problem (4.146) into

$$\max_{\mathbf{Q}_j \in \mathscr{Q}_j,\, \mu \ge 0,\, \psi \ge 0} \quad -\mu\epsilon_{\mathbf{g}_{je}}^2 - \psi + \overline{\mathbf{g}}_{je}\mathbf{Q}_j\overline{\mathbf{g}}_{je}^{\mathrm{H}}, \tag{4.150a}$$

$$\text{subject to} \begin{cases} \begin{bmatrix} \mu\mathbf{I}_{N_t} + \mathbf{Q}_j & \mathbf{Q}_j\overline{\mathbf{g}}_{je}^{\mathrm{H}} \\ \overline{\mathbf{g}}_{je}\mathbf{Q}_j & \psi \end{bmatrix} \succeq \mathbf{0}, & (4.150\text{b}) \\ \mathbf{g}_{jn}\mathbf{Q}_j\mathbf{g}_{jn}^{\mathrm{H}} = 0. & (4.150\text{c}) \end{cases}$$

Problem (4.150) is an SDP and its optimal solution $\mathbf{Q}_j^*$ can be efficiently obtained.

- **Step 2**: The optimal robust covariance $\mathbf{Q}_j^*$ depends on the "hidden" worst-case $\hat{\mathbf{g}}_{je}^*$, which can be explicitly stated by the following problem:

$$\min_{\hat{\mathbf{g}}_{je}} \quad (\overline{\mathbf{g}}_{je} + \hat{\mathbf{g}}_{je})\mathbf{Q}_j^*(\overline{\mathbf{g}}_{je} + \mathbf{e}_{\mathbf{g}_{je}})^{\mathrm{H}}, \tag{4.151a}$$

$$\text{subject to} \quad \|\hat{\mathbf{g}}_{je}\|^2 \le \varepsilon_{\mathbf{g}_{je}}^2. \tag{4.151b}$$

Since this problem is convex, strong duality holds for (4.151) and for its dual. Thus, via LR and SC, explained respectively in Sections 4.2.2 and 4.2.1.3 in this chapter, the worst channel mismatch for problem (4.151) is $\hat{\mathbf{g}}_{je}^* = -\overline{\mathbf{g}}_{je}\mathbf{Q}_j^*(\lambda\mathbf{I} + \mathbf{Q}_j^*)$, where λ is the solution to the following SDP problem:

$$\max_{\lambda \geq 0} \quad \gamma, \tag{4.152a}$$

$$\text{subject to} \quad \begin{bmatrix} \mu\mathbf{I}_{U_t} + \mathbf{Q}_j^* & \mathbf{Q}_j^*\overline{\mathbf{g}}_{je}^{\mathrm{H}} \\ \overline{\mathbf{g}}_{je}\mathbf{Q}_j^* & -\lambda\epsilon_{\mathbf{g}_{je}}^2 - \gamma + \overline{\mathbf{g}}_{je}\mathbf{Q}_j^*\overline{\mathbf{g}}_{je}^{\mathrm{H}} \end{bmatrix} \succeq \mathbf{0}. \tag{4.152b}$$

- **Step 3**: Using $\mathbf{Q}_j^*$ and $\hat{\mathbf{g}}_{je}^*$, maximization of the secrecy rate over $\mathbf{Q}_s$ for the worst channel mismatch $\hat{\mathbf{g}}_{ne}$ in the bounded set $\mathscr{R}_{\mathbf{g}_{ne}}$ is equivalent to

$$\max_{\mathbf{Q}_s \in \mathscr{Q}_s} \min_{\hat{\mathbf{g}}_{ne} \in \mathscr{R}_{\mathbf{g}_{ne}}} \frac{\sigma^2 + \mathbf{h}_{nn}\mathbf{Q}_s\mathbf{h}_{nn}^{\mathrm{H}}}{\sigma^2 + (\overline{\mathbf{g}}_{ne} + \hat{\mathbf{g}}_{ne})\mathbf{Q}_s(\overline{\mathbf{g}}_{ne} + \hat{\mathbf{g}}_{ne})^{\mathrm{H}} + (\overline{\mathbf{g}}_{je} + \hat{\mathbf{g}}_{je}^*)\mathbf{Q}_j^*(\overline{\mathbf{g}}_{je} + \hat{\mathbf{g}}_{je}^*)^{\mathrm{H}}}. \tag{4.153}$$

This problem is nonconvex, and the difficulty of solving it results from the inner minimization. To solve this problem, we use the SP to transform it into a solvable quasi-convex optimization problem and get

$$\min_{\mu \geq 0,\ \psi \geq 0,\ \mathbf{Q}_s \in \mathscr{Q}_s} \frac{\sigma^2 + \mu\epsilon_{\mathbf{g}_{je}}^2 + \psi + \mathrm{tr}(\mathbf{Q}_s\overline{\mathbf{g}}_{ne}^{\mathrm{H}}\overline{\mathbf{g}}_{ne}) + (\overline{\mathbf{g}}_{je} + \hat{\mathbf{g}}_{je}^*)\mathbf{Q}_j^*(\overline{\mathbf{g}}_{je} + \hat{\mathbf{g}}_{je}^*)^{\mathrm{H}}}{\sigma^2 + \mathrm{tr}(\mathbf{Q}_s\overline{\mathbf{h}}_{nn}^{\mathrm{H}}\overline{\mathbf{h}}_{nn})}, \tag{4.154a}$$

$$\text{subject to} \quad \begin{bmatrix} \mu\mathbf{I} - \mathbf{Q}_s & -\mathbf{Q}_s\overline{\mathbf{g}}_{ne}^{\mathrm{H}} \\ -\overline{\mathbf{g}}_{ne}\mathbf{Q}_s & \psi \end{bmatrix} \succeq \mathbf{0}. \tag{4.154b}$$

To transform this problem into an efficiently solvable SDP form, one can use CCT by letting $\mu = \frac{\mu'}{\gamma}$, $\psi = \frac{\psi'}{\gamma}$, and $\mathbf{Q}_s = \frac{\mathbf{Q}_s'}{\gamma}$ for some $\gamma > 0$ and rewrite problem (4.154) as

$$\min_{\mu' \geq 0,\ \psi' \geq 0,\ \mathbf{Q}_s' \in \mathscr{Q}_s',\ \gamma > 0} \quad t, \tag{4.155a}$$

$$\text{subject to} \begin{cases} \begin{bmatrix} \mu'\mathbf{I} - \mathbf{Q}_s' & -\mathbf{Q}_s'\overline{\mathbf{g}}_{ne}^{\mathrm{H}} \\ -\overline{\mathbf{g}}_{ne}\mathbf{Q}_s' & \psi' \end{bmatrix} \succeq \mathbf{0}, & (4.155b) \\ \sigma^2\gamma + \gamma(\overline{\mathbf{g}}_{je} + \hat{\mathbf{g}}_{je}^*)\mathbf{Q}_j^*(\overline{\mathbf{g}}_{je} + \hat{\mathbf{g}}_{je}^*)^{\mathrm{H}} + \mu'\epsilon_{\mathbf{g}_{je}}^2 + \mathrm{tr}(\mathbf{Q}_s'\overline{\mathbf{g}}_{ne}^{\mathrm{H}}\overline{\mathbf{g}}_{ne}) + \psi' \leq t, & (4.155c) \\ \sigma^2\gamma + \mathrm{tr}(\mathbf{Q}_s'\overline{\mathbf{h}}_{nn}^{\mathrm{H}}\overline{\mathbf{h}}_{nn}) = 1, & (4.155d) \end{cases}$$

where $\mathscr{Q}_s' = \{\mathbf{Q}_s' : \mathbf{Q}_s' \succeq 0, \mathrm{tr}(\mathbf{Q}_s') \leq \gamma p_s^{\max}\}$.

- **Step 4**: Although $\hat{\mathbf{g}}_{ne}$ does not explicitly appear in (4.155), the optimal robust covariance $\mathbf{Q}_s^*$ depends on the “hidden” worst-case $\hat{\mathbf{g}}_{ne}^*$, which is explicitly stated in the following problem:

$$\max_{\hat{\mathbf{g}}_{ne}} \quad (\overline{\mathbf{g}}_{ne} + \hat{\mathbf{g}}_{ne})\mathbf{Q}_s^*(\overline{\mathbf{g}}_{ne} + \hat{\mathbf{g}}_{ne})^{\mathrm{H}}, \tag{4.156a}$$

$$\text{subject to} \quad \|\hat{\mathbf{g}}_{ne}\|^2 \le \varepsilon^2_{\mathbf{g}_{ne}}. \tag{4.156b}$$

Despite the fact that this problem is nonconvex, its optimal solution can be easily obtained because it is a trust region subproblem [54], for which strong duality holds. Thus, using LR and SC, the worst channel mismatch for problem (4.156) is

$$\hat{\mathbf{g}}^*_{je} = \begin{cases} \overline{\mathbf{g}}_{ne}\mathbf{Q}_s^*(\lambda\mathbf{I} - \mathbf{Q}_s^*)^{\dagger}, & \overline{\mathbf{g}}_{ne} \neq \mathbf{0}, \\ \varepsilon^2_{\mathbf{g}_{ne}}\mathscr{P}(\mathbf{Q}_s^*), & \text{otherwise}, \end{cases} \tag{4.157a}$$

where $\mathscr{P}(\cdot)$ is the operator that returns the normalized eigenvector corresponding to the largest eigenvalue, and λ is obtained by solving the SDP

$$\max_{\lambda \ge 0} \quad \gamma, \tag{4.158a}$$

$$\text{subject to} \quad \begin{bmatrix} \lambda\mathbf{I} - \mathbf{Q}_s^* & \mathbf{Q}_s^*\overline{\mathbf{g}}_{ne}^{\mathrm{H}} \\ \overline{\mathbf{g}}_{ne}\mathbf{Q}_s^* & -\lambda\epsilon^2_{\mathbf{g}_{ne}} - \gamma - \overline{\mathbf{g}}_{ne}\mathbf{Q}_s^*\overline{\mathbf{g}}_{ne}^{\mathrm{H}} \end{bmatrix} \succeq \mathbf{0}. \tag{4.158b}$$

4.3.1.2 Numerical Example

Here, we present the numerical results on the worst-case secrecy rate of the proposed system in [6] discussed in Section 4.3.1.1 in this chapter. The system model is shown in Fig. 4.1. We assume the legitimate transmitter and the friendly jammer have four transmit antennas, that is, $U_{\mathrm{s}} = U_{\mathrm{j}} = 4$. The channel matrices are assumed to be composed of independent, zero-mean Gaussian random variables with unit variance. We perform Monte Carlo simulations consisting of 500 independent trials. The normalized background noise power is considered to be the same at the legitimate receiver and eavesdropper, and we assume $\sigma_d^2 = \sigma_e^2 = 1$ dB as in [6].

In Fig. 4.2, the worst-case secrecy rate is plotted as a function of the uncertainty bounds, $\varepsilon^2_{\mathbf{g}_{ne}}$ and $\varepsilon^2_{\mathbf{g}_{je}}$ for different values of the maximum allowable transmit power of the legitimate transmitter and the friendly jammer $p_s^{\max}$ and $p_j^{\max}$, respectively. Note that increasing the transmit power increases the worst-case secrecy rate, and increasing the uncertainty bounds calls for more transmit power for each transmitter to reach the higher worst-case secrecy rate.

4.3.1.3 Overview of Other Works on Robust Secure Transmission

Two full duplex sources, each with multiple transmit antennas and a single receive antenna in the presence of an eavesdropper with a single antenna are considered in [55]. It is assumed that CSI values are uncertain and bounded to an elliptical region.

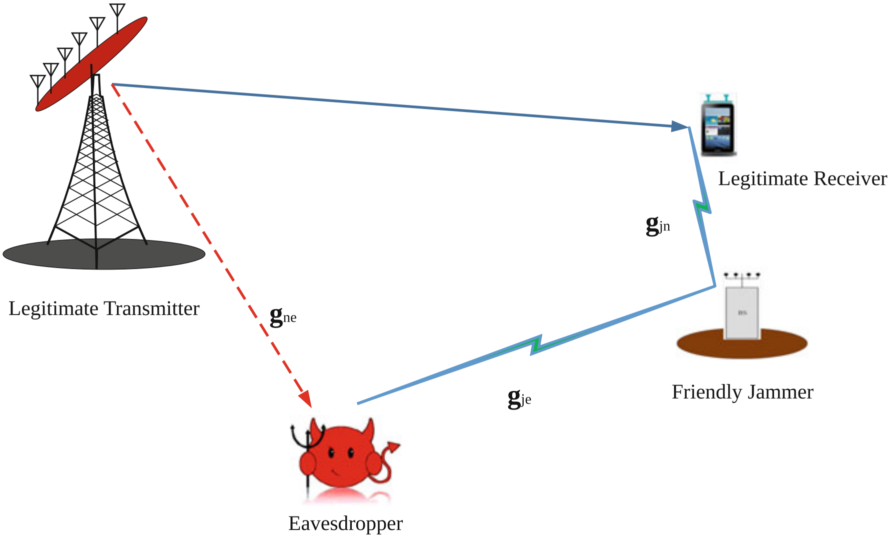

Fig. 4.1 System model for partial CSI feedback with bounded uncertainty

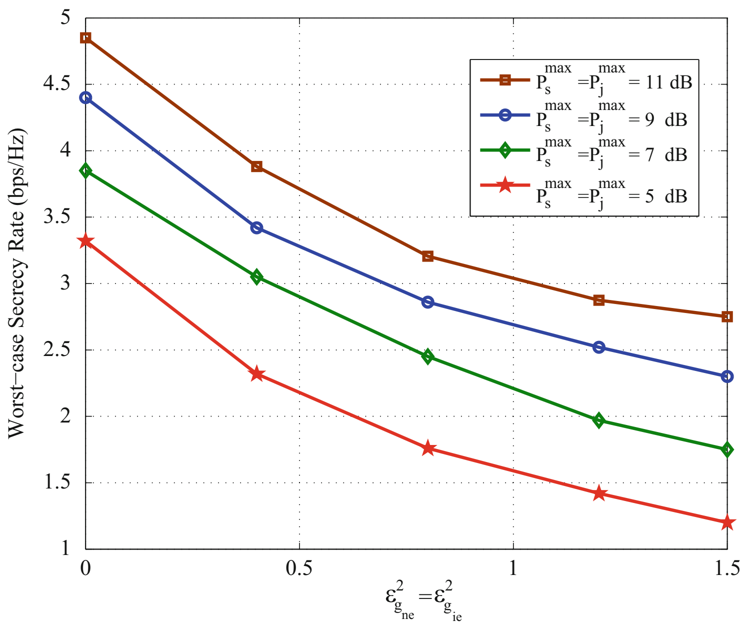

Fig. 4.2 Worst-case secrecy rate vs. uncertainty bounds $\varepsilon^2_{\mathbf{g}_{ne}} = \varepsilon^2_{\mathbf{g}_{je}}$ for different maximum allowable transmit power levels of legitimate transmitter and friendly jammer $p_s^{\mathtt{max}} = p_j^{\mathtt{max}}$

The objective of the robust secure beamforming is to maximize the worst-case sum secrecy rate with the transmit power constraint. Since the objective function includes both convex and concave terms, the convex terms are transformed into linear functions, and the problem is decomposed into four optimization subproblems. Subsequently, using relaxation methods, including SDR, SP, and LR, the locally optimal solution is obtained.

In [48], the robust resource allocation for a multiple-input multiple-output (MIMO) wiretap channel is investigated where a friendly jammer is deployed to improve secure communication. A multi-antenna transmitter establishes a secure link with its multi-antenna receiver in the presence of a multi-antenna eavesdropper, where a multi-antenna cooperative jammer transmits a jamming signal to the eavesdropper to disrupt its operation. The objective is to minimize the transmit power levels of both the legitimate transmitter and the friendly jammer subject to the constraint on the worst-case secrecy rate. The optimization problem is not jointly convex with respect to the transmit covariance matrices of the legitimate transmitter and the friendly jammer. To tackle this issue, the optimization problem is divided into two subproblems. The objective of the first subproblem is to minimize the transmit power of the legitimate transmitter subject to the constraint on the approximate worst-case secrecy rate. In doing so, the transmit covariance matrix of the legitimate transmitter is designed for a fixed transmit power covariance matrix of the jammer. This robust power minimization problem is reformulated into an SDP using the SP. The objective of the second subproblem is to minimize the transmit power of the friendly jammer for a fixed transmit power covariance matrix of the legitimate transmitter, subject to the constraint on the approximate worst-case secrecy rate. This problem is also reformulated into an SDP using the SP and DI. To make both subproblems tractable and convex, the constraint on the worst-case secrecy rate is approximated by a convex constraint obtained from its Taylor series expansion. The two subproblems are iteratively solved until an acceptable solution is obtained.

A wireless broadcast system consisting of a legitimate transmitter, a legitimate information-decoding receiver, an energy-harvesting receiver, and multiple eavesdroppers is considered in [56]. The legitimate transmitter is equipped with multiple antennas, and each of the other nodes is equipped with a single antenna. All CSI values are assumed to be uncertain but bounded in the worst-case sense. The objective is to maximize the worst-case secrecy rate under the transmit power constraint and the worst-case energy-harvesting constraint. Using relaxation methods, including the EF, the CCT, and SP, the nonconvex optimization problem is transformed into a SDP, which can be solved via any existing solver.

The physical-layer secrecy of an amplify-and-forward (AF) relay network that consists of a source, multiple trusted relays, a destination, and multiple eavesdroppers equipped with multiple antennas is considered in [57]. It is assumed that there is no direct link between the source and the destination, that is, first the source broadcasts its data to the relays, and then all relays transmit the confidential message to the destination using cooperative beamformaing while employing cooperative jamming to confuse the eavesdroppers. The objective is to maximize the worst-case secrecy rate subject to the total transmit power constraint for all the relays and the individual per-relay transmit power constraint. A joint cooperative beamforming and cooperative jamming is designed, which is robust against CSI uncertainty between the eavesdroppers and relays. Using relaxation methods, including SDR, SP, VNTI, the EF, and SPCA, the nonconvex problem is transformed into a sequence of convex approximation problems. The relaxation methods guarantee convergence to a KKT solution for the original problem.

Robust ergodic resource allocation for an uplink-secure transmission in an OFDMA-based and decode-and-forward (DF) relay-assisted CRN consisting of multiple secondary users (SUs) and primary users (PUs) and a secondary base station in the presence of multiple eavesdroppers is studied in [58]. It is assumed that all CSI values are uncertain. To control the secondary network's interference on PUs, two methods are proposed: (1) hard protection, by which the worst-case average interference of the secondary network is considered as a constraint, and (2) soft protection, by which the outage probability of PUs due to interference from the secondary network is treated as a constraint. The objective is to maximize the worst-case ergodic sum secrecy rate of secondary users subject to their average transmit power constraint, channel allocation limitation, relaying constraint and the soft or hard protection constraint. The relaying constraint means that the worst-case average secure data rate from the secondary base station to the relay station should be equal to the worst-case average secure data rate from the relay station to secondary users. The optimization problems are nonconvex and intractable. To overcome this difficulty, the constraint that each channel is assigned to at most one user is relaxed by using the time-sharing factor. In doing so, instead of a binary decision variable, a continuous variable between 0 and 1 (time-sharing factor) is considered, which indicates the portion of time that each channel is assigned to each user. In addition, for the hard protection case, an upper bound for the ellipsoid uncertainty is obtained. To develop more tractable formulas, NA is used. For the soft protection case, three approaches are proposed: (1) BA, (2) MI, and (3) SPCA. The approximate optimization problems are solved by the LR method. Simulations indicate that the iterative method has the best performance, but with higher complexity.

Table 4.1 summarizes the aforementioned existing works on nonconvex optimization problems on secure transmissions.

4.3.1.4 Robust Transmission in Relay-Assisted and Beamforming Systems

Example 1 A secondary network and a primary network are considered in [23], where the secondary base station with U_t antennas is communicating with N SUs and also acts as a relay for primary transmitter–receiver pairs. The channel gain vector between a primary transmitter and the secondary base station is $\mathbf{g}_1 \in \mathbb{C}^{U_t}$, and the channel gain vector between the secondary base station and the primary receiver is $\mathbf{g}_2 \in \mathbb{C}^{U_t}$. It is assumed that only their respective estimates, denoted

Table 4.1 Summary of existing works on robust secure transmission: worst-case optimization

Reference	SP	SDR	EF	BT-TB	CCT	BT-DI	LR	SPCA	BT-VNTI	BT-NA	BT-BA	BT-MI
[5]	√	–	√	–	√	√	√	–	–	–	–	–
[6]	√	–	√	–	√	√	√	–	–	–	–	–
[48]	√	–	–	√	–	√	–	–	–	–	–	–
[55]	√	√	√	–	–	–	√	–	–	–		
[56]	√	–	√	–	√	–	–	–	–	–	–	–
[57]	√	√	√	–	–	–	–	√	√	–	–	–
[58]	–	–	–	–	–	–	√	√	–	√	√	√

by $\overline{\mathbf{g}}_1$ and $\overline{\mathbf{g}}_2$, are available. The channel error vectors for $\mathbf{g}_1$ and $\mathbf{g}_2$ are $\hat{\mathbf{g}}_1$ and $\hat{\mathbf{g}}_2$, respectively, and are assumed to be bounded, that is, $\|\hat{\mathbf{g}}_1\| \leq \epsilon_1$ and $\|\hat{\mathbf{g}}_2\| \leq \epsilon_2$. Noise at the secondary user n, the secondary base station, and the primary receiver are assumed to be zero-mean circular complex Gaussian variables with variances σ_{cn}^2, σ_{cb}^2, and σ_p^2, respectively.

For the AF relaying scenario, the beamforming weight vectors toward the secondary and primary receivers are $\mathbf{w}_c$ and $\mathbf{w}_p$, respectively. The data rate of SU n under the worst-case condition of error is

$$R_n = \min_{\|\hat{\mathbf{g}}_1\| \leq \epsilon_1} \log\left(1 + \frac{|\mathbf{w}_c^{\mathrm{H}}\mathbf{h}_n|^2}{\sigma_{cn}^2 + \mathbf{w}_p^{\mathrm{H}}\mathbf{B}_n\mathbf{w}_p}\right), \qquad \forall n = 1, \ldots, N, \tag{4.159}$$

where $\mathbf{h}_n \in \mathbb{C}^{U_t}$ is the channel gain between SU n and its base station, and $\mathbf{B}_n = ((\overline{\mathbf{g}}_1 + \hat{\mathbf{g}}_1) \odot \mathbf{h}_n)((\overline{\mathbf{g}}_1 + \hat{\mathbf{g}}_1) \odot \mathbf{h}_n)^{\mathrm{H}} + \sigma_{cb}^2 \mathrm{diag}\{|h_{n1}|^2, \ldots, |h_{nU_t}|^2\}$.

The worst-case data rate of the primary receiver for AF relaying by the secondary base station is

$$R_p = \min_{\|\hat{\mathbf{g}}_1\| \leq \epsilon_1,\, \|\hat{\mathbf{g}}_2\| \leq \epsilon_2} \frac{1}{2} \log\left(1 + \frac{|\mathbf{w}_p^{\mathrm{H}}(\overline{\mathbf{g}}_1 + \hat{\mathbf{g}}_1) \odot (\overline{\mathbf{g}}_2 + \hat{\mathbf{g}}_2)|^2}{\sigma_p^2 + \mathbf{w}_p^{\mathrm{H}}\mathbf{C}\mathbf{w}_p + |\mathbf{w}_c^{\mathrm{H}}(\overline{\mathbf{g}}_2 + \hat{\mathbf{g}}_2)|^2}\right), \tag{4.160}$$

where $\mathbf{C} = \sigma_{cb}^2 \mathrm{diag}\{|\overline{g}_{21} + \hat{g}_{21}|^2, \ldots, |\overline{g}_{2U_t} + \hat{g}_{2U_t}|^2\}$. The objective is to maximize the minimum worst-case data rate of SUs subject to the PU's worst-case data rate, and the constraint on the total transmit power of secondary base station, which is formulated as

$$\max_{\mathbf{w}_c,\, \mathbf{w}_p} \quad \min_{n=1,\ldots,N} R_n, \tag{4.161a}$$

$$\text{subject to} \begin{cases} R_p \geq R^{\mathtt{min}}, & (4.161\text{b}) \\ \min\limits_{\|\hat{\mathbf{g}}_1\| \leq \epsilon_1} \mathbf{w}_p^{\mathrm{H}}\mathbf{A}\mathbf{w}_p + \|\mathbf{w}_c\|^2 \leq p^{\max}, & (4.161\text{c}) \end{cases}$$

where $\mathbf{A} = \mathbf{G}_1\mathbf{G}_1^{\mathrm{H}} + \sigma_{cb}^2\mathbf{I}$ and $\mathbf{G}_1 = \mathrm{diag}\{\bar{\mathbf{g}}_1 + \hat{\mathbf{g}}_1\}$. The minimum required data rate for the PU and its maximum transmit power are $R^{\min}$ and $p^{\max}$. Due to uncertainty in the objective function and constraints, this problem is intractable. To obtain its solution, the following steps are proposed in [23].

Step 1: To reduce the computational complexity, the TI and Cauchy–Schwarz inequality are used. The worst-case data rate for SU n in (4.159) is approximated by

$$\tilde{R}_n = \log\left(1 + \frac{|\mathbf{w}_c^{\mathrm{H}}\mathbf{h}_n|^2}{\sigma_{cn}^2 + \max_{\|\hat{\mathbf{g}}_1\|\le\epsilon_1} \mathbf{w}_p^{\mathrm{H}}\mathbf{B}_n\mathbf{w}_p}\right). \tag{4.162}$$

By applying the TI, $|\mathbf{w}_p^{\mathrm{H}}(\bar{\mathbf{g}}_1 + \hat{\mathbf{g}}_1) \odot \mathbf{h}_n)|$ is upper bounded to

$$|\mathbf{w}_p^{\mathrm{H}}(\bar{\mathbf{g}}_1 + \hat{\mathbf{g}}_1) \odot \mathbf{h}_n)| \le |\mathbf{w}_p^{\mathrm{H}}(\bar{\mathbf{g}}_1 \odot \mathbf{h}_n)| + |\mathbf{w}_p^{\mathrm{H}}(\hat{\mathbf{g}}_1 \odot \mathbf{h}_n)|. \tag{4.163}$$

Since $\|\hat{\mathbf{g}}_1\| \le \epsilon_1$, by applying Cauchy–Schwarz inequality, the upper bound of $|\mathbf{w}_p^{\mathrm{H}}(\hat{\mathbf{g}}_1 \odot \mathbf{h}_n)|$ is

$$|\mathbf{w}_p^{\mathrm{H}}(\hat{\mathbf{g}}_1 \odot \mathbf{h}_n)| \le \|\mathbf{w}_p^{\mathrm{H}}\|\|\hat{\mathbf{g}}_1 \odot \mathbf{h}_n\| \le \epsilon_1\|\mathbf{w}_p^{\mathrm{H}}\|\|\mathbf{h}_n\|. \tag{4.164}$$

Let

$$\begin{aligned}\max_{\|\hat{\mathbf{g}}_1\|\le\epsilon_1} |\mathbf{w}_p^{\mathrm{H}}(\bar{\mathbf{g}}_1 + \hat{\mathbf{g}}_1) \odot \mathbf{h}_n)|^2 &\le |\mathbf{w}_p^{\mathrm{H}}(\bar{\mathbf{g}}_1 \odot \mathbf{h}_n)|^2 + 2|\mathbf{w}_p^{\mathrm{H}}(\hat{\mathbf{g}}_1 \odot \mathbf{h}_n)|\epsilon_1\|\mathbf{w}_p\|\|\mathbf{h}_n\| \\ &\quad + \epsilon_1^2\|\mathbf{w}_p^{\mathrm{H}}\|^2\|\mathbf{h}_n\|^2 = \mathbf{w}_p^{\mathrm{H}}\mathbf{F}_n\mathbf{w}_p,\end{aligned} \tag{4.165a}$$

where $\mathbf{F}_n = (\bar{\mathbf{g}}_1 \odot \mathbf{h}_n)(\bar{\mathbf{g}}_1 \odot \mathbf{h}_n)^{\mathrm{H}} + \epsilon_1\big(\epsilon_1\|\mathbf{h}_n\|^2 + 2\|\mathbf{h}_n\|(\bar{\mathbf{g}}_1 \odot \mathbf{h}_n)\big)\mathbf{I}$. Using (4.165), we can simplify (4.162) to

$$\hat{R}_n = \log\left(1 + \frac{|\mathbf{w}_c^{\mathrm{H}}\mathbf{h}_n|^2}{\sigma_{cn}^2 + \mathbf{w}_p^{\mathrm{H}}\mathbf{H}_n\mathbf{w}_p}\right), \quad \forall n = 1, \ldots, N, \tag{4.166}$$

where $\mathbf{H}_n = \mathbf{F}_n + \sigma_{cb}^2\mathrm{diag}\{|h_{n1}|^2, \ldots, |h_{nU_t}|^2\}$.

Step 2: Using Cauchy–Schwarz inequality and $\|\hat{\mathbf{g}}_1\| \le \epsilon_1$, the total transmit power constraint is replaced by

$$\mathbf{w}_p^{\mathrm{H}}\mathbf{A}\mathbf{w}_p + \|\mathbf{w}_c\|^2 \le \mathbf{w}_p^{\mathrm{H}}\tilde{\mathbf{A}}\mathbf{w}_p + \|\mathbf{w}_c\|^2 \le p^{\max}, \tag{4.167}$$

where $\tilde{\mathbf{A}} = \mathrm{diag}\{\bar{g}_{11} + \epsilon_1, \ldots, \bar{g}_{1U_t} + \epsilon_1\}$.
Step 3: The worst-case data rate of the PU can be relaxed as

$$\tilde{R}_p = \frac{1}{2}\log\left(1 + \frac{\min_{\|\hat{\mathbf{g}}_1\|\le\epsilon_1,\ \|\hat{\mathbf{g}}_2\|\le\epsilon_2} |\mathbf{w}_p^{\mathrm{H}}(\bar{\mathbf{g}}_1 + \hat{\mathbf{g}}_1) \odot (\bar{\mathbf{g}}_2 + \hat{\mathbf{g}}_2)|^2}{\sigma_p^2 + \max_{\|\hat{\mathbf{g}}_2\|\le\epsilon_2} \mathbf{w}_p^{\mathrm{H}}\mathbf{C}\mathbf{w}_p + \max_{\|\hat{\mathbf{g}}_2\|\le\epsilon_2,} |\mathbf{w}_c^{\mathrm{H}}(\bar{\mathbf{g}}_2 + \hat{\mathbf{g}}_2)|^2}\right). \tag{4.168}$$

Similar to the relaxation of the SU's worst-case data rate, the upper bound of $\max_{\|\hat{\mathbf{g}}_2\| \le \epsilon_2,} |\mathbf{w}_c^{\mathrm{H}}(\bar{\mathbf{g}}_2 + \hat{\mathbf{g}}_2)|^2$ is

$$\max_{\|\hat{\mathbf{g}}_2\| \le \epsilon_2,} |\mathbf{w}_c^{\mathrm{H}}(\bar{\mathbf{g}}_2 + \hat{\mathbf{g}}_2)|^2 \le |\mathbf{w}_c^{\mathrm{H}}\bar{\mathbf{g}}_2|^2 + \epsilon_2(2\|\bar{\mathbf{g}}_2\| + \epsilon_2)\|\mathbf{w}_c\|^2 = \mathbf{w}_c^{\mathrm{H}}\mathbf{M}_1\mathbf{w}_c, \tag{4.169}$$

where $\mathbf{M_1} = \bar{\mathbf{g}}_2\bar{\mathbf{g}}_2^{\mathrm{H}} + \epsilon_2^2(1 + 2\sqrt{\bar{\mathbf{g}}_2\bar{\mathbf{g}}_2^{\mathrm{H}}})$. Similarly, the numerator of (4.168) is lower bounded to

$$\min_{\|\hat{\mathbf{g}}_1\| \le \epsilon_1,\ \|\hat{\mathbf{g}}_2\| \le \epsilon_2} |\mathbf{w}_p^{\mathrm{H}}(\bar{\mathbf{g}}_1 + \hat{\mathbf{g}}_1) \odot (\bar{\mathbf{g}}_2 + \hat{\mathbf{g}}_2)|^2 \ge \mathbf{w}_p^{\mathrm{H}}\mathbf{L}\mathbf{w}_p, \tag{4.170}$$

where

$$\begin{aligned} \mathbf{L} = \big(&\epsilon_2\|\bar{\mathbf{g}}_1\| - \epsilon_1\|\bar{\mathbf{g}}_2\| - (\epsilon_2\epsilon_1)^2 - \|\bar{\mathbf{g}}_1 \odot \bar{\mathbf{g}}_2\|(\epsilon_2\|\bar{\mathbf{g}}_1\| - \epsilon_1\|\bar{\mathbf{g}}_2\| - \epsilon_1\epsilon_2)\big)\mathbf{I} \\ &+ (\bar{\mathbf{g}}_1 \odot \bar{\mathbf{g}}_2)(\bar{\mathbf{g}}_1 \odot \bar{\mathbf{g}}_2)^{\mathrm{H}}. \end{aligned} \tag{4.171a}$$

In a similar manner, $\max_{\|\hat{\mathbf{g}}_2\| \le \epsilon_2,} |\mathbf{w}_p^{\mathrm{H}}(\bar{\mathbf{g}}_2 + \hat{\mathbf{g}}_2)|^2$ is upper bounded to

$$\max_{\|\hat{\mathbf{g}}_2\| \le \epsilon_2,} |\mathbf{w}_p^{\mathrm{H}}(\bar{\mathbf{g}}_2 + \hat{\mathbf{g}}_2)|^2 \le \mathbf{w}_p^{\mathrm{H}}\mathbf{M}_2\mathbf{w}_p, \tag{4.172}$$

where $\mathbf{M}_2 = \mathrm{diag}\{\bar{g}_{21} + \epsilon_2, \ldots, \bar{g}_{2U_t} + \epsilon_2\}$. Substituting (4.169), (4.170), and (4.172) into (4.168), we rewrite the approximate PU's worst-case data rate:

$$\hat{R}_p = \log\left(1 + \frac{\mathbf{w}_p^{\mathrm{H}}\mathbf{L}\mathbf{w}_p}{\sigma_p^2 + \mathbf{w}_p^{\mathrm{H}}\mathbf{M}_2\mathbf{w}_p + \mathbf{w}_c^{\mathrm{H}}\mathbf{M}_1\mathbf{w}_c}\right). \tag{4.173}$$

Step 4: From the preceding step, problem (4.161) is approximated by

$$\max_{\mathbf{w}_c, \mathbf{w}_p} \quad \min_{n=1,\ldots,N} \hat{R}_n, \tag{4.174a}$$

$$\text{subject to} \begin{cases} \hat{R}_p \ge R^{\mathtt{min}}, & (4.174\text{b}) \\ \mathbf{w}_p^{\mathrm{H}}\tilde{\mathbf{A}}\mathbf{w}_p + \|\mathbf{w}_c\|^2 \le p^{\mathtt{max}}, & (4.174\text{c}) \end{cases}$$

which is still nonconvex. To transform problem (4.174) into a convex one, we use the SDR and epigraph methods and get

$$\max_{\mathbf{X}, \mathbf{Y}} \quad t, \tag{4.175a}$$

$$\text{subject to} \begin{cases} (2^t - 1)(\sigma_{cn}^2 + \mathrm{tr}(\mathbf{H}_n\mathbf{Y})) \leq \mathrm{tr}(\mathbf{h}_n\mathbf{h}_n^{\mathrm{H}}\mathbf{X}), & (4.175\mathrm{b}) \\ \mathrm{tr}(\mathbf{X}) + \mathrm{tr}(\mathbf{M}_1\mathbf{Y}) \leq p^{\max}, & (4.175\mathrm{c}) \\ (2^{2R^{\min}} - 1)(\sigma_p^2 + \mathrm{tr}(\mathbf{M}_2\mathbf{Y}) + \mathrm{tr}(\mathbf{M}_1\mathbf{X})) \leq \mathrm{tr}(\mathbf{L}\mathbf{Y}) & (4.175\mathrm{d}) \\ \mathbf{X} \succeq \mathbf{0}, \ \ \mathrm{rank}(\mathbf{X}) = 1, & (4.175\mathrm{e}) \\ \mathbf{Y} \succeq \mathbf{0}, \ \ \mathrm{rank}(\mathbf{Y}) = 1, & (4.175\mathrm{f}) \end{cases}$$

where $\mathbf{X} = \mathbf{w}_c\mathbf{w}_c^{\mathrm{H}}$ and $\mathbf{Y} = \mathbf{w}_p\mathbf{w}_p^{\mathrm{H}}$. Following the same approach as in Section 4.2.1.5, problem (4.175) can be transformed into SDP form.

Example 2 A system consisting of a source, a destination, and N_{r} AF relays is studied in [8]. It is assumed that there is no direct link between the source and the destination, and relay nodes cooperate with each other to produce virtual beamforming to the destination, where ω_r is the complex coefficient of the virtual beamforming for relay r. The channel gain between the source and relay r is h_r, and the channel gain between relay r and the destination is h'_r. It is assumed that the exact value of $\mathbf{h}$ is available to the source, but $\mathbf{h}'$ is uncertain. In practice, $\mathbf{h}'$ is estimated by the relays and sent to the source. The uncertain parameter is modeled by

$$\mathbf{h}' = \overline{\mathbf{h}}' + \hat{\mathbf{h}}', \tag{4.176}$$

where $\overline{\mathbf{h}}' = [\overline{h}'_1, \ldots, \overline{h}'_{N_{\mathrm{r}}}]^{\mathrm{T}}$ is the estimated CSI and $\hat{\mathbf{h}}' = [\hat{h}'_1, \ldots, \hat{h}'_{N_{\mathrm{r}}}]$ is the error vector. The CSI's uncertainty set is

$$\mathscr{R}_{\mathbf{h}'} = \{\hat{\mathbf{h}}' \in \mathbb{C}^{N_{\mathrm{r}}} | \|\hat{\mathbf{h}}'\|^2 \leq N_{\mathrm{r}}\epsilon_{\mathbf{h}'}^2\}, \tag{4.177}$$

where $\epsilon_{\mathbf{h}'}$ is a known constant. The worst-case SINR at the destination is

$$\Gamma = \frac{|\sum_{r=1}^{N_{\mathrm{r}}} h'_r h_r l_r w_r|^2 p_s}{\sum_{r=1}^{N_{\mathrm{r}}} |h'_r|^2 |l_r|^2 |w_r|^2 \sigma_r^2 + \sigma_0^2}, \tag{4.178}$$

where p_s is the transmit power of the source, σ_0^2 and σ_r^2 are the noise power at the destination and at relay r, respectively, and l_r is the scaling factor, obtained by

$$l_r = (|h_r|^2 p_s + \sigma_r^2)^{-1/2}. \tag{4.179}$$

The objective is to maximize the worst-case SINR at the receiver subject to the individual relays' power constraints, that is,

$$\max_{|\mathbf{w}|^2 \preceq \mathbf{p}^{\max}} \ \min_{\hat{\mathbf{h}}' \in \mathscr{R}_{\mathbf{h}'}} \ \Gamma, \tag{4.180}$$

where $\mathbf{p}^{\max} = [p_1^{\max}, \ldots, p_{N_r}^{\max}]$ is the vector of maximum allowable transmit power levels of relays. Using the epigraph method, problem (4.180) can be rewritten as

$$\max_{|\mathbf{w}|^2 \preceq \mathbf{p}^{\max}} \; t, \tag{4.181a}$$

$$\text{subject to} \quad \min_{\hat{\mathbf{h}}' \in \mathscr{R}_{\mathbf{h}'}} \frac{|\sum_{r=1}^{N_r} h'_r h_r l_r w_r|^2 p_s}{\sum_{r=1}^{M} |h'_r|^2 |l_r|^2 |w_r|^2 \sigma_r^2 + \sigma_0^2} \geq t. \tag{4.181b}$$

Since (4.181) is quasi-convex, for some given t it can be solved by solving the following problem:

$$\min_{|\mathbf{w}|^2 \preceq \mathbf{p}^{\max}} \; \|\mathbf{w}\|^2 \tag{4.182a}$$

$$\text{subject to} \quad \min_{\hat{\mathbf{h}}' \in \mathscr{R}_{\mathbf{h}'}} \frac{|\sum_{r=1}^{N_r} h'_r h_r l_r w_r|^2 p_s}{\sum_{r=1}^{N_r} |h'_r|^2 |l_r|^2 |w_r|^2 \sigma_r^2 + \sigma_0^2} \geq t. \tag{4.182b}$$

Let $\tilde{w}_r = w_r h_r l_r$ and $\mathbf{H} = \text{diag}\{\frac{1}{|h_1|^2 l_1^2}, \ldots, \frac{1}{|h_{N_r}|^2 l_{N_r}^2}\}$. Problem (4.182) is equivalent to

$$\min_{\tilde{\mathbf{w}} \in \mathbb{C}^{N_r}} \quad \tilde{\mathbf{w}}^{\mathrm{H}} \mathbf{H} \tilde{\mathbf{w}}, \tag{4.183a}$$

$$\text{subject to} \begin{cases} \min_{\hat{\mathbf{h}}' \in \mathscr{R}_{\mathbf{h}'}} \dfrac{|\sum_{r=1}^{N_r} (\overline{h}'_r + \hat{h}'_r) \tilde{w}_r|^2 p_s}{\sum_{r=1}^{N_r} \frac{|\overline{h}'_r + \hat{h}'_r|^2}{|h_r|^2} |\tilde{w}_r|^2 \sigma_r^2 + \sigma_0^2} \geq t, & (4.183b) \\ p_r^{\max} |h_r|^2 |l_r|^2 \geq |\tilde{w}_r|^2, \qquad \forall r. & (4.183c) \end{cases}$$

To obtain the optimal solution for (4.183), we use the SP and SDR methods. In doing so, we rewrite constraint (4.183b) as

$$(\overline{\mathbf{h}}' + \hat{\mathbf{h}}')^{\mathrm{H}} \mathbf{Q} (\overline{\mathbf{h}}' + \hat{\mathbf{h}}') \geq 0, \quad \forall \hat{\mathbf{h}}' \in \mathscr{R}_{\mathbf{h}'}, \tag{4.184}$$

where

$$\mathbf{Q} = p_s \mathbf{v}\mathbf{v}^{\mathrm{H}} - t\,\text{diag}\left\{ \frac{|v_1|^2 \sigma_1^2}{|h_1|^2}, \ldots, \frac{|v_{N_r}|^2 \sigma_{N_r}^2}{|h_{N_r}|^2} \right\}, \tag{4.185}$$

and $\mathbf{v}$ is the element-wise phase-shifted version of $\tilde{\mathbf{w}}$ such that $\tilde{w}_r = \frac{v_r (\overline{h}'_r)^*}{|\overline{h}'_r|}$, $\tilde{\mathbf{h}}' = |\overline{\mathbf{h}}'|$ is the real-valued estimated CSI, and $\tilde{\mathbf{h}} = \left[\frac{\hat{h}'_1 (\overline{h}'_1)^*}{|(\overline{h}'_1)^*|}, \ldots, \frac{\hat{h}'_{N_r} (\overline{h}'_{N_r})^*}{|(\overline{h}'_{N_r})^*|} \right]^{\mathrm{T}}$ is an element-wise phase-shifted CSI uncertainty vector.

Using the SP, we can express (4.184) equivalently by

$$\begin{bmatrix} (\tilde{\mathbf{h}}')^{\mathrm{T}}\mathbf{Q}\tilde{\mathbf{h}}' - t\sigma_0^2 - s\sigma_r^2\epsilon_{\mathbf{h}'}^2 & (\tilde{\mathbf{h}}')^{\mathrm{T}}\mathbf{Q} \\ \mathbf{Q}\tilde{\mathbf{h}}' & \mathbf{Q} + s\mathbf{I} \end{bmatrix} \succeq \mathbf{0}, \qquad \exists s \geq 0, \tag{4.186}$$

and using the SDR method, we can transform (4.183) into SDP form:

$$\min_{\mathbf{V}\succeq\mathbf{0},s} \quad \mathrm{tr}(\mathbf{VG}), \tag{4.187a}$$

$$\text{subject to} \begin{cases} \begin{bmatrix} (\tilde{\mathbf{h}}')^{\mathrm{T}}\mathbf{Q}\tilde{\mathbf{h}}' - t\sigma_0^2 - sN_r\epsilon_{\mathbf{h}}^2 & (\tilde{\mathbf{h}}')^{\mathrm{T}}\mathbf{Q} \\ \mathbf{Q}\tilde{\mathbf{h}}' & \mathbf{Q} + s\mathbf{I} \end{bmatrix} \succeq \mathbf{0}, & (4.187\text{b}) \\ p_r^{\max}|h_r|^2|l_r|^2 \geq [V]_{r,r}, \quad \forall r, & (4.187\text{c}) \end{cases}$$

where $\mathbf{V} = \mathbf{v}\mathbf{v}^{\mathrm{H}}$, and $[V]_{r,r}$ is the (r, r)th element of $\mathbf{V}$.

Similarly, using the SP and SDR relaxation, the worst-case SINR maximization problem (4.181) is converted to SDP form:

$$\max_{\mathbf{V}\succeq\mathbf{0},s} \quad t, \tag{4.188a}$$

$$\text{subject to} \begin{cases} \begin{bmatrix} (\tilde{\mathbf{h}}^2)^{\mathrm{T}}\mathbf{Q}\tilde{\mathbf{h}}' - t\sigma_0^2 - sN_r\epsilon_{\mathbf{h}'}^2 & (\tilde{\mathbf{h}}')^{\mathrm{T}}\mathbf{Q} \\ \mathbf{Q}\tilde{\mathbf{h}}' & \mathbf{Q} + s\mathbf{I} \end{bmatrix} \succeq \mathbf{0}, & (4.188\text{b}) \\ p_r^{\max}|h_r|^2|l_r|^2 \geq [V]_{r,r} \quad \forall r. & (4.188\text{c}) \end{cases}$$

The optimal solution to (4.188) can be efficiently obtained via the iterative algorithm that utilizes bisection search. For more details, the interested reader is referred to Section 2 in Chapter 4 in [1].

In [8], Theorem 1, it is shown that when $\tilde{\mathbf{h}}^2 \succ \sqrt{N_r\epsilon_{\mathbf{h}'}^2}\mathbf{1}$, problem (4.187) always has a rank 1 optimal solution. In this case, the optimal weight at the rth relay is

$$w_r = \frac{v_r(\overline{h}_r' h_r)^*}{l_r|\overline{h}_r' h_r|}. \tag{4.189}$$

4.3.1.5 Overview of Other Works on Robust Transmission in Relay-Assisted and Beamforming Systems

A wireless network consisting of multiple sources and multiple destinations that communicate via multiple relays is studied in [46]. The objective is to minimize the total transmit power of relays subject to the SINR constraint at destinations, where relaying coefficients are the optimization variables. It is assumed that all the CSIs are imperfect, modeled by ellipsoid uncertainty. The optimization problem is nonconvex and intractable, and three relaxation methods, SDR, BTs, and SP, are utilized to make the problem tractable and convex. The upper bound on the objective function and the lower bounds on the achievable SINRs are derived, which are used to relax the SINR constraint via the SP to transform the approximated optimization problem into SDP form. Subsequently, randomization is utilized to obtain the "best" rank 1 solution.

A wireless network with multiple single-antenna transmitters and receivers supported by multiple MIMO relays is studied in [59]. In doing so, two optimization problems are considered: (1) minimizing the total transmit power of the relays while satisfying SINR requirements for all receivers and (2) maximizing the minimum of SINRs in all receivers while satisfying the transmit power constraint of each relay. It is assumed that the covariance matrices of the channels between MIMO relays and receivers are uncertain, modeled by the worst-case method. It is shown that the robust optimization problems are nonconvex but can be solved via the relaxation methods, including the SDR, LR, EF, and Cauchy–Schwarz inequality.

Robust relay beamforming for two-way relay networks is studied in [60]. It is assumed that two nodes communicate via multiple two-way relays, and the objective is to maximize the minimum worst-case SINR of the two nodes subject to the constraint on the total transmit power of relays. The problem is nonconvex and intractable and is decomposed into a series of robust relay power minimization problems using the bisection search [1]. Subsequently, relaxation methods, including the SP, EF, and SDR, are utilized to transform each power minimization problem into SDP form. In this manner, a suboptimal solution to the original problem is efficiently obtained.

Resource allocation under channel uncertainty for relay-aided device-to-device (D2D) communication underlaying LTE-A cellular networks is studied in [61]. It is assumed that there are multiple fixed relay nodes by which the traffic of D2D and cellular users are transmitted. Relay selection is done at higher layers, and channels are exclusively assigned to relays using a binary decision variable. The objective is to maximize the minimum achievable rate over two hops for relay-assisted D2D communication while maintaining the QoS (i.e., the minimum rate) requirement for cellular and D2D users under the total transmit power constraint for relays and users, satisfying the constraint on channel allocation, and also maintaining the interference caused in each relay below a given threshold. The side information, including channel gains, and interference levels are uncertain, modeled by the worst-case method. The optimization problem is a mixed-integer nonlinear program (MINLP), which is computationally intractable. A common approach to solving such problems is to relax the constraint on the exclusivity of channels for each user by introducing the time-sharing factor. In this way, the binary decision variable is replaced by a continuous variable between 0 and 1, which denotes the portion of time that each channel is assigned to each user. To obtain a more tractable formula, NA is applied. The relaxed optimization problem satisfies the time-sharing constraint, and the solution of the problem relaxed by the dual decomposition method is asymptotically optimal.

Table 4.2 Summary of existing works on robust relay-assisted and beamforming systems

Reference	SP	SDR	EF	LR	BT-TI	BT-Cauchy–Schwarz inequality	BT-TTMI	BT-NA
[8]	√	√	√	–	–	–	–	–
[23]	–	√	√	–	√	√	–	–
[46]	√	√	√	–	–	–	√	–
[59]	–	√	√	√	–	√	–	–
[60]	√	√	√	–	–	–	–	–
[61]	–	–	–	√	–	–	–	√

In Table 4.2, the aforementioned existing works on robust transmission in relay-assisted and beamforming systems are summarized. Interestingly, epigraph form (EF) and SDR are the most widely used methods in the literature.

4.3.1.6 Robust Transmission in Cognitive Radio Networks

Example 1 The downlink transmission in CRNs where the secondary base station is equipped with multiple antennas is studied in [7]. The objective is to maximize the minimum SINR for SUs while the interference caused by the secondary base station on PUs is kept below a given threshold. Uncertain parameters include channel gains between the secondary base station and users (both primary and secondary). The number of PUs is Q, and the number of SUs is N. The secondary base station has U_t antennas, and each SU has only one antenna. The instantaneous SINR for SU n is

$$\mathrm{SINR}_n = \frac{\mathbf{w}_n^{\mathrm{H}}(\mathbf{h}_n\mathbf{h}_n^{\mathrm{H}})\mathbf{w}_n}{\sigma_0^2 + \sum\limits_{m=1,\, m\neq n}^{N} \mathbf{w}_m^{\mathrm{H}}(\mathbf{h}_n\mathbf{h}_n^{\mathrm{H}})\mathbf{w}_m}, \tag{4.190}$$

where $\mathbf{w}_n \in \mathbb{C}^{1\times U_t}$ is the transmit beamforming vector of user n, and $\mathbf{h}_n$ is the channel between SU n and the secondary base station.

The constraint on interference for PU q is

$$\sum_{n=1}^{N} |\mathbf{g}_q^{\mathrm{H}}\mathbf{w}_n|^2 \leq IT_q, \qquad \forall q = 1, \ldots, Q, \tag{4.191}$$

where $\mathbf{g}_q$ is the channel gain between the secondary base station and PU q, and IT_q is the maximum allowable interference on PUs.

Assuming that the secondary base station has neither perfect CSI nor statistical knowledge of the uncertainty, in the worst-case robust optimization, we have

$$\mathbf{h}_n = \bar{\mathbf{h}}_n + \hat{\mathbf{h}}_n, \quad \forall n, \tag{4.192}$$

$$\mathbf{g}_q = \bar{\mathbf{g}}_q + \hat{\mathbf{g}}_q, \quad \forall q, \tag{4.193}$$

where $\overline{\mathbf{h}}_n$ and $\overline{\mathbf{g}}_q$ are the exact CSI by the secondary base station, and $\hat{\mathbf{h}}_n$ and $\hat{\mathbf{g}}_q$ are the CSI errors that are bounded to ellipsoid regions, that is,

$$\mathscr{R}_{\mathbf{h}_n} = \{\hat{\mathbf{h}}_n : \hat{\mathbf{h}}_n^{\mathrm{H}} \mathbf{C}_n \hat{\mathbf{h}}_n \leq 1\}, \quad \forall n, \tag{4.194}$$

$$\mathscr{R}_{\mathbf{g}_q} = \{\hat{\mathbf{g}}_q : \hat{\mathbf{g}}_q^{\mathrm{H}} \mathbf{D}_q \hat{\mathbf{g}}_q \leq 1\}, \quad \forall q, \tag{4.195}$$

where $\mathbf{C}_n \succ \mathbf{0}$ and $\mathbf{D}_q \succ \mathbf{0}$ determine the quality of the CSI and are assumed to be known. The CSI is perfect if $\mathbf{C}_n$ and $\mathbf{D}_q$ approach infinity and the CSI is the worst if they are zero [7].

The objective is to maximize the minimum of the worst-case SINR for SUs subject to constraints on the worst-case interference on PUs and the constraint on the transmit power of the secondary base station. By utilizing the EF, the relaxed worst-case optimization problem is

$$\max_{\{\mathbf{w}_n, \forall n\}} \quad s, \tag{4.196a}$$

$$\text{subject to} \begin{cases} \sum_{n=1}^{N} \|\mathbf{w}_n\|^2 \leq p^{\max}, & \text{(4.196b)} \\ \sum_{n=1}^{N} |\mathbf{g}_q^{\mathrm{H}} \mathbf{w}_n|^2 \leq IT_q, \quad \forall \hat{\mathbf{g}}_q \in \mathscr{R}_{\mathbf{g}_q}, \; q = 1, \ldots, Q, & \text{(4.196c)} \\ \dfrac{\mathbf{w}_n^{\mathrm{H}} (\mathbf{h}_n \mathbf{h}_n^{\mathrm{H}}) \mathbf{w}_n}{\sigma_0^2 + \sum\limits_{m=1, m \neq n}^{N} \mathbf{w}_m^{\mathrm{H}} (\mathbf{h}_n \mathbf{h}_n^{\mathrm{H}}) \mathbf{w}_m} \geq s, \quad \forall \hat{\mathbf{h}}_n \in \mathscr{R}_{\mathbf{h}_n}, \; n = 1, \ldots, N. & \text{(4.196d)} \end{cases}$$

For further simplification, let $\mathbf{W}_n = \mathbf{w}_n \mathbf{w}_n^{\mathrm{H}}$, $\mathbf{T} = \sum\limits_{n=1}^{N} \mathbf{W}_n$, and $\mathbf{Q}_n = \mathbf{W}_n - s \sum\limits_{m=1, m \neq n}^{N} \mathbf{W}_m$ for all PUs and SUs. Problem (4.196) can be written

$$\max_{\{\mathbf{W}_n, \forall n\}} \quad s, \tag{4.197a}$$

$$\text{subject to} \begin{cases} \mathrm{tr}(\mathbf{T}) \leq p^{\max}, & \text{(4.197b)} \\ \mathrm{tr}(\mathbf{T} \mathbf{g}_q \mathbf{g}_q^{\mathrm{H}}) \leq IT_q, \quad \forall \hat{\mathbf{g}}_q \in \mathscr{R}_{\mathbf{g}_q}, \; q = 1, \ldots, Q, & \text{(4.197c)} \\ \mathrm{tr}(\mathbf{Q}_n \mathbf{h}_n \mathbf{h}_n^{\mathrm{H}}) \geq s\sigma_0^2, \quad \forall \hat{\mathbf{h}}_n \in \mathscr{R}_{\mathbf{h}_n}, \; n = 1, \ldots, N, & \text{(4.197d)} \\ \mathrm{rank}(\mathbf{W}_n) = 1, \quad \forall n = 1, \ldots, N. & \text{(4.197e)} \end{cases}$$

Now, using the SP, we can express problem (4.197) equivalently as

$$\max_{\{\mathbf{T}_n,\, u_n,\, \forall n\},\, \{r_q,\, \forall q\}} \quad s \tag{4.198a}$$

$$\text{subject to} \begin{cases} \operatorname{tr}(\mathbf{T}) \le p^{\max}, & (4.198b) \\ \begin{bmatrix} -\bar{\mathbf{g}}_q^{\mathrm{H}} \mathbf{T} \bar{\mathbf{g}}_q + IT_q - r_q & -\bar{\mathbf{g}}_l^{\mathrm{H}} \mathbf{T} \\ -\mathbf{T} \bar{\mathbf{g}}_q^{\mathrm{H}} & -\mathbf{T} + r_q \mathbf{D}_q \end{bmatrix} \succeq \mathbf{0}, \quad \exists r_q \ge 0, \ \forall q, & (4.198c) \\ \begin{bmatrix} \bar{\mathbf{h}}_n^{\mathrm{H}} \mathbf{Q}_n \bar{\mathbf{h}}_n - s\sigma_0^2 - u_n & \bar{\mathbf{h}}_n^{\mathrm{H}} \mathbf{Q}_n \\ \mathbf{Q}_n \bar{\mathbf{h}}_n^{\mathrm{H}} & \mathbf{Q}_n + u_n \mathbf{C}_n \end{bmatrix} \succeq \mathbf{0}, \quad \exists u_n \ge 0, \ \forall n, & (4.198d) \\ \operatorname{rank}(\mathbf{W}_n) = 1, \quad \forall n = 1, \ldots, N. & (4.198e) \end{cases}$$

Due to constraint (4.198e), problem (4.198) is still nonconvex but can be transformed into a quasi-convex problem by using SDR.

Example 2 In [62], a robust energy-efficient algorithm for an underlay CRN consisting of multiple SUs and PUs in multiple bands with uncertainty in CSI values is proposed. The worst-case optimization problem belongs to maxi-min problems with infinite constraints, which are nontrivial to solve. This is because the outer-maximization problem is nonconvex, and the inner-minimization problem is concave, which is NP-hard in general. To solve this problem, the infinite constraint is transformed into its equivalent convex constraint to handle uncertainty in channel gains from the secondary base station to PUs, and a closed-formed solution for the uncertainty in interference caused by the primary base station on SUs is obtained. Subsequently, the outer-maximization problem is solved via the fractional programming technique, and the inner-minimization problem is solved using the Lagrange dual method, leading to a globally optimal solution.

In this setup, the downlink of the secondary network consisting of one secondary base station and N SUs that coexist with Q PUs over K channels is studied. The channel gain from the secondary base station to SU n on channel k is h_n^k. SU n occupies K_n channels, and $\mathbf{h}_n = [h_n^1, \ldots, h_n^{K_n}]^{\mathrm{T}}$ is the channel gain vector between the secondary base station and SU n. The power allocation vector for the secondary base station is $\mathbf{p} = [p_1, \ldots, p_K]^{\mathrm{T}}$, where the element p_k is the transmit power of the secondary base station on channel k. The SINR of SU n on sub-channel k is

$$\gamma_n^k = \frac{p_k h_n^k}{I_k + \sigma^2}, \quad \forall n, \tag{4.199}$$

where I_k is the aggregated interference caused by all primary base stations on channel k, and σ^2 is the noise power at the SU's receiver.

Let P_C and ζ denote the constant circuit power consumption and the amplifier coefficient for the cognitive (secondary) base station, respectively. The total power dissipation by the cognitive base station is $P_C + \zeta \sum_{k=1}^{K} p_k$. The vector of channel gains between the cognitive base station and PU q is $\mathbf{g}_q = [g_q^1, \ldots, g_q^K]^{\mathrm{T}}$, where g_q^k is the channel gain between the cognitive base station and PU q on channel k. The channel gain between the cognitive base station and SU n is modeled by $\mathbf{h}_n = \overline{\mathbf{h}}_n + \hat{\mathbf{h}}_n$, where $\overline{\mathbf{h}}_n$ is the exact value and $\hat{\mathbf{h}}_n$ the error in the channel gain. The uncertainty region is modeled by

$$\mathscr{R}_{\mathbf{h}_n} = \{\mathbf{h}_n | \|\mathbf{W}_n(\mathbf{h}_n - \overline{\mathbf{h}}_n)\| \le \epsilon_{\mathbf{h}_n}\}, \tag{4.200}$$

where $\mathbf{W}_n$ is an invertible $\mathbb{R}^{K_n \times K_n}$ weighting matrix, and $\epsilon_{\mathbf{h}_n}$ is the bound on the uncertainty region.

Similarly, the uncertainty region of $\mathbf{g}_q$ is modeled by

$$\mathscr{R}_{\mathbf{g}_q} = \{\mathbf{g}_q | \|\mathbf{M}_q(\mathbf{g}_q - \overline{\mathbf{g}}_q)\| \le \epsilon_{\mathbf{g}_q}\}, \tag{4.201}$$

where $\mathbf{M}_q$ is an invertible $\mathbb{R}^{K \times K}$ weighting matrix, and $\epsilon_{\mathbf{g}_q}$ is the bound on the uncertainty region. Also, $\overline{\mathbf{g}}_q$ is the exact value and $\hat{\mathbf{g}}_q$ is the error in the channel gain. The interference from the primary base station on the SU in channel k is I_k, and its uncertainty region is modeled by

$$\mathscr{R}_{I_k} = \{I_k | \|z_k(I_k - \overline{I}_k)\| \le \epsilon_{I_k}\}, \tag{4.202}$$

where z_k and ϵ_{I_k} are the weight factor and the bound on the uncertainty region, respectively, $\overline{I}_k$ is the exact value, and $\hat{I}_k$ is the error in I_k.

Now, the robust energy-efficient SINR maximization problem is

$$\max_{\mathbf{p}} \min_{\mathbf{h}_n, I_k} \quad \frac{\sum_{n=1}^{N} \sum_{k=1}^{K_n} \log(1 + \gamma_n^k)}{P_C + \zeta \sum_{k=1}^{K} p_k}, \tag{4.203a}$$

$$\text{subject to} \begin{cases} \sum_{k=1}^{K} p_k \le p^{\max}, & (4.203\text{b}) \\ \mathbf{p}^{\mathrm{T}} \cdot \mathbf{g}_q \le IT_q, \quad \forall \mathbf{g}_q \in \mathscr{R}_{\mathbf{g}_q}, \quad q = 1, \ldots, Q, & (4.203\text{c}) \\ \mathbf{h}_n \in \mathscr{R}_{\mathbf{h}_n}, \quad \forall n = 1, \ldots, N, & (4.203\text{d}) \\ I_k \in \mathscr{R}_{I_k}, \quad \forall k, & (4.203\text{e}) \end{cases}$$

where IT_q is the allowable interference threshold for PU q, and $p^{\max}$ is the maximum transmit power of the secondary transmitter over all K channels.

Due to the existence of an infinite number of constraints in (4.203c) and (4.203d), problem (4.203) is a semi-infinite programming problem, which is difficult to solve.

To tackle this issue, the infinite constraints are replaced by their equivalent convex constraints. To satisfy constraint (4.203b) $\forall \mathbf{g}_q \in \mathscr{R}_{\mathbf{g}_q}$, it is rewritten as

$$\max_{\mathbf{g}_q \in \mathscr{R}_{\mathbf{g}_q}} \mathbf{p}^{\mathrm{T}} \cdot \mathbf{g}_q \leq IT_q,$$

which can be stated by utilizing the protection function as

$$\begin{aligned}\max_{\mathbf{g}_q \in \mathscr{R}_{\mathbf{g}_q}} \mathbf{p}^{\mathrm{T}} \cdot \mathbf{g}_q &= \mathbf{p}^{\mathrm{T}} \cdot \overline{\mathbf{g}}_q + \epsilon_{\mathbf{g}_q} \max_{\mathbf{g}_q \in \{\mathbf{g}_q \mid \|\frac{1}{\epsilon_{\mathbf{g}_q}} \mathbf{M}_q(\mathbf{g}_q - \overline{\mathbf{g}}_q)\|_2 \leq 1\}} \mathbf{p}^{\mathrm{T}} \cdot \left(\frac{1}{\epsilon_{\mathbf{g}_q}} \mathbf{M}_q^{-1} \mathbf{M}_q (\mathbf{g}_q - \overline{\mathbf{g}}_q) \right) \\ &= \mathbf{p}^{\mathrm{T}} \cdot \overline{\mathbf{g}}_q + \epsilon_{\mathbf{g}_q} \|\mathbf{M}_q^{-1} \cdot \mathbf{p}\|_2^*, \end{aligned} \tag{4.204a}$$

where $\|\cdot\|^*$ is the dual norm of $\|\cdot\|$. For more details, the interested reader is referred to Section 1.3 in Chapter 1 in this book. From (4.204), constraint (4.203b) is equivalent to

$$\mathbf{p}^{\mathrm{T}} \cdot \overline{\mathbf{g}}_q + \epsilon_{\mathbf{g}_q} \|\mathbf{M}_q^{-1} \cdot \mathbf{p}\|_2^* \leq IT_q, \tag{4.205}$$

which is a convex constraint since the dual norm is a convex function, as was discussed in Chapter 1. Similarly to tackling the computational complexity of considering I_k, we use the protection function to rewrite the inner-minimization problem (4.203) as

$$I_k^* = \arg \min_{I_k \in \mathscr{R}_{I_k}} \sum_{n=1}^{N} \sum_{k=1}^{K_n} \log(1 + \gamma_n^k) = \arg \min_{I_k \in \mathscr{R}_{I_k}} \frac{h_n^k p_k}{I_k \sigma^2} = \arg \max_{I_k \in \mathscr{R}_{I_k}} I_k = \bar{I}_k + \frac{\epsilon_{I_k}}{z_k}, \tag{4.206}$$

$$\max_{\mathbf{p}} \min_{\mathbf{h}_n} \frac{\sum_{n=1}^{N} \sum_{k=1}^{K_n} \log\left(1 + \frac{z_k p_k h_n^k}{z_k \bar{I}_k + z_k \sigma^2 + \epsilon_{I_k}}\right)}{P_C + \zeta \sum_{k=1}^{K} p_k}, \tag{4.207a}$$

$$\text{subject to} \begin{cases} \mathbf{p}^{\mathrm{T}} \cdot \overline{\mathbf{g}}_q + \epsilon_{\mathbf{g}_q} \|_2 \mathbf{M}_q^{-1} \cdot \mathbf{p}\|_2^* \leq IT_q, & (4.207\text{b}) \\ \mathbf{h}_n \in \mathscr{R}_{\mathbf{h}_n}, \quad \forall n = 1, \ldots, N, & (4.207\text{c}) \\ \sum_{k=1}^{K} p_k \leq p^{\max}. & (4.207\text{d}) \end{cases}$$

Note that the preceding optimization problem is still nonconvex. To address this issue, [62] focuses on the important special case of $K_n = 1$, where a closed-form solution for $\mathbf{h}_n$ uncertainty is derived. Subsequently, problem (4.207) is efficiently solved using the fractional programming method, which was explained in Section 4.2.1.7 in this chapter. In doing so, it is assumed that each SU occupies only one channel for its own transmission, that is, we have at most $N = K$ SUs. Without loss of generality, it is also assumed that SU n occupies channel n. Now, the inner-minimization problem in (4.207) is expressed as

$$\min_{\mathbf{h}_n} \quad \frac{\sum_{k=1}^{K} \log\left(1 + \frac{z_k p_k h_n^k}{z_k \bar{I}_k + z_k \sigma^2 + \epsilon_{I_k}}\right)}{P_C + \zeta \sum_{k=1}^{K} p_k}, \tag{4.208a}$$

$$\text{subject to} \quad \mathbf{h}_n \in \mathscr{R}_{\mathbf{h}_n}. \tag{4.208b}$$

Since uncertainties in $\mathbf{h}_n$ among all channels are independent, problem (4.208) can be decomposed into multiple subproblems. The subproblem for SU n is

$$\min_{h_n} \quad \frac{\log\left(1 + \frac{z_k p_k h_k^n}{z_k \bar{I}_k + z_k \sigma^2 + \epsilon_{I_k}}\right)}{P_C + \zeta \sum_{k=1}^{K} p_k}, \tag{4.209a}$$

$$\text{subject to} \quad h_n \in \mathscr{R}_{h_n}, \tag{4.209b}$$

where the uncertainty region for h_n is $\mathscr{R}_{h_n} = \{h_n | \|w_n(h_n - \bar{h}_n)\|_2 \leq \epsilon_{h_n}\}, \quad \forall n$. The optimal solution h_n^* to problem (4.208) is

$$h_n^* = \arg \min_{h_n \in \mathscr{R}_{h_n}} \log\left(1 + \frac{z_k p_k h_n^k}{z_k \bar{I}_k + z_k \sigma^2 + \epsilon_{I_k}}\right) = \arg \min_{h_n \in \mathscr{R}_{h_n}} \frac{z_k p_k h_n^k}{z_k \bar{I}_k + z_k \sigma^2 + \epsilon_{I_k}}$$
$$= \arg \min_{h_n \in \mathscr{R}_{h_n}} h_n = \bar{h}_n - \frac{\epsilon_{h_n}}{w_n}.$$

Substituting h_n^* into (4.207), we get

$$\max_{\mathbf{p}} \quad \frac{\sum_{k=1}^{K} \log\left(1 + \frac{z_k p_k (w_k \bar{h}_k - \epsilon_{h_k})}{w_k (z_k \bar{I}_k + z_k \sigma^2 + \epsilon_{I_k})}\right)}{P_C + \zeta \sum_{k=1}^{K} p_k}, \tag{4.211a}$$

$$\text{subject to} \begin{cases} \mathbf{p}^{\mathrm{T}} \cdot \hat{\mathbf{g}}_q + \epsilon_{\mathbf{g}_q} \|\mathbf{M}_q^{-1} \cdot \mathbf{p}\|^* \leq IT_q, & (4.211b) \\ \sum_{k=1}^{K} p_k \leq P^{\max}. & (4.211c) \end{cases}$$

Since both constraints (4.211b) and (4.211c) are convex and the objective function has a fractional form, problem (4.211) is a fractional programming problem. Therefore, using the nonlinear fractional programming relaxation method, which was explained in Section 4.2.1.7 in this chapter, we can obtain the optimal solution.

The proposed approach for $K_n = 1$ can be extended to the general case where each SU can occupy multiple channels, provided that uncertainties in h_n for all K_n channels are independent. In this case, the uncertainty region for each h_n^k in $\mathbf{h_n}$ is $\mathscr{R}_{h_n^k} = \{h_n^n | \|w_n^k(h_n^k - \overline{h}_n^k)\|_2 \le \epsilon_{h_n^k}\}, \quad k = 1, \ldots, K_n, \quad n = 1, \ldots, N$. Similarly, the approach to tackling uncertainties in $\mathbf{g}_q, \forall q$ and $I_k, \forall k$ in (4.204) and (4.206) can be used to tackle uncertainties in $\mathbf{g}_q, \forall q$ and I_k for the general case $K_n > 1$. Optimization problem (4.207) in the general case is quite different from that in the important special case of $K_n = 1$. This is because the variables $\mathbf{p}$ and $\mathbf{h}_n$ in (4.207) are coupled with each other in the general case, which makes problem (4.207) nontrivial. To tackle this issue, an alternative iterative algorithm is proposed in which the optimal value of $\mathbf{p}$ for a given feasible value of $\mathbf{h}_n$ is derived, which is the solution to the outer-maximization problem in (4.207). Subsequently, the optimal value of $\mathbf{h}_n$ is obtained for the optimal value of $\mathbf{p}$ in the outer-maximization problem in the previous iteration, which is the solution to the inner-minimization problem in (4.207). These steps are interchangeably repeated until either the difference of the optimal energy efficiency values in the outer-maximization problem between two subsequent iterations becomes less than a predefined threshold or the maximum number of iterations is reached.

Example 3 The downlink of a CRN with K SUs and one secondary base station coexisting with Q PUs is studied in [24], where the CSI values are uncertain, modeled by the worst-case method. Both PUs and SUs are equipped with single antennas, and the secondary base station has U transmit antennas. The channel gain from the secondary base station to each SU n is modeled by a complex-valued vector $\mathbf{h}_n \in \mathbb{C}^{U\times 1}$, which is uncertain, whose exact value is $\overline{\mathbf{h}}_n$. The uncertainty region is modeled by

$$\mathscr{R}_{\mathbf{h}_n} = \{\mathbf{h}_n \mid \|\mathbf{a}_n\|_2 \le \epsilon_{\mathbf{h}_n}\}, \qquad n = 1, \ldots, N, \tag{4.212}$$

where $\mathbf{a}_n = \mathbf{h}_n - \overline{\mathbf{h}}_n$, and $\epsilon_{\mathbf{h}_n}$ is a known constant.

The channel gain from the secondary base station to PU q is also a complex-valued vector $\mathbf{g}_q \in \mathbb{C}^{U\times 1}$, assumed to be uncertain, whose exact value is $\overline{\mathbf{g}}_q$. The uncertainty region for $\mathbf{g}_q$ is

$$\mathscr{R}_{\mathbf{g}_q} = \{\mathbf{g}_q \mid \|\mathbf{b}_q\| \le \epsilon_{\mathbf{g}_q}\}, \quad q = 1, \ldots, Q, \tag{4.213}$$

where $\mathbf{b}_q = \mathbf{g}_q - \overline{\mathbf{g}}_q$, and $\epsilon_{\mathbf{g}_q}$ is a known constant.

The SINR for SU n is

$$\mathrm{SINR}_n = \frac{|\mathbf{w}_n^{\mathrm{H}}\mathbf{h}_n|^2}{\sigma_n^2 + \sum\limits_{i=1,\, i\neq n}^{N} |\mathbf{w}_i^{\mathrm{H}}\mathbf{h}_n|^2}, \tag{4.214}$$

where σ^2 is the noise power and $\mathbf{w}_n \in \mathbb{C}^{U\times 1}$ the precoding weight vector for SU n. It is assumed that the CRN is far from the primary transmitters, that is, interference on SU n from the primary network is much less than the interference from other SUs and is treated as noise.

The objective is to minimize the transmit power of the secondary base station while simultaneously targeting a lower bound on the received SINR for the SUs and imposing an upper limit on the interference level on PUs. Mathematically, this problem is

$$\min_{\mathbf{w}_n, n=1,\ldots,N} \quad \sum_{n=1}^{N} \|\mathbf{w}_n\|^2, \tag{4.215a}$$

$$\text{subject to} \begin{cases} \mathrm{SINR}_n \geq \gamma_n, & \forall \mathbf{h}_n \in \mathscr{R}_{\mathbf{h}_n}, \quad n = 1,\ldots,N, \quad (4.215\text{b}) \\ \sum\limits_{n=1}^{N} |\mathbf{w}_n^{\mathrm{H}}\mathbf{g}_q|^2 \leq IT_q, & \forall \mathbf{g}_q \in \mathscr{R}_{\mathbf{g}_q}, \quad q = 1,\ldots,Q, \quad (4.215\text{c}) \end{cases}$$

where γ_n is the minimum required SINR of SU n, and IT_q is the maximum allowable interference level on PU q by the secondary base station.

Due to the existence of an infinite number of constraints in (4.215b) and (4.215c), the preceding problem is intractable. To deal with this issue,the worst-case approach is adopted, where for the worst channel realizations, the minimum and maximum values of the SINR and interference are considered. In this way, (4.215) is reformulated as

$$\min_{\mathbf{w}_n,\, n=1,\ldots,N} \quad \sum_{n=1}^{N} \|\mathbf{w}_n\|^2, \tag{4.216a}$$

$$\text{subject to} \begin{cases} \min\limits_{\mathbf{h}_n \in \mathscr{R}_{\mathbf{h}_n}} \mathrm{SINR}_n \geq \gamma_n, & \forall n = 1,\ldots,N, \quad (4.216\text{b}) \\ \max\limits_{\mathbf{g}_q \in \mathscr{R}_{\mathbf{g}_q}} \sum\limits_{n=1}^{N} |\mathbf{w}_n^{\mathrm{H}}\mathbf{g}_q|^2 \leq IT_q, & \forall q = 1,\ldots,Q. \quad (4.216\text{c}) \end{cases}$$

It is mathematically appealing to express $|\mathbf{w}_n^{\mathrm{H}}\mathbf{h}_n|^2$ in quadratic form as

$$|\mathbf{w}_n^{\mathrm{H}}\mathbf{h}_n|^2 = \mathbf{w}_n^{\mathrm{H}}(\overline{\mathbf{h}}_n + \mathbf{a}_n)(\overline{\mathbf{h}}_n + \mathbf{a}_n)^{\mathrm{H}}\mathbf{w}_n = \mathbf{w}_n^{\mathrm{H}}(\overline{\mathbf{H}}_n + \mathbf{A}_n)\mathbf{w}_n, \tag{4.217}$$

where $\overline{\mathbf{H}}_n = \overline{\mathbf{h}}_n \overline{\mathbf{h}}_n^{\mathrm{H}}$ and $\mathbf{A}_n = \overline{\mathbf{h}}_n \mathbf{a}_n^{\mathrm{H}} + \mathbf{a}_n \overline{\mathbf{h}}_n^{\mathrm{H}} + \mathbf{a}_n \mathbf{a}_n^{\mathrm{H}}$. Using the TI and Cauchy–Schwarz inequality, we obtain an upper bound on $\|\mathbf{A}_n\|$, denoted by η_n:

$$\|\mathbf{A}_n\| \leq \|\overline{\mathbf{h}}_n \mathbf{a}_n^{\mathrm{H}}\| + \|\mathbf{a}_n \overline{\mathbf{h}}_n^{\mathrm{H}}\| + \|\mathbf{a}_n \mathbf{a}_n^{\mathrm{H}}\| \leq \|\overline{\mathbf{h}}_n\|\|\mathbf{a}_n^{\mathrm{H}}\| + \|\mathbf{a}_n\|\|\overline{\mathbf{h}}_n^{\mathrm{H}}\| + \|\mathbf{a}_n\|^2 \tag{4.218a}$$

$$= \epsilon_{\mathbf{h}_n}^2 + 2\epsilon_{\mathbf{h}_n}\|\overline{\mathbf{h}}_n\| = \eta_n. \tag{4.218b}$$

The identity $\mathbf{x}\mathbf{A}_n\mathbf{x}^{\mathrm{H}} = \mathrm{tr}(\mathbf{A}_n\mathbf{x}\mathbf{x}^{\mathrm{H}})$ is used to simplify the quadratic expression to

$$|\mathbf{w}_n^{\mathrm{H}}\mathbf{h}_n|^2 = \mathrm{tr}((\overline{\mathbf{H}}_n + \mathbf{A}_n)\mathbf{W}_n), \tag{4.219}$$

where $\mathbf{W}_n = \mathbf{w}_n\mathbf{w}_n^{\mathrm{H}}$. Using a similar formulation, $|\mathbf{w}_n^{\mathrm{H}}\mathbf{g}_q|^2$ is

$$|\mathbf{w}_n^{\mathrm{H}}\mathbf{g}_q|^2 = \mathrm{tr}((\overline{\mathbf{G}}_q + \mathbf{B}_q)\mathbf{W}_n), \tag{4.220}$$

where $\overline{\mathbf{G}}_q = \overline{\mathbf{g}}_q\overline{\mathbf{g}}_q^{\mathrm{H}}$, $\mathbf{B}_q$ is the norm-bounded uncertainty matrix $\mathbf{B}_q \leq \xi_q$, and $\xi_q = \epsilon_{\mathbf{g}_q}^2 + 2\epsilon_{\mathbf{g}_q}\|\overline{\mathbf{g}}_q\|$.

Using the preceding expressions, we rewrite problem (4.216) as

$$\min_{\mathbf{W}_n,\, n=1,\ldots,N} \quad \sum_{n=1}^{N} \mathrm{tr}(\mathbf{W}_n), \tag{4.221a}$$

$$\text{subject to} \begin{cases} \min\limits_{\|\mathbf{A}_n\|\leq\eta_n} \dfrac{\mathrm{tr}((\overline{\mathbf{H}}_n + \mathbf{A}_n)\mathbf{W}_n)}{\sigma_n^2 + \sum\limits_{i=1,\, i\neq n}^{N} \mathrm{tr}((\overline{\mathbf{H}}_n + \mathbf{A}_n)\mathbf{W}_i)} \geq \gamma_n, \quad \forall n = 1,\ldots,N, & (4.221\text{b}) \\[2ex] \max\limits_{\|\mathbf{B}_q\|\leq\xi_q} \sum\limits_{n=1}^{N} \mathrm{tr}((\overline{\mathbf{G}}_q + \mathbf{B}_q)\mathbf{W}_n) \leq IT_q, \quad \forall q = 1,\ldots,Q. & (4.221\text{c}) \end{cases}$$

In [24], three approaches are proposed for solving this problem. In the first approach, the loose upper and lower bounds on the terms appearing in the numerator and denominator of the worst-case SINR are obtained, which leads to an SDP optimization problem. In the second approach, the exact upper and lower bounds on the aforementioned terms are obtained, leading to a non-SDP but convex optimization problem, which can be efficiently solved. In the third approach, an exact minimum of the SINR is obtained, leading to a convex optimization problem. In what follows, the aforementioned three approaches are described.

Loosely Bounded Robust Solution To obtain an approximate solution to (4.221), constraint (4.221b) is approximated as

$$\min_{\|\mathbf{A}_n\|\leq\eta_n} \mathrm{tr}((\overline{\mathbf{H}}_n + \mathbf{A}_n)\mathbf{W}_n) - \gamma_n \sum_{i=1,i\neq n}^{N} \max_{\|\mathbf{A}_n\|\leq\eta_n} \mathrm{tr}((\overline{\mathbf{H}}_n + \mathbf{A}_n)\mathbf{W}_i) \geq \gamma_n\sigma_n^2. \tag{4.222}$$

The minimization and maximization in (4.222) can be approximated as

$$\mathrm{tr}((\overline{\mathbf{H}}_n - \mathbf{I}\eta_n)\mathbf{W}_n) - \gamma_n \sum_{i=1, i\neq n}^{N} \mathrm{tr}((\overline{\mathbf{H}}_n + \mathbf{I}\eta_n)\mathbf{W}_i) \geq \gamma_n \sigma_n^2. \tag{4.223}$$

Similarly, constraint (4.221c) is approximated as

$$\sum_{n=1}^{N} \mathrm{tr}((\overline{\mathbf{G}}_q + \mathbf{I}\epsilon_{\mathbf{g}_q})\mathbf{W}_n) \leq IT_q. \tag{4.224}$$

Using the preceding expressions, we rewrite (4.221):

$$\min_{\mathbf{W}_n,\, n=1,\ldots,N} \quad \sum_{n=1}^{N} \mathrm{tr}(\mathbf{W}_n), \tag{4.225a}$$

$$\text{subject to} \begin{cases} \mathrm{tr}((\overline{\mathbf{H}}_n - \mathbf{I}\eta_n)\mathbf{W}_n) & \\ -\gamma_n \sum_{i=1, i\neq n}^{N} \mathrm{tr}((\overline{\mathbf{H}}_n + \mathbf{I}\eta_n)\mathbf{W}_i) \geq \gamma_n \sigma_n^2, & \forall n = 1, \ldots, N, \quad (4.225\text{b}) \\ \sum_{n=1}^{N} \mathrm{tr}((\overline{\mathbf{G}}_q + \mathbf{I}\epsilon_{\mathbf{g}_q})\mathbf{W}_n) \leq IT_q, & \forall q = 1, \ldots, Q, \quad (4.225\text{c}) \\ \mathbf{W}_n = \mathbf{W}_n^{\mathrm{H}}, & \forall n = 1, \ldots, N, \quad (4.225\text{d}) \\ \mathbf{W}_n \succeq \mathbf{0}, & \forall n = 1, \ldots, N. \quad (4.225\text{e}) \end{cases}$$

Using the SDR method, we can obtain a locally optimal solution for (4.225a).

Strictly Bounded Robust Solution To find tight approximations for constraints (4.221b) and (4.221c), LR is used in [24] to find the exact maximum and exact minimum for each term. The maximum and minimum values of $\mathrm{tr}((\overline{\mathbf{H}}_n + \mathbf{A}_n)\mathbf{W}_n)$ with respect to $\mathbf{A}_n$ are

$$\mathbf{A}_n^{\mathtt{max}} = \eta_n \frac{\mathbf{W}_n^{\mathrm{H}}}{\|\mathbf{W}_n\|}, \quad \mathbf{A}_n^{\mathtt{min}} = -\eta_n \frac{\mathbf{W}_n^{\mathrm{H}}}{\|\mathbf{W}_n\|}. \tag{4.226}$$

Using the preceding expressions, constraint (4.221b) is approximated by

$$\mathrm{tr}\left(\overline{\mathbf{H}}_n\left(\mathbf{W}_n - \gamma_n \sum_{i=1, i\neq n}^{N} \mathbf{W}_i\right)\right) - \eta_n\left(\|\mathbf{W}_n\| + \gamma_n \sum_{i=1, i\neq n}^{N} \|\mathbf{W}_i\|\right) \geq \gamma_n \sigma_n^2. \tag{4.227}$$

Similarly, constraint (4.221c) is approximated by

$$\sum_{n=1}^{N}\left(\operatorname{tr}(\overline{\mathbf{G}}_q\mathbf{W}_n)+\epsilon_{\mathbf{g}_q}\|\mathbf{W}_n\|\right)\leq IT_q. \tag{4.228}$$

Finally, the main problem is approximated by

$$\min_{\mathbf{W}_n,\ n=1,\ldots,N}\quad \sum_{n=1}^{N}\operatorname{tr}(\mathbf{W}_n), \tag{4.229a}$$

$$\text{subject to}\begin{cases}\operatorname{tr}\left(\overline{\mathbf{H}}_n\left(\mathbf{W}_n-\gamma_n\sum\limits_{i=1,i\neq n}^{N}\mathbf{W}_i\right)\right)-\eta_n\left(\|\mathbf{W}_n\|\right.\\ \left.+\gamma_n\sum\limits_{i=1,i\neq n}^{N}\|\mathbf{W}_i\|\right)\geq\gamma_n\sigma_n^2, & \forall n=1,\ldots,N, \quad (4.229\text{b})\\ \sum\limits_{n=1}^{N}\left(\operatorname{tr}(\overline{\mathbf{G}}_q\mathbf{W}_n)+\epsilon_{\mathbf{g}_q}\|\mathbf{W}_n\|\right)\leq IT_q, & \forall q=1,\ldots,Q, \quad (4.229\text{c})\\ \mathbf{W}_n=\mathbf{W}_n^{\mathrm{H}}, & \forall n=1,\ldots,N, \quad (4.229\text{d})\\ \mathbf{W}_n\succeq\mathbf{0}, & \forall n=1,\ldots,N. \quad (4.229\text{e})\end{cases}$$

This problem is convex and can be solved using any standard numerical optimization package such as CVX.

Exact Robust Method The aforementioned two approaches consider the problem of minimizing the uncertain SINR using two conservative approaches, while in this method, the worst-case channel realization is considered instead. Here, constraint (4.221b) is written

$$\min_{\|\mathbf{A}_n\|\leq\eta_n}\left[\operatorname{tr}((\overline{\mathbf{H}}_n+\mathbf{A}_n)\mathbf{W}_n)-\gamma_n\sum_{i=1,i\neq n}^{N}\operatorname{tr}((\overline{\mathbf{H}}_n+\mathbf{A}_n)\mathbf{W}_i)\right]\geq\gamma_n\sigma_n^2. \tag{4.230}$$

Using LR, the solution to the preceding minimization problem is

$$\mathbf{A}_n^{\mathtt{min}}=-\eta_n\frac{\left(\mathbf{W}_n-\gamma_n\sum\limits_{i=1,i\neq n}^{N}\mathbf{W}_i\right)^{\mathrm{H}}}{\|\mathbf{W}_n-\gamma_n\sum\limits_{i=1,i\neq n}^{N}\mathbf{W}_i\|}. \tag{4.231}$$

Finally, the main problem is reformulated as

$$\min_{\mathbf{W}_n,\, n=1,\ldots,N} \quad \sum_{n=1}^{N} \mathrm{tr}(\mathbf{W}_n), \tag{4.232a}$$

$$\text{subject to} \begin{cases} \mathrm{tr}\Big(\overline{\mathbf{H}}_n\Big(\mathbf{W}_n - \gamma_n \sum\limits_{i=1, i\neq n}^{N} \mathbf{W}_i\Big)\Big) - \eta_n \|\mathbf{W}_n \\ - \gamma_n \sum\limits_{i=1,i\neq n}^{N} \mathbf{W}_i\| \geq \gamma_n \sigma_n^2, & \forall n = 1, \ldots, N, \quad (4.232\text{b}) \\ \sum\limits_{n=1}^{N}\Big(\mathrm{tr}(\overline{\mathbf{G}}_q \mathbf{W}_n) + \epsilon_{\mathbf{g}_q} \|\mathbf{W}_n\|\Big) \leq IT_q, & \forall q = 1, \ldots, Q, \quad (4.232\text{c}) \\ \mathbf{W}_n = \mathbf{W}_n^{\mathrm{H}}, & \forall n = 1, \ldots, N, \quad (4.232\text{d}) \\ \mathbf{W}_n \succeq \mathbf{0}, & \forall n = 1, \ldots, N. \quad (4.232\text{e}) \end{cases}$$

Similar to (4.229), this problem is convex and can be solved using any existing problem solvers such as CVX.

4.3.1.7 Overview of Other Works on Robust Transmission in Cognitive Radio Networks

Robust ergodic resource allocation for the uplink in an OFDMA-based CRN consisting of multiple SUs and PUs and a secondary base station is studied in [63], where all CSI values are assumed to be uncertain. In doing so, two problems are considered. In the first problem, the objective is to maximize the worst-case ergodic sum rate of SUs subject to a constraint on their average transmit power, a constraint on the worst-case average interference from SUs on PUs, and a constraint on channel allocation. In the second problem, the average-based constraints in the first problem are replaced by their corresponding outage probability constraints. The first optimization problem is nonconvex and intractable. To reduce the computational complexity and transform the problem into a tractable and convex one, the following steps are taken. The constraint on channel exclusivity for each user is relaxed using the time-sharing factor. In doing so, the binary decision variable is replaced by a continuous variable between 0 and 1, which indicates the portion of time that each channel is assigned to each user. Second, to relax the constraint on the worst-case average interference, two approaches are proposed: (1) a method based on BA and (2) a method utilizing an upper bound on the outage probability using the ellipsoid uncertainty region. Since the Bernstein method gives a tight approximation, its performance is better than the second approach. Third, to obtain more tractable formulas for the two proposed approaches, NA is applied. Finally, the approximated optimization problems are solved via LR. For the second problem, similar steps

Table 4.3 Summary of existing works on robust transmission in cognitive radio networks

Reference	SP	SDR	NLFP	LR	BT-TI	BT-Cauchy–Schwarz inequality	EF	BT-NA	BT-BA	SPCA	BT-MI
[7]	√	√	–	√	–	–	√	–	–	–	–
[24]	–	√	–	√	√	√	–	–	–	–	–
[34]	–	–	–	√	–	–	–	√	√	–	–
[62]	–	–	√	√	–	–	–	–	–	–	
[63]	–	–	–	√	–	–	–	√	√	√	√

are proposed, except that in the second step the following three approaches are proposed: (1) BT based on MI, (2) BT based on BAs, and (3) an iterative approach based on SPCA. Similar to the first problem, the final approximated optimization problems are solved via LR. Simulations show that the third approach has better performance, but its computational complexity is higher.

In [34], robust resource allocation for the uplink of an OFDMA-based CRN is investigated. The objective is to maximize the weighted sum rate subject to constraints on channel assignment, the limit on the transmit power of SUs, and the outage probability of PUs due to interference from SUs. It is assumed that CSI values between the primary and secondary networks are uncertain and the respective error is bounded. Since the problem is nonconvex and intractable, three relaxation methods, NA, BA, and LR, are used to transform the problem into a convex and tractable one.

Table 4.3 summarizes the aforementioned existing works on nonconvex optimization problems on robust transmission in CRNs.

4.3.2 Partial CSI Feedback: Stochastic Uncertainty

We now present the application of relaxation methods for stochastic robust optimization problems, where probabilistic constraints often have no closed-form expressions and in general are not convex, which makes the outage-based constrained problem hard to solve.

4.3.2.1 Robust Transmission in Relay-Assisted and Beamforming Systems

Example 1 The stochastic robust downlink multiuser MIMO transceiver design with arbitrary distribution of channel uncertainty is studied in [31], where it is assumed that the distribution of interference plus noise is not known, and the error in channel estimation via linear minimum mean square error (LMMSE) estimator has an arbitrary distribution. Hence, the QoS requirements are considered for the worst-case distribution of the error in channel estimation.

The system consists of one base station equipped with N transmit antennas and K users where user k is equipped with M_k antennas such that $\sum_{k=1}^{K} M_k = M$. It is assumed that L_k independent data streams are transmitted to user k, where $\sum_{k=1}^{K} L_k = L$. To guarantee data recovery by users, it is necessary that $L_k \leq M_k$ and $L \leq \min\{M, N\}$. The $M_k \times 1$ vector of interference plus noise for user k is $\mathbf{n}_k$. It is assumed that the interference plus noise has an arbitrary distribution, where only its first two moments are known, that is, $\mathbf{n}_k \backsim A(\mathbf{0}, \mathbf{R}_k)$, where A is an arbitrary distribution. The total MSE of user k's signal is

$$\text{MSE}_k = \|\mathbf{F}_k(\overline{\mathbf{H}}_k + \hat{\mathbf{H}}_k)\mathbf{G} - \mathbf{D}_k\|_F^2 + \text{tr}(\mathbf{F}_k\mathbf{R}_k\mathbf{F}_k^{\text{H}}), \tag{4.233}$$

where $\mathbf{F}_k$ is an $L_k \times M_k$ equalizer matrix deployed at the receiver and $\overline{\mathbf{H}}_k$ and $\hat{\mathbf{H}}_k$ are the $M_k \times N$ matrices of channel gain and error, respectively. The precoding $N \times L$ matrix at the base station is $\mathbf{G}$, and the matrix $\mathbf{D}_k = [\mathbf{0}_{L_k \times \sum_{k=1}^{k-1} L_k} \quad \mathbf{I}_{L_k} \quad \mathbf{0}_{L_k \times \sum_{k=k+1}^{K} L_k}]$ is used to select the data stream for user k. Note that the MSE distribution depends on the distribution of $\hat{\mathbf{H}}_k$. In addition, $\mathbf{G}$ and $\mathbf{F}_k$ are unknown and in general depend on the statistics of $\hat{\mathbf{H}}_k$. Thus, the MSE distribution cannot be obtained or approximated in advance. The optimization problem for transceiver design is formulated as

$$\min_{\mathbf{G},\, \mathbf{F}_k,\, \forall k} \quad \text{tr}(\mathbf{G}\mathbf{G}^{\text{H}}) \tag{4.234a}$$

$$\text{subject to} \quad \sup_{\text{vec}(\hat{\mathbf{H}}_k)\sim A(\mathbf{0}, \Sigma_k)} \Pr\{\text{MSE}_k \geq \varepsilon_k\} \leq \zeta_k, \quad k = 1, \ldots, K, \tag{4.234b}$$

where ε_k and ζ_k are the maximum allowable MSE and the minimum predefined outage probability at receiver k, respectively. Since the statistics of $\hat{\mathbf{H}}_k$ is unknown, the supremum of the outage probability in (4.234b) is used to meet the QoS in the worst case, which makes problem (4.234) intractable and nonconvex. To tackle this difficulty, two relaxation methods are considered, MI and the duality method. Simulations show that the duality method has better performance, but the Markov method has less computational complexity. In the sequel, these methods are explained.

Markov Method When MI is used, an upper bound on the outage probability is

$$\Pr\{\text{MSE}_k \geq \varepsilon_k\} \leq \frac{\mathbb{E}\{\text{MSE}_k\}}{\varepsilon_k}, \tag{4.235}$$

where $\mathbb{E}\{\text{MSE}_k\} = \|[\text{vec}((\mathbf{G}^{\text{T}} \otimes \mathbf{F}_k)\Sigma_k^{\frac{1}{2}})^{\text{T}}\text{vec}(\mathbf{F}_k\overline{\mathbf{H}}_k\mathbf{G} - \mathbf{D}_k)^{\text{T}}\text{vec}(\mathbf{R}_k^{\frac{1}{2}}\mathbf{F}_k^{\text{H}})^{\text{T}}]\|_2^2$ and $\text{vec}(\hat{\mathbf{H}}_k) \sim A(\mathbf{0}, \Sigma_k)$. Accordingly, problem (4.234) can be approximated as

$$\min_{\mathbf{G},\, \mathbf{F}_k,\, \forall k} \quad \text{tr}(\mathbf{G}\mathbf{G}^{\text{H}}), \tag{4.236a}$$

$$\text{subject to} \quad \|[\text{vec}((\mathbf{G}^{\text{T}} \otimes \mathbf{F}_k)\Sigma_k^{\frac{1}{2}})^{\text{T}}\text{vec}(\mathbf{F}_k\overline{\mathbf{H}}_k\mathbf{G} - \mathbf{D}_k)^{\text{T}}\text{vec}(\mathbf{R}_k^{\frac{1}{2}}\mathbf{F}_k^{\text{H}})^{\text{T}}]\|_2^2 \leq \varepsilon_k\zeta_k, \quad \forall k. \tag{4.236b}$$

Although the preceding problem is tractable, it is still nonconvex. To find a near-optimal solution, it is divided into two convex subproblems, and each subproblem is iteratively solved until convergence. In the first subproblem, it is assumed that all equalizer matrices are fixed, and in the second subproblem, for a given precoder matrix, the equalizer matrices are obtained. In the first subproblem, for a given equalizer matrix, by using the epigraph method, problem (4.236) can be written as the following SOCP problem:

$$\min_{\mathbf{G}} \quad t, \tag{4.237a}$$

$$\text{subject to} \begin{cases} \mathrm{tr}(\mathbf{G}\mathbf{G}^{\mathrm{H}}) \le t, & (4.237\mathrm{b}) \\ \|[\mathrm{vec}((\mathbf{G}^{\mathrm{T}} \otimes \mathbf{F}_k)\Sigma_k^{\frac{1}{2}})^{\mathrm{T}} \mathrm{vec}(\mathbf{F}_k \overline{\mathbf{H}}_k \mathbf{G} - \mathbf{D}_k)^{\mathrm{T}} & (4.237\mathrm{c}) \\ \mathrm{vec}(\mathbf{R}_k^{\frac{1}{2}} \mathbf{F}_k^{\mathrm{H}})^{\mathrm{T}}]\|_2 \le \sqrt{\varepsilon_k \zeta_k}, \quad \forall k, & \end{cases}$$

where t is the slack variable. In the second subproblem, it is enough to minimize the left-hand side of (4.236b) with respect to the equalizer matrix, which is formulated as

$$\min_{\mathbf{F}_k} \quad \|\mathbf{G}^{\mathrm{T}} \otimes \mathbf{F}_k)\Sigma_k^{\frac{1}{2}})\|_F^2 + \|\mathbf{F}_k \overline{\mathbf{H}}_k \mathbf{G} - \mathbf{D}_k\|_F^2 + \|\mathbf{R}_k^{\frac{1}{2}} \mathbf{F}_k^{\mathrm{H}}\|_F^2. \tag{4.238}$$

The first term of the preceding cost function is rewritten as

$$\|\mathbf{G}^{\mathrm{T}} \otimes \mathbf{F}_k)\Sigma_k^{\frac{1}{2}})\|_F^2 = \mathrm{tr}\Big(\sum_{i=1}^{N}\sum_{j=1}^{N} g_{ij}\Sigma_k^{ji}\mathbf{F}_k^{\mathrm{H}}\mathbf{F}_k\Big), \tag{4.239}$$

where g_{ij} is the (i, j)th element of the matrix $\mathbf{G}^*\mathbf{G}^{\mathrm{T}}$, and Σ_k^{ji} is the (j, i)th $M_k \times M_k$ subblock of matrix Σ_k. Substituting (4.239) into (4.237) and taking the derivative of the cost function with respect to $\mathbf{F}_k$, we obtain the optimal equalizer

$$\mathbf{F}_k = (\overline{\mathbf{H}}_k \mathbf{G}\mathbf{D}_k^{\mathrm{H}})^{\mathrm{H}} \left(\overline{\mathbf{H}}_k \mathbf{G}\mathbf{G}^{\mathrm{H}} \overline{\mathbf{H}}_k^{\mathrm{H}} + \mathbf{R}_k + \sum_{i=1}^{N}\sum_{j=1}^{N} g_{ij}\Sigma_k^{ji} \right)^{-1}. \tag{4.240}$$

Markov Method for Robust Transceiver Design

Step 0: Initialization: Choose a feasible solution $[\mathbf{G}(0), \mathbf{F}_1(0), \ldots, \mathbf{F}_K(0)]$ to problem (4.236).
For $l = 1, \ldots$
Step 1: Update $\mathbf{G}(l)$ by solving the convex problem (4.237);
Step 2: Update $\mathbf{F}_k(l), \quad \forall k$ by solving the convex problem (4.238);
Step 3: Stop if $\mathbf{G}(l-1)^{\mathrm{H}}\mathbf{G}(l-1) - \mathbf{G}(l)^{\mathrm{H}}\mathbf{G}(l) \le \epsilon$, where ϵ is the threshold.

Duality Method The MSE is a function of unknown $\mathbf{F}_k$ and $\mathbf{G}$, and both depend on $\hat{\mathbf{H}}_k$. Thus, the MSE is a sum of correlated elements. Although uncertainty in channel estimation $\hat{\mathbf{H}}_k$ is arbitrarily distributed, from the generalized weak-convergence theorem [64], the MSE is in fact not arbitrarily distributed. Hence, the Markov method is conservative and MI is not tight. To find a near-optimal solution, the duality method is used, and the outage probability constraint is reformulated as

$$\sup_{f(\mathbf{x}_k)} \quad \Pr\{\psi(\mathbf{x}_k) \geq \varepsilon_k\}, \tag{4.241a}$$

$$\text{subject to} \begin{cases} \displaystyle\int_{\mathbf{x}_k \in \mathbb{C}^{NM_k}} f(\mathbf{x}_k) d\mathbf{x}_k = 1, & (4.241\text{b}) \\ \mathbf{E}\{\mathbf{x}_k\} = \mathbf{0}, & (4.241\text{c}) \\ \mathbf{E}\{\mathbf{x}_k \mathbf{x}_k^{\mathrm{H}}\} = \Sigma_k, & (4.241\text{d}) \end{cases}$$

where $\psi_k(\mathbf{x}_k) = \mathrm{MSE}_k$, $\mathbf{x}_k = \mathrm{vec}(\hat{\mathbf{H}}_k)$, and $f(\mathbf{x}_k)$ is the probability density function (pdf) of $\mathbf{x}_k$. The Lagrangian function for this problem is

$$L_k(f(\mathbf{x}_k), \alpha_k, \eta_k, \Xi_k) = \Pr\{\psi(\mathbf{x}_k) \geq \varepsilon_k\} + \alpha_k \left(1 - \int_{\mathbf{x}_k \in \mathbb{C}^{NM_k}} f(\mathbf{x}_k) d\mathbf{x}_k\right) - \eta_k^{\mathrm{H}} \mathbf{E}\{\mathbf{x}_k\}$$
$$+ \mathrm{tr}(\Xi_k(\Sigma_k - \mathbf{E}\{\mathbf{x}_k \mathbf{x}_k^{\mathrm{H}}\})), \tag{4.242a}$$

where α_k, η_k, Ξ_k are Lagrangian multipliers and $\Xi_k = \Xi_k^{\mathrm{H}}$. The Lagrangian dual function is

$$g_k(\alpha_k, \eta_k, \Xi_k) = \sup_{f(\mathbf{x}_k) \geq 0} L_k(f(\mathbf{x}_k), \alpha_k, \eta_k, \Xi_k) \tag{4.243a}$$

$$= \begin{cases} \alpha_k + \mathrm{tr}(\Xi_k^{\mathrm{H}} \Sigma_k), & \text{if } A_k \geq 0, \forall \mathbf{x}_k : \psi_k(\mathbf{x}_k) < \varepsilon_k \text{ and } A_k > 1, \forall \mathbf{x}_k : \psi_k(\mathbf{x}_k) \geq \varepsilon_k \\ +\infty, & \text{otherwise}, \end{cases} \tag{4.243b}$$

where $A_k = \alpha_k + \eta_k^{\mathrm{H}} \mathbf{x}_k + \mathrm{tr}(\Xi_k^{\mathrm{H}} \mathbf{x}_k \mathbf{x}_k^{\mathrm{H}})$. The corresponding dual problem of (4.241) is

$$\min_{\alpha_k, \eta_k, \Xi_k} \quad \alpha_k + \mathrm{tr}(\Xi_k^{\mathrm{H}} \Sigma_k), \tag{4.244a}$$

$$\text{subject to} \begin{cases} \alpha_k + \eta_k^{\mathrm{H}} \mathbf{x}_k + \mathrm{tr}(\Xi_k^{\mathrm{H}} \mathbf{x}_k \mathbf{x}_k^{\mathrm{H}}) \geq 0, & \forall \mathbf{x}_k : \mathbf{x}_k \in \mathbb{C}^{NM_k}, \quad (4.244\text{b}) \\ \alpha_k + \eta_k^{\mathrm{H}} \mathbf{x}_k + \mathrm{tr}(\Xi_k^{\mathrm{H}} \mathbf{x}_k \mathbf{x}_k^{\mathrm{H}}) > 1, & \forall \mathbf{x}_k : \psi(\mathbf{x}_k) \geq \varepsilon_k, \quad (4.244\text{c}) \\ \Xi_k = \Xi_k^{\mathrm{H}}. & (4.244\text{d}) \end{cases}$$

The compact form of (4.244) is

$$\min_{\mathbf{Z}_k} \quad \mathrm{tr}(\mathbf{Z}_k \tilde{\Sigma}_k), \tag{4.245a}$$

$$\text{subject to} \begin{cases} \mathbf{Z}_k \succeq \mathbf{0}, & (4.245\text{b}) \\ \mathbf{u}_k^{\mathrm{H}} \mathbf{Z}_k \mathbf{u}_k - 1 > 0, \ \ \forall \ \hat{\mathbf{H}}_k : \|\mathbf{F}_k(\overline{\mathbf{H}}_k + \hat{\mathbf{H}}_k)\mathbf{G} - \mathbf{D}_k\|_F^2 + \mathrm{tr}(\mathbf{F}_k \mathbf{R}_k \mathbf{F}_k^{\mathrm{H}}) \geq \varepsilon_k, & (4.245\text{c}) \end{cases}$$

where $\mathbf{Z}_k = \begin{bmatrix} \Xi_k^{\mathrm{H}} & \frac{1}{2}\eta_k \\ \frac{1}{2}\eta_k^{\mathrm{H}} & \alpha_k \end{bmatrix}$, $\tilde{\Sigma}_k = \begin{bmatrix} \Sigma_k & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}$, and $\mathbf{u}_k = [\mathbf{x}_k^{\mathrm{T}} \ \ 1]^{\mathrm{T}}$. Since in (4.245c) there are infinitely many possible channel realizations, it is an infinite constraint. Consequently, although problem (4.245) has no probabilistic constraints, it is still an infinite constrained problem, which is difficult to solve and should be transformed into a finite constrained problem. To do so, the Frobenius norm in (4.245c) is replaced by the spectral norm,

$$\|\mathbf{F}_k(\overline{\mathbf{H}}_k + \hat{\mathbf{H}}_k)\mathbf{G} - \mathbf{D}_k\|_F^2 = \mathbf{u}_k^{\mathrm{H}} \mathbf{Q}_k^{\mathrm{H}} \mathbf{Q}_k \mathbf{u}_k, \tag{4.246}$$

where $\mathbf{Q}_k = [(\mathbf{G}^{\mathrm{T}} \otimes \mathbf{F}_k)\mathrm{vec}(\mathbf{F}_k \overline{\mathbf{H}}_k \mathbf{G} - \mathbf{D}_k)]$. Using (4.246), the S-lemma, and SC, we transform (4.245c) into LMI form:

$$\exists \beta_k > 0 : \begin{bmatrix} \beta_k \mathbf{Z}_k + \mathrm{diag}\{\mathbf{0}, \varepsilon_k - \mathrm{tr}(\mathbf{F}_k \mathbf{R}_k \mathbf{F}_k^{\mathrm{H}}) - \beta_k\} & \mathbf{Q}_k^{\mathrm{H}} \\ \mathbf{Q}_k & \mathbf{I}_{LL_k} \end{bmatrix} \succeq \mathbf{0}. \tag{4.247}$$

Using the preceding expressions, we convert stochastic problem (4.241) into the following deterministic finite constrained problem:

$$\min_{\beta_k, \tilde{\mathbf{Z}}_k} \quad \frac{\mathrm{tr}(\tilde{\mathbf{Z}}_k \tilde{\Sigma}_k)}{\beta_k}, \tag{4.248a}$$

$$\text{subject to} \begin{cases} \tilde{\mathbf{Z}}_k \succeq \mathbf{0}, \beta_k > 0, & (4.248\text{b}) \\ \begin{bmatrix} \tilde{\mathbf{Z}}_k + \mathrm{diag}\{\mathbf{0}, \varepsilon_k - \mathrm{tr}(\mathbf{F}_k \mathbf{R}_k \mathbf{F}_k^{\mathrm{H}}) - \beta_k\} & \mathbf{Q}_k^{\mathrm{H}} \\ \mathbf{Q}_k & \mathbf{I}_{LL_k} \end{bmatrix} \succeq \mathbf{0}, & (4.248\text{c}) \end{cases}$$

where $\tilde{\mathbf{Z}}_k = \mathbf{Z}_k \beta_k$. Consequently, bilevel optimization problem (4.234) can be equivalently replaced by the following single-level optimization problem:

$$\min_{\mathbf{G}, \mathbf{F}_k, \beta_k, \tilde{\mathbf{Z}}_k, \forall k} \quad \mathrm{tr}(\mathbf{G}\mathbf{G}^{\mathrm{H}}), \tag{4.249a}$$

$$\text{subject to} \begin{cases} \tilde{\mathbf{Z}}_k \succeq \mathbf{0}, \beta_k > 0, \dfrac{\mathrm{tr}(\tilde{\mathbf{Z}}_k \tilde{\Sigma}_k)}{\beta_k} \leq \zeta_k, \forall k, & (4.249\text{b}) \\ \begin{bmatrix} \beta_k \tilde{\mathbf{Z}}_k + \mathrm{diag}\{\mathbf{0}, \varepsilon_k - \mathrm{tr}(\mathbf{F}_k \mathbf{R}_k \mathbf{F}_k^{\mathrm{H}}) - \beta_k\} & \mathbf{Q}_k^{\mathrm{H}} \\ \mathbf{Q}_k & \mathbf{I}_{LL_k} \end{bmatrix} \succeq \mathbf{0}, \ \ \forall k. & (4.249\text{c}) \end{cases}$$

As with the Markov method, problem (4.249) is solved by convergence-guaranteed iterative algorithms applied to two convex subproblems. Note that the global optimality of the two aforementioned approaches for multiuser MIMO is not achievable, but the authors prove that for the special case of single-user MIMO, the robust optimization problem is convex, and global optimality is attainable.

Example 2 In [29], a network with K source–destination pairs is considered that communicates through a set of L distributed relays. It is assumed that each node has a single antenna, and there is no direct link between each pair. It is also assumed that the CSI values of the second hop are uncertain and the error in channel estimation is a Gaussian random variable with known distribution. Robust beamforming vectors are designed to minimize the average transmit power at relays subject to SINR outage probability constraints. The SINR at destination k is

$$\mathrm{SINR}_k = \frac{\left|\sum_{l=1}^{L} h_{ki} w_i g_{ik}\right|^2 p_k}{\sigma^2 + \sum_{l=1}^{L} |h_{ki} w_i \sigma|^2 + \sum_{l=1, l\neq k}^{K} \left|\sum_{i=1}^{L} h_{ki} w_i g_{il}\right|^2 p_l}, \tag{4.250}$$

where g_{ik} is the Rayleigh flat fading channel coefficient from source k to relay i, p_k is the transmit power of source k, h_{ki} is the Rayleigh flat fading channel coefficient from relay i to destination k, w_i is the beamforming weight for relay i, and σ^2 is the noise power.

Channel gains for the second hop are assumed to be imperfect, modeled by $\mathbf{h}_k = \overline{\mathbf{h}}_k + \hat{\mathbf{h}}_k$, where $\mathbf{h}_k = [h_{k1}, \ldots, h_{kL}]$, $\overline{\mathbf{h}}_k$ is the exact value of $\mathbf{h}_k$, and $\hat{\mathbf{h}}_k \sim CN(\mathbf{0}, \mathbf{Q}_k)$, where $\mathbf{Q}_k$ is the variance of $\hat{\mathbf{h}}_k$. The robust optimization problem is

$$\min_{\mathbf{w}} \quad \sum_{i=1}^{L} |w_i|^2 \left(\sum_{l=1}^{K} |g_{il}|^2 p_l + \sigma^2 \right) \tag{4.251a}$$

$$\text{subject to} \quad \Pr\left\{ \mathrm{SINR}_k \geq \gamma_k \right\} \geq 1 - \rho_k, \quad k = 1, \ldots, K, \tag{4.251b}$$

where γ_k is the acceptable SINR threshold, and ρ_k is the maximum allowable outage probability. Since the closed-form expression for the pdf and cumulative distribution function (CDF) of SINR_k in (4.250) are very difficult to obtain, the closed-form expression for the nonoutage probability constraint (4.251b) cannot be easily obtained. Therefore, problem (4.251) is intractable. To tackle this issue via relaxation methods, two approaches are proposed. In the first approach, using the Bernstein-type inequality, a lower bound on the nonoutage probability constraint is obtained. Then, via the SDR method, the approximated problem is transformed into SDP form. In the second method, instead of the Bernstein-type inequality, the SP is used.

Using the Bernstein-type inequality and SDR and introducing auxiliary variables ζ_k, x_k, and y_k, the relaxed problem for (4.251) is

$$\min_{\mathbf{W}} \quad \operatorname{tr}\left(\mathbf{W}\left(\sum_{k=1}^{K} \mathbf{G}_k \mathbf{G}_k^{\mathrm{H}} + \sigma^2 \mathbf{I}\right)\right), \tag{4.252a}$$

$$\text{subject to} \begin{cases} 1 - e^{-\zeta_k} \geq 1 - \rho_k, & \forall k = 1, \ldots, K, \quad (4.252\text{b}) \\ x_k + \zeta_k y_k \leq \operatorname{tr}(\mathbf{A}_k) + c_k - \sigma^2, & \forall k = 1, \ldots, K, \quad (4.252\text{c}) \\ \sqrt{2\zeta_k}\sqrt{\|\mathbf{A}_k\|_F^2 + 2\|\mathbf{z}_k\|^2} \leq x_k, & \forall k = 1, \ldots, K, \quad (4.252\text{d}) \\ y_k \mathbf{I} + \mathbf{A}_k \succeq \mathbf{0}, & \forall k = 1, \ldots, K, \quad (4.252\text{e}) \\ \operatorname{rank}(\mathbf{W}) = 1, & (4.252\text{f}) \end{cases}$$

where $\mathbf{G}_k = \sqrt{p_k}\operatorname{diag}\{g_{1k}, \ldots, g_{Lk}\}$, $\mathbf{W} = \mathbf{w}\mathbf{w}^{\mathrm{H}}$, $\mathbf{A}_k = \mathbf{Q}_k^{\frac{1}{2}} \tilde{\mathbf{A}}_k \mathbf{Q}_k^{\frac{1}{2}}$, $\mathbf{z}_k = \mathbf{Q}_k^{\frac{1}{2}} \tilde{\mathbf{A}}_k \overline{\mathbf{h}}_k$, $c_k = \overline{\mathbf{h}}_k^{\mathrm{H}} \tilde{\mathbf{A}}_k \overline{\mathbf{h}}_k$, $\tilde{\mathbf{A}}_k = \frac{\mathbf{v}_k \mathbf{v}_k^{\mathrm{H}}}{\gamma_k} - \sum_{l=1, l \neq k} \mathbf{v}_{\mathrm{a}} \mathbf{v}_{\mathrm{a}}^{\mathrm{H}} - \sigma^2 \operatorname{diag}\{W_{11}, \ldots, W_{LL}\}$, and $\mathbf{v}_k = \mathbf{G}_k \mathbf{w}$. In (4.252b), the smallest ζ_k that minimizes the transmit power is $\zeta_k = \ln \rho_k$. Due to rank constraint, problem (4.252) is still nonconvex. To cope with this issue, the rank constraint is removed and problem (4.252) is transformed into SDP form. Then, using randomization, the rank 1 solution is obtained.

In the second approach, using the SP, a closed-form lower bound on the nonoutage probability is obtained. Subsequently, following rank relaxation, a conservative reformulation of problem (4.251) is

$$\min_{\mathbf{W}} \quad \operatorname{tr}\left(\mathbf{W}\left(\sum_{k=1}^{K} \mathbf{G}_k \mathbf{G}_k^{\mathrm{H}} + \sigma^2 \mathbf{I}\right)\right), \tag{4.253a}$$

$$\text{subject to} \quad \begin{bmatrix} \mathbf{A}_k + \lambda_k \mathbf{I} & \mathbf{z}_k \\ \mathbf{z}_k^{\mathrm{H}} & \tilde{c}_k \end{bmatrix} \succeq \mathbf{0}, \quad \forall k = 1, \ldots, K, \tag{4.253b}$$

where $\tilde{c}_k = \overline{\mathbf{h}}_k^{\mathrm{H}} \tilde{\mathbf{A}}_k \overline{\mathbf{h}}_k - \lambda_k \tilde{r}_k^2 - \sigma^2$, $\tilde{r}_k = \sqrt{\frac{\mathrm{CDF}^{-1}(1-\rho_k)}{2}}$ in which $\mathrm{CDF}^{-1}(x)$ is the inverse CDF of a chi-squared random variable, and λ_k is a nonnegative auxiliary random variable. Simulation results show that the first method outperforms the second one and yields tighter bounds.

4.3.2.2 Overview of Other Works on Robust Relay-Assisted and Beamforming Systems

An outage-constrained robust transmit power optimization for multiuser MISO downlink, where a multi-antenna base station communicates with multiple single-antenna users, is studied in [37]. It is assumed that each user decodes its signal and other users' signals are considered interference. In addition, CSI values are uncertain and error is modeled by a Gaussian random variable. The objective is to minimize the total transmit power of the base station subject to the constraint on each user's rate outage, where the covariance matrix of the transmitted signal by each user is the optimization variable. The optimization problem is nonconvex and intractable, and an effort is made to find a tight, convex, and efficiently computable upper bound for the constraint on the rate outage. Using relaxation methods, including the Bernstein-type inequality, SP, SDP, and LDI, three convex approaches are proposed. The first approach is based on the worst-case robust method, the second approach uses Bernstein-type inequality, and the third approach uses LDI. Simulations show that the second method has superior performance, but its computational complexity is high, and the third approach outperforms the first one.

Power allocation strategies for the downlink of a multi-antenna base station with multiple single-antenna users is studied in [42], where the CSI error is assumed to have a Gaussian distribution. The objective is to minimize the transmit power subject to MSE outage probability and the constraint on the maximum transmit power. Using the VPI, the stochastic uncertainty is eliminated in a conservative manner, and the problem is converted into a deterministic one. The approximated problem is solved using the theory of interference functions [65].

Table 4.4 summarizes the aforementioned works on robust transmission in relay-assisted and beamforming systems.

4.3.2.3 Robust Transmission in Cognitive Radio Networks

Example 1 A two-tier heterogeneous network that includes a macro-cell and a femto-cell is considered in [27]. The macro base station and the closed-access femto base station are equipped with N_{M} and N_{F} antennas, respectively. The macro-cell shares the downlink spectrum with the femto-cell and serves a single-antenna macro-user with QoS guarantee. The femto-cell serves a single-antenna femto-user with "best effort." The femto-user's SINR is

$$\mathrm{SINR}_{\mathrm{F}} = \frac{|\mathbf{h}_{\mathrm{FF}}^{\mathrm{H}} \mathbf{w}_{\mathrm{F}}|^2}{\sigma_{\mathrm{F}}^2 + |\mathbf{h}_{\mathrm{FM}}^{\mathrm{H}} \mathbf{w}_{\mathrm{M}}|^2}, \tag{4.254}$$

Table 4.4 Summary of existing works on robust relay-assisted and beamformning systems

Reference	SP	SDR	BT-BTI	LR	BT-MI	SC	EF	BT-LDI	BT-VPI
[29]	√	√	√	–	–	–	–	–	–
[31]	–	–	–	√	√	√	√	–	–
[37]	√	√	√	–	–	–	–	√	–
[42]	–	–	–	–	–	–	–	–	√

where $\mathbf{h}_{\mathrm{FF}} \in \mathbb{C}^{N_{\mathrm{F}}}$ and $\mathbf{h}_{\mathrm{FM}} \in \mathbb{C}^{N_{\mathrm{M}}}$ are the channel gains between the femto base station and the femto-user and the channel gain between the macro base station and the femto-user, respectively, $\mathbf{w}_{\mathrm{F}} \in \mathbb{C}^{N_{\mathrm{F}}}$ and $\mathbf{w}_{\mathrm{M}} \in \mathbb{C}^{N_{\mathrm{M}}}$ are the beamforming vectors for the femto-user and the macro-user, respectively, and σ_{F}^2 is the additive noise power at the femto-user.

Depending on the availability of the CSI at the femto base station, two cases are considered. In the first case, called "no CSI feedback," it is assumed that the femto base station only has the channels' statistics. In the second case, it is assumed that uncertain values of $\mathbf{h}_{\mathrm{FF}}$ and $\mathbf{h}_{\mathrm{FM}}$, denoted by $\overline{\mathbf{h}}_{\mathrm{FF}}$ and $\overline{\mathbf{h}}_{\mathrm{FM}}$, respectively, are available to the femto base station. When partial CSI is available, the performance is better than that of no CSI feedback. The stochastic robust problem is

$$\min_{\mathbf{w}_{\mathrm{F}}} \quad \|\mathbf{w}_{\mathrm{F}}\|^2, \tag{4.255a}$$

$$\text{subject to} \begin{cases} \Pr\left\{\mathrm{SINR}_{\mathrm{F}} \geq \gamma_{\mathrm{F}}\right\} \geq 1-\rho_{\mathrm{F}}, & (4.255\mathrm{b}) \\ \Pr\left\{|\mathbf{h}_{\mathrm{MF}}^{\mathrm{H}}\mathbf{w}_{\mathrm{F}}|^2 \leq \varepsilon_{\mathrm{M}}\right\} \geq 1-\rho_{\mathrm{M}}, & (4.255\mathrm{c}) \end{cases}$$

where γ_{F} is the minimum allowable SINR of the femto-user, ε_{M} is the maximum tolerable interference caused by the femto base station on the macro-user, and ρ_{F} and ρ_{M} are the maximum allowable outage probabilities for the SINR and the interference, respectively.

Since there is no feedback from the macro-user to the femto base station, it is assumed that only the statistics of the channel between the macro-user and the femto base station, denoted by $\mathbf{h}_{\mathrm{MF}} \in \mathbb{C}^{N_{\mathrm{F}}}$, are available to the femto-cell, and $\mathbf{h}_{\mathrm{FF}}$ and $\mathbf{h}_{\mathrm{FM}}$ are modeled by

$$\mathbf{h}_{\mathrm{FF}} = \overline{\mathbf{h}}_{\mathrm{FF}} + \hat{\mathbf{h}}_{\mathrm{FF}}, \quad \text{and} \quad \mathbf{h}_{\mathrm{FM}} = \overline{\mathbf{h}}_{\mathrm{FM}} + \hat{\mathbf{h}}_{\mathrm{FM}}, \tag{4.256}$$

where $\overline{\mathbf{h}}_{\mathrm{FF}} \in \mathbb{C}^{N_{\mathrm{F}}}$ and $\overline{\mathbf{h}}_{\mathrm{FM}} \in \mathbb{C}^{N_{\mathrm{M}}}$ are the exact vectors, and $\hat{\mathbf{h}}_{\mathrm{FF}} \in \mathbb{C}^{N_{\mathrm{F}}}$ and $\hat{\mathbf{h}}_{\mathrm{FM}} \in \mathbb{C}^{N_{\mathrm{M}}}$ are the corresponding error vectors. Let $\hat{\mathbf{h}}_{\mathrm{FF}} \sim \mathcal{CN}(\mathbf{0}, \mathbf{C}_{\mathrm{e,FF}})$ and $\hat{\mathbf{h}}_{\mathrm{FM}} \sim \mathcal{CN}(\mathbf{0}, \mathbf{C}_{\mathrm{e,FM}})$, where the channel error covariance matrices, denoted by $\mathbf{C}_{\mathrm{e,FF}}$ and $\mathbf{C}_{\mathrm{e,FM}}$, are positive definite. Now, problem (4.255) can be written as

$$\min_{\mathbf{w}_{\mathrm{F}}} \quad \|\mathbf{w}_{\mathrm{F}}\|^2, \tag{4.257a}$$

$$\text{subject to} \begin{cases} \Pr\left\{ \dfrac{|(\overline{\mathbf{h}}_{\mathrm{FF}} + \hat{\mathbf{h}}_{\mathrm{FF}})^{\mathrm{H}}\mathbf{w}_{\mathrm{F}}|^2}{\sigma_{\mathrm{F}}^2 + |(\overline{\mathbf{h}}_{\mathrm{FM}} + \hat{\mathbf{h}}_{\mathrm{FM}})^{\mathrm{H}}\mathbf{w}_{\mathrm{M}}|^2} \geq \gamma_{\mathrm{F}} \right\} \geq 1-\rho_{\mathrm{F}}, & (4.257\mathrm{b}) \\ \Pr\left\{ |\mathbf{h}_{\mathrm{MF}}^{\mathrm{H}}\mathbf{w}_{\mathrm{F}}|^2 \leq \varepsilon_{\mathrm{M}} \right\} \geq 1-\rho_{\mathrm{M}}. & (4.257\mathrm{c}) \end{cases}$$

This problem is difficult to solve. To tackle this issue, the SDR method is applied, and the reformulated problem is conservatively approximated using a Bernstein-type inequality. Define

$$\hat{\mathbf{h}}_{\mathrm{FF}} = \mathbf{C}_{\mathrm{e,FF}}^{\frac{1}{2}}\mathbf{v}_{\mathrm{FF}}, \quad \text{and} \quad \hat{\mathbf{h}}_{\mathrm{FM}} = \mathbf{C}_{\mathrm{e,FM}}^{\frac{1}{2}}\mathbf{v}_{\mathrm{FM}}, \tag{4.258}$$

where $\mathbf{v}_{\mathrm{FF}} \sim CN(\mathbf{0}, \mathbf{I}_{N_{\mathrm{F}}})$ and $\mathbf{v}_{FM} \sim CN(\mathbf{0}, \mathbf{I}_{N_{\mathrm{M}}})$, and let

$$\bar{\mathbf{h}} = [\bar{\mathbf{h}}_{\mathrm{FF}} \quad \bar{\mathbf{h}}_{\mathrm{FM}}]^{\mathrm{T}}, \quad \mathbf{v} = [\mathbf{v}_{\mathrm{FF}} \quad \mathbf{v}_{\mathrm{FM}}]^{\mathrm{T}}, \quad \text{and} \quad \mathbf{C}^{\frac{1}{2}} = \mathrm{diag}\{\mathbf{C}_{\mathrm{e,FF}}^{\frac{1}{2}}, \ \mathbf{C}_{\mathrm{e,FM}}^{\frac{1}{2}}\}, \tag{4.259}$$

where $\mathbf{v} \sim CN(\mathbf{0}, \mathbf{I}_{N_{\mathrm{F}}+N_{\mathrm{FM}}})$. By utilizing SDR, problem (4.257) is rewritten as

$$\min_{\mathbf{W}_{\mathrm{F}} \in \mathbb{H}_{++}} \quad \mathrm{tr}(\mathbf{W}_{\mathrm{F}}), \tag{4.260a}$$

$$\text{subject to} \begin{cases} \Pr\left\{\mathbf{v}^{\mathrm{H}}\mathbf{A}\mathbf{v} + 2\mathrm{Re}\{\mathbf{v}^{\mathrm{H}}\mathbf{B}\} \geq s\right\} \geq 1 - \rho_{\mathrm{F}}, & (4.260b) \\ \mathrm{tr}(\mathbf{C}_{\mathrm{h,MF}}\mathbf{W}_{\mathrm{F}}) \leq \dfrac{\varepsilon_{\mathrm{M}}}{\ln \frac{1}{\rho_{\mathrm{M}}}}, & (4.260c) \end{cases}$$

where $\mathbf{A} = \mathbf{C}^{\frac{1}{2}}\mathbf{W}\mathbf{C}^{\frac{1}{2}}$, $\mathbf{B} = \mathbf{C}^{\frac{1}{2}}\mathbf{W}\bar{\mathbf{h}}$, $s = \sigma_{\mathrm{F}}^2 - \bar{\mathbf{h}}^{\mathrm{H}}\mathbf{W}\bar{\mathbf{h}}$, $\mathbf{W} = \mathrm{diag}\{\frac{1}{\gamma_F}\mathbf{W}_{\mathrm{F}}, \ -\mathbf{w}_{\mathrm{M}}\mathbf{w}_{\mathrm{M}}^{\mathrm{H}}\}$, and $\mathbf{C}_{\mathrm{h,FF}}$ is the channel covariance matrix of $\mathbf{h}_{\mathrm{FF}}$. A closed form for the pdf and CDF of $\mathbf{v}^{\mathrm{H}}\mathbf{A}\mathbf{v} + 2\mathrm{Re}\{\mathbf{v}^{\mathrm{H}}\mathbf{B}\}$ is difficult to obtain, and the probability in (4.260b) does not have a closed-form expression. Thus, problem (4.260) is still intractable. A tractable approximation to problem (4.257), obtained using Bernstein-type inequality, is

$$\min_{\mathbf{W}_{\mathrm{F}} \in \mathbb{H}_{++}^{N_{\mathrm{F}}}, \, x \in \mathbb{R}} \quad \mathrm{tr}(\mathbf{W}_{\mathrm{F}}), \tag{4.261a}$$

$$\text{subject to} \begin{cases} \dfrac{1}{\gamma_{\mathrm{F}}}\mathrm{tr}\left(\left(\mathbf{C}_{\mathrm{e,FF}} + \bar{\mathbf{h}}_{\mathrm{FF}}\bar{\mathbf{h}}_{\mathrm{FF}}^{\mathrm{H}}\right)\mathbf{W}_{\mathrm{F}}\right) - \sqrt{2\delta}x \geq \sigma_{\mathrm{F}}^2 & (4.261b) \\ \quad + \mathbf{w}_{\mathrm{M}}^{\mathrm{H}}\left((1+\delta)\mathbf{C}_{\mathrm{e,FF}} + \bar{\mathbf{h}}_{\mathrm{FM}}\bar{\mathbf{h}}_{\mathrm{FM}}^{\mathrm{H}}\right)\mathbf{w}_{\mathrm{M}}, & \\ \mathrm{tr}(\mathbf{C}_{\mathrm{h,MF}}\mathbf{W}_{\mathrm{F}}) \leq \dfrac{\varepsilon_{\mathrm{M}}}{\ln \frac{1}{\rho_{\mathrm{M}}}}, & (4.261c) \\ \dfrac{1}{\gamma_{\mathrm{F}}}\|\mathbf{D}\| \leq x, & (4.261d) \end{cases}$$

where $\delta = -\ln \rho_{\mathrm{F}}$, $\mathbf{D} = \left[\mathrm{vec}\left(\mathbf{C}_{\mathrm{e,FF}}^{\frac{1}{2}}\mathbf{W}_F\mathbf{C}_{\mathrm{e,FF}}^{\frac{1}{2}}\right) \quad \sqrt{2}\mathrm{vec}\left(\mathbf{C}_{\mathrm{e,FF}}^{\frac{1}{2}}\mathbf{W}_F\bar{\mathbf{h}}_{\mathrm{FF}}\right) \quad \zeta_{\mathrm{FM}}\right]^{\mathrm{T}}$, and $\zeta_{\mathrm{FM}} = \gamma_{\mathrm{F}}\sqrt{\|\mathbf{C}_{\mathrm{e,FM}}^{\frac{1}{2}}\mathbf{w}_M\mathbf{w}_M^{\mathrm{H}}\mathbf{C}_{\mathrm{e,FM}}^{\frac{1}{2}}\|_{\mathrm{F}}^2 + 2\|\mathbf{C}_{\mathrm{e,FM}}^{\frac{1}{2}}\mathbf{w}_{\mathrm{M}}\mathbf{w}_{\mathrm{M}}^{\mathrm{H}}\bar{\mathbf{h}}_{\mathrm{FM}}\|^2}$. Problem (4.261) is convex and can be efficiently solved to obtain the globally optimal solution. In [27], a special case in which $\hat{\mathbf{h}}_{\mathrm{FF}}$ and $\mathbf{h}_{\mathrm{MF}}$ are independent and identically distributed (i.i.d.) is investigated. It is shown that the optimal solution to (4.257) can be obtained. The optimization problem for this case is

$$\min_{\iota_{\mathrm{F}} \geq 0} \quad \iota_{\mathrm{F}}, \tag{4.262a}$$

$$\text{subject to} \begin{cases} \frac{1}{2} \int_{0}^{+\infty} Q_1 \left(\frac{\sqrt{2}\|\bar{\mathbf{h}}_{\mathrm{FF}}\|}{\sigma_{\mathrm{e,FF}}}, \sqrt{\frac{\gamma_{\mathrm{F}}\left(\mathbf{w}_{\mathrm{M}}^{H}\mathbf{C}_{\mathrm{e,FM}}^{H}\mathbf{w}_{\mathrm{M}}x_{\mathrm{FM}} + 2\sigma_{\mathrm{F}}^{2}\right)}{\iota_{\mathrm{F}}\sigma_{\mathrm{e,FF}}^{2}}} \right) & (4.262b) \\ \exp\left(-\frac{x_{\mathrm{FM}} + \zeta_{\mathrm{FM}}}{2}\right) I_0\left(\sqrt{\zeta_{\mathrm{FM}} x_{\mathrm{FM}}}\right) dx_{\mathrm{FM}} \geq 1 - \rho_{\mathrm{F}}, \\ \sigma_{\mathrm{h,MF}}^{2}\iota_{\mathrm{F}} \leq \frac{\varepsilon_{\mathrm{M}}}{\ln \frac{1}{\rho_{\mathrm{M}}}}, & (4.262c) \end{cases}$$

where $\iota_{\mathrm{F}} = \|\mathbf{w}_{\mathrm{F}}\|^2$ and $\zeta_{\mathrm{FM}} = \frac{2|\bar{\mathbf{h}}_{\mathrm{FM}}^{H}\mathbf{w}_{\mathrm{M}}|^2}{\mathbf{w}_{\mathrm{M}}^{H}\mathbf{C}_{\mathrm{e,FM}}\mathbf{w}_{\mathrm{M}}}$. In constraint (4.262b), $Q_1(x, y)$ is the first-order Marcum's Q-function defined as [66]

$$Q_1(x, y) = \int_{y}^{+\infty} u \exp\left(-\frac{u^2 + x^2}{2}\right) I_0(xu)dx, \tag{4.263}$$

where $I_0(\cdot)$ is the zero-order modified Bessel function of the first kind. The optimal solution can be obtained by the simple and computationally efficient bisection method [1]. The no CSI case is discussed subsequently in Section 4.3.3 in this chapter.

Table 4.5 summarizes the aforementioned work on robust transmission in CRNs.

4.3.3 No CSI Feeback

In this section, we discuss the application of relaxation methods for resource allocation problems when there is no CSI feedback. The complexity of deploying this scheme is less than other schemes with some loss in performance.

Table 4.5 Summary of existing work on robust transmission in cognitive radio networks

Reference	SP	SDR	BT-BTI	LR	BT-TI	BT-Cauchy–Schwarz inequality	EF
[27]	–	√	√	–	–	–	–

Example 1 A cooperative network with L radio access points and K mobile users is considered in [28]. It is assumed that access point l has N_l antennas and each mobile user has a single antenna. It is also assumed that only channel statistics in the form of channel distribution information is available, that is, no CSI feedback. The objective is to minimize the total downlink transmit power subject to the constraint on the probability of nonoutage SINR. This is achieved via a general stochastic coordinated beamforming framework. To solve the optimization problem, the stochastic DC programming relaxation method is used, which makes the nonconvex optimization problem tractable and guarantees optimality. The SINR for mobile user k is

$$\Gamma_k = \frac{|\mathbf{h}_k^{\mathrm{H}}\mathbf{v}_k|^2}{\sum\limits_{i\neq k}|\mathbf{h}_k^{\mathrm{H}}\mathbf{v}_i|^2 + \sigma_k^2}, \tag{4.264}$$

where $\mathbf{h}_k = [\mathbf{h}_{k1}^{\mathrm{T}}, \ldots, \mathbf{h}_{kL}^{\mathrm{T}}]$ is the CSI vector for mobile user k, whose element $\mathbf{h}_{kl} \in \mathbb{C}^{N_l}$ is the CSI between radio access point l and mobile user k, and $\mathbf{v}_k = [\mathbf{v}_{1k}^{\mathrm{T}}, \ldots, \mathbf{v}_{Lk}^{\mathrm{T}}]$ is a vector whose element $\mathbf{v}_{lk} \in \mathbb{C}^{N_l}$ is the transmit beamforming vector from radio access unit l to mobile user k. The additive Gaussian noise power at mobile user k is σ_k^2.

The robust optimization problem is

$$\min_{\mathbf{v}\in\mathscr{V}} \quad \sum_{l=1}^{L}\sum_{k=1}^{K}\|\mathbf{v}_{lk}\|^2, \tag{4.265a}$$

$$\text{subject to} \quad \Pr\left\{\Gamma_k \geq \gamma_k, \quad \forall k = 1, \ldots, K\right\} \geq 1 - \varepsilon, \tag{4.265b}$$

where γ_k is the minimum required SINR, and ε is the maximum allowable outage probability. The convex feasible set of the beamforming vector is

$$\mathscr{V} = \left\{\mathbf{v}_{lk} \in \mathbb{C}^{N_l} : \sum_{k=1}^{K}\|\mathbf{v}_{lk}\|^2 \leq p_l^{\max}, \qquad \forall l, k\right\}, \tag{4.266}$$

where $p_l^{\max}$ is the maximum transmit power for radio access point l. Because of chance constraint (4.265b), problem (4.265) is intractable. To circumvent this difficulty, constraint (4.265b) is relaxed by way of DC approximation, and DC programming is used. Accordingly, problem (4.265) is rewritten as

$$\min_{\mathbf{v}\in\mathscr{V}} \quad \sum_{l=1}^{L}\sum_{k=1}^{K}\|\mathbf{v}_{lk}\|^2, \tag{4.267a}$$

$$\text{subject to} \quad \inf_{\nu>0} \frac{1}{\nu}[u(\mathbf{v}, \nu) - u(\mathbf{v}, 0)] \leq \varepsilon, \tag{4.267b}$$

where $u(\mathbf{v}, \nu) = \mathbb{E}\big\{\max_{k=1,\ldots,K+1}\{s_k(\mathbf{v}, \mathbf{h}, \nu)\}\big\}$, $s_k(\mathbf{v}, \mathbf{h}, \nu) = \nu + \sum_{i \neq k} \mathbf{v}_i^{\mathrm{H}} \mathbf{h}_k \mathbf{h}_k^{\mathrm{H}} \mathbf{v}_i + \sigma_k^2 + \sum_{i \neq k} \frac{1}{\gamma_i} \mathbf{v}_i^{\mathrm{H}} \mathbf{h}_i \mathbf{h}_i^{\mathrm{H}} \mathbf{v}_i$, $\forall k = 1, \ldots, K$, and $s_{K+1}(\mathbf{v}, \mathbf{h}, \nu) = \sum_{i=1}^{K} \frac{1}{\gamma_i} \mathbf{v}_i^{\mathrm{H}} \mathbf{h}_i \mathbf{h}_i^{\mathrm{H}} \mathbf{v}_i$. Note that relaxation by DC approximation does not affect the optimality of the solution, and the relaxed problem is equivalent to the original one.

Due to the existence of an infimum function in constraint (4.267b), it is difficult to solve the foregoing problem directly. However, since $\frac{1}{\nu}[u(\mathbf{v}, \nu) - u(\mathbf{v}, 0)]$ is nondecreasing in $\nu > 0$, the following κ-approximation problem can be solved instead of problem (4.267):

$$\min_{\mathbf{v} \in \mathscr{V}} \quad \sum_{l=1}^{L} \sum_{k=1}^{K} \|\mathbf{v}_{lk}\|^2, \tag{4.268a}$$

$$\text{subject to} \quad [u(\mathbf{v}, \kappa) - \kappa \varepsilon] - u(\mathbf{v}, 0) \leq 0, \tag{4.268b}$$

where $\kappa > 0$ is any fixed parameter small enough to approximate the original problem. Although a very small $\kappa > 0$ results in numerical stability and performance improvement, it imposes more computational complexity. To get around this difficulty, κ is treated as an optimization variable, and (4.268a) is rewritten as

$$\min_{\mathbf{v} \in \mathscr{V}, \kappa > 0} \quad \sum_{l=1}^{L} \sum_{k=1}^{K} \|\mathbf{v}_{lk}\|^2, \tag{4.269a}$$

$$\text{subject to} \quad [u(\mathbf{v}, \kappa) - \kappa \varepsilon] - u(\mathbf{v}, 0) \leq 0. \tag{4.269b}$$

To solve (4.269), an iterative SCA method is used, and a suboptimal solution is obtained using a Bernstein-type inequality, which provides a closed-form approximation for the chance constraint. This method is applicable when the CSI distribution is complex Gaussian, but it is not robust for other distributions.

Example 2 A no CSI scenario is also studied in [27], in addition to the partial CSI scenario that was explained in Section 4.3.2.3. In the no CSI scenario, problem (4.255) is formulated as a convex optimization problem using SDR. In doing so, a closed-form expression for probability functions in (4.255b) and (4.255c) is obtained. Assume

$$\mathbf{h}_{\mathrm{FF}} \sim CN(\mathbf{0}, \mathbf{C}_{\mathrm{h,FF}}), \quad \mathbf{h}_{\mathrm{FM}} \sim CN(\mathbf{0}, \mathbf{C}_{\mathrm{h,FM}}), \quad \text{and} \quad \mathbf{h}_{\mathrm{MF}} \sim CN(\mathbf{0}, \mathbf{C}_{\mathrm{h,MF}}), \tag{4.270}$$

where channel covariance matrices $\mathbf{C}_{\mathrm{h,FF}}$, $\mathbf{C}_{\mathrm{h,FM}}$, and $\mathbf{C}_{\mathrm{h,MF}}$ are positive definite. Thus, the random variables $|\mathbf{h}_{\mathrm{FF}}^{\mathrm{H}} \mathbf{w}_{\mathrm{F}}|^2$, $|\mathbf{h}_{\mathrm{FM}}^{\mathrm{H}} \mathbf{w}_{\mathrm{F}}|^2$, and $|\mathbf{h}_{\mathrm{FM}}^{\mathrm{H}} \mathbf{w}_{\mathrm{M}}|^2$ are independently and exponentially distributed with parameters $\frac{1}{\mathbf{w}_{\mathrm{F}}^{\mathrm{H}} \mathbf{C}_{\mathrm{h,FF}} \mathbf{w}_{\mathrm{F}}}$, $\frac{1}{\mathbf{w}_{\mathrm{F}}^{\mathrm{H}} \mathbf{C}_{\mathrm{h,MF}} \mathbf{w}_{\mathrm{F}}}$, and $\frac{1}{\mathbf{w}_{\mathrm{M}}^{\mathrm{H}} \mathbf{C}_{\mathrm{h,FM}} \mathbf{w}_{\mathrm{M}}}$, respectively. Now problem (4.255) is rewritten as

$$\min_{\mathbf{w}_{\mathrm{F}} \in \mathbb{C}^{N_F}} \quad \|\mathbf{w}_{\mathrm{F}}\|^2, \tag{4.271a}$$

$$\text{subject to} \begin{cases} \exp\left(\frac{-\gamma_F \sigma_F^2}{\mathbf{w}_F^H \mathbf{C}_{h,FF} \mathbf{w}_F}\right) \frac{\mathbf{w}_F^H \mathbf{C}_{h,FF} \mathbf{w}_F}{\mathbf{w}_F^H \mathbf{C}_{h,FF} \mathbf{w}_F + \gamma_F \mathbf{w}_M^H \mathbf{C}_{h,FM} \mathbf{w}_M} \geq 1 - \rho_F, & (4.271b) \\ \mathbf{w}_F^H \mathbf{C}_{h,MF} \mathbf{w}_F \leq \frac{\varepsilon_M}{\ln \frac{1}{\rho_M}}. & (4.271c) \end{cases}$$

Due to the nonconvexity of constraint (4.271b), problem (4.271) remains nonconvex. Using the SDR method, this problem is converted into a convex one. In doing so, a positive semidefinite matrix $\mathbf{W}_F = \mathbf{w}_F \mathbf{w}_F^H$ is defined, and the rank 1 constraint is relaxed. Consequently, problem (4.271) is relaxed to

$$\min_{\mathbf{W}_F \in \mathbb{H}_{++}^{N_F}} \quad \text{tr}(\mathbf{W}_F), \tag{4.272a}$$

$$\text{subject to} \begin{cases} \text{tr}(\mathbf{C}_{h,FF} \mathbf{W}_F) \exp\left(\frac{\gamma_F \sigma_F^2}{\text{tr}(\mathbf{C}_{h,FF} \mathbf{W}_F)}\right) & (4.272b) \\ + \gamma_F \text{tr}(\mathbf{C}_{h,FM} \mathbf{W}_M) \exp\left(\frac{\gamma_F \sigma_F^2}{\text{tr}(\mathbf{C}_{h,FF} \mathbf{W}_F)}\right) \leq \frac{\text{tr}(\mathbf{C}_{h,FF} \mathbf{W}_F)}{1 - \rho_F}, \\ \text{tr}(\mathbf{C}_{h,MF} \mathbf{W}_F) \leq \frac{\varepsilon_M}{\ln \frac{1}{\rho_M}}. & (4.272c) \end{cases}$$

The relaxed problem (4.272) is convex and can be efficiently solved using CVX or SeDuMi. A special case in which $\mathbf{h}_{FF}$ and $\mathbf{h}_{MF}$ are i.i.d. variables is studied in [27], where the channel covariance matrices are $\mathbf{C}_{h,FF} = \sigma_{h,FF}^2 \mathbf{I}_{N_F}$ and $\mathbf{C}_{h,MF} = \sigma_{h,MF}^2 \mathbf{I}_{N_F}$. Problem (4.271) is simplified to

$$\min_{\iota_F \geq 0} \quad \iota_F, \tag{4.273a}$$

$$\text{subject to} \begin{cases} \exp\left(\frac{-\gamma_F \sigma_F^2}{\sigma_{h,FF}^2 \iota_F}\right) \frac{\sigma_{h,FF}^2 \iota_F}{\sigma_{h,FF}^2 \iota_F + \gamma_F \mathbf{w}_M^H \mathbf{C}_{h,MF} \mathbf{w}_M} \geq 1 - \rho_F, & (4.273b) \\ \sigma_{h,MF}^2 \iota_F \leq \frac{\varepsilon_M}{\ln \frac{1}{\rho_M}}, & (4.273c) \end{cases}$$

where $\iota_F = \|\mathbf{w}_F\|^2$. The optimal solution to (4.273) can obtained using the simple and computationally efficient bisection method [1].

Table 4.6 summarizes the aforementioned works on robust resource allocation with no CSI.

Table 4.6 Summary of existing works on robust resource allocation: no CSI feedback

Reference	BT-BTI	DCA	SCA	LR	SDR	BT-Cauchy–Schwarz inequality	EF
[27]	–	–	–	–	√	–	–
[28]	√	√	√	–	–	–	–

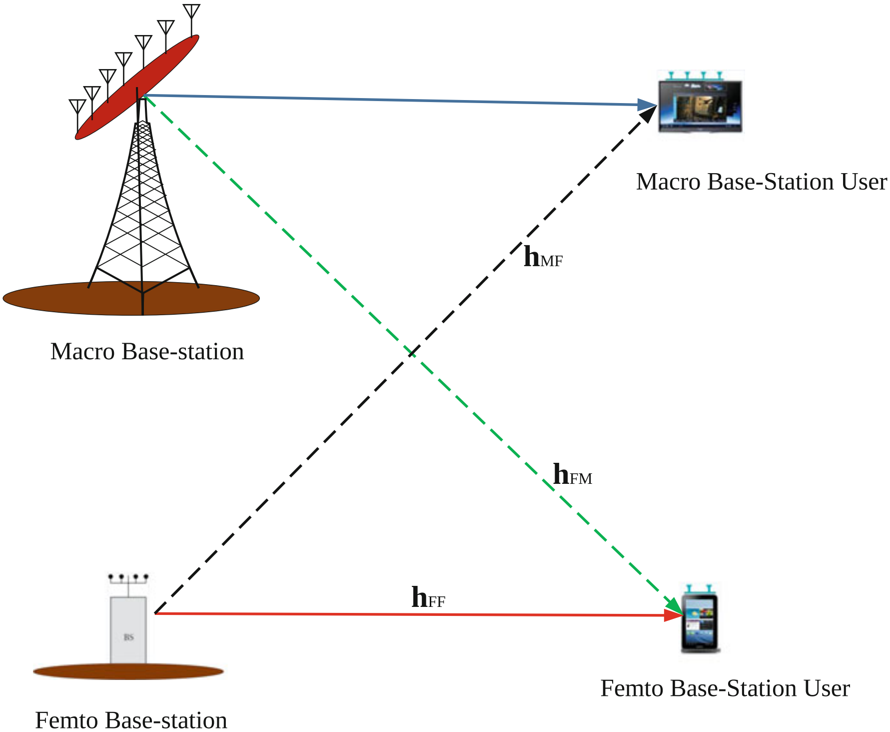

Fig. 4.3 System model for partial CSI feedback with stochastic uncertainty and for no CSI feedback

4.3.3.1 Numerical Example

Here we present the numerical results on the transmit power of the proposed system in [27], discussed in Section 4.3.2.3 in this chapter, as an example of partial CSI feedback with stochastic uncertainty and as an example of no CSI feedback. The system model is shown in Fig. 4.3.

The macro base station and closed-access femto base station are equipped with four transmit antennas, that is, $N_{\mathrm{M}} = N_{\mathrm{F}} = 4$. The maximum allowable outage probability for the SINR and for the interference are 0.1, that is, $\rho_{\mathrm{M}} = \rho_{\mathrm{F}} = 0.1$. The additive noise power at the femto-user is $\sigma_{\mathrm{F}}^2 = 0.01$, and the maximum tolerable interference caused by the femto base station on the macro-user is $\varepsilon_M = -3\,\mathrm{dB}$. We uniformly generate the macro base station beamforming vector on the unit-norm sphere $\|\mathbf{w}_{\mathrm{M}}\| = 1$. In simulations, the SDR-based problems are solved using CVX, and if the solutions are not of rank 1, the Gaussian randomization method (explained in Section 4.2.1.5 in this chapter) is used. We perform Monte Carlo simulations consisting of 500 channel realizations.

The transmit power is also investigated for the spatially correlated channels. Let $\mathbf{C}_{\mathrm{h,FF}} = \sigma_{\mathrm{h,FF}}^2\mathbf{C}_{\mathrm{F}}$, $\mathbf{C}_{\mathrm{h,MF}} = \sigma_{\mathrm{h,MF}}^2\mathbf{C}_{\mathrm{h}}$, and $\mathbf{C}_{\mathrm{h,FM}} = \sigma_{\mathrm{h,FM}}^2\mathbf{C}_{\mathrm{h}}$,

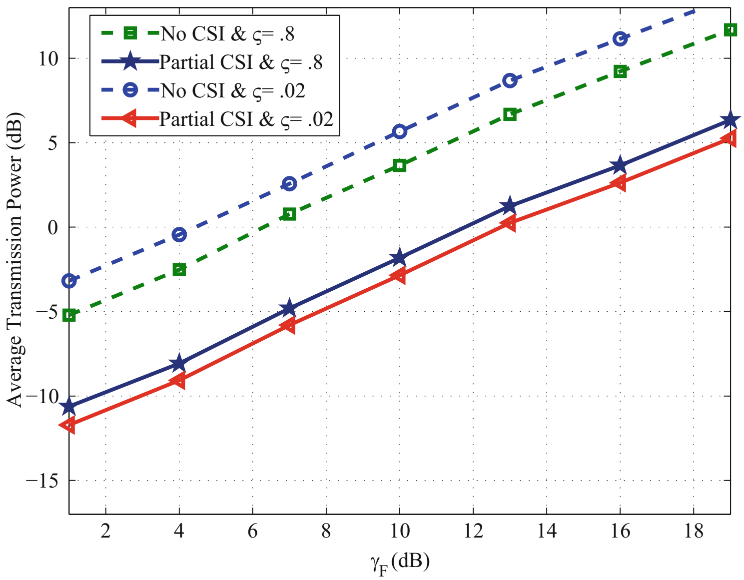

Fig. 4.4 Average transmit power versus minimum required SINR of femto-user for the two cases, that is, partial CSI feedback and no CSI feedback for $\varsigma = 0.02,\ 0.8$

$$[\mathbf{C}_{\mathrm{h}}]_{m,n} = \varsigma^{|m-n|}, \tag{4.274}$$

where $\sigma^2_{\mathrm{h,FF}} = 1$ and $\sigma^2_{\mathrm{h,FM}} = \sigma^2_{\mathrm{h,MF}} = 0.01$.

In Fig. 4.4, the average transmit power versus the minimum allowable SINR of the femto-user for the two cases, that is, partial CSI feedback and no CSI feedback for $\varsigma = 0.02, 0.8$, is shown. As can be seen, the difference in the transmit power of the femto base station in the two cases for $\varsigma = 0.02$ is around 7 dB, showing that the partial feedback CSI with stochastic uncertainty is significantly more power efficient than the no CSI case. Moreover, we observe that the difference in the transmit power is higher when channels are spatially more uncorrelated (i.e., smaller ς).

4.4 Concluding Remarks

In this chapter, we presented a taxonomy of relaxation methods for solving nonconvex and intractable robust optimization problems that can be utilized for allocating resources in many emerging wireless networks. Using these methods, the intractable and NP-hard robust optimization problems can be approximately or equivalently reformulated into tractable robust optimization problems that can be easily solved. Utilizing the methods discussed in this chapter, although globally optimal solutions may not be obtained, in general, locally optimal or near-optimal solutions can be obtained via efficient and tractable algorithms. In addition, we presented and reviewed some recent and noteworthy cases in the literature on using relaxation methods in robust optimization problems in communication systems. In particular, we covered important topics in future wireless networks, namely, beamforming, cooperative relaying, CRNs, and physical-layer secure communications. The examples were categorized into (1) partial CSI feedback with bounded uncertainty, (2) partial CSI feedback with stochastic uncertainty, and (3) no CSI feedback.

Appendices

Appendix 1: Proof of Corollary 4.1

Let $\hat{\mathbf{W}}$ be the optimal solution to problem (4.136), which is unique and of rank 1 by Proposition 4.1. By Lemma 4.3, $\hat{\mathbf{W}}$ is feasible for problem (4.128), which implies that $\mathrm{tr}(\hat{\mathbf{W}}) \geq \mathrm{tr}(\mathbf{W}^*)$, where $\mathbf{W}^*$ is the optimal solution to (4.128). Since we also have [cf., (4.136)] $\mathrm{tr}(\hat{\mathbf{W}}) \leq \mathrm{tr}(\mathbf{W}^*)$, we conclude that $\mathrm{tr}(\hat{\mathbf{W}}) = \mathrm{tr}(\mathbf{W}^*)$, that is, $\hat{\mathbf{W}}$ is the optimal solution to problem (4.128). From Lemma 4.3, $\mathbf{W}^*$ is a feasible solution to problem (4.136). Now, $\mathrm{tr}(\hat{\mathbf{W}}) = \mathrm{tr}(\mathbf{W}^*)$ further implies that $\mathbf{W}^*$ is the optimal solution to problem (4.136) as well. Hence, from Proposition 4.1, $\mathbf{W}^*$ should be unique and of rank 1.

Appendix 2: Tightness of Solution to Relaxed Problem

Consider the dual of (4.139) as

$$\min_{\mathbf{W}\succeq\mathbf{0}} \quad \mathrm{tr}(\mathbf{W}) \tag{4.275a}$$

$$\text{subject to} \quad \frac{\max_{\mathbf{G}_e\in\mathscr{R}_{\mathbf{G}_e}} \; 1+\mathrm{tr}(\mathbf{G}_e^{\mathrm{H}}\mathbf{W}\mathbf{G}_e)}{\min_{\mathbf{h}\in\mathscr{R}_{\mathbf{h}}} \; 1+\mathbf{h}^{\mathrm{H}}\mathbf{W}\mathbf{h}} \leq \gamma_{\mathrm{relax}}^*, \qquad \forall e=1,\ldots,E, \tag{4.275b}$$

where $R = \log(\frac{1}{\gamma_{\mathrm{relax}}^*})$. Since problem (4.275) is similar to problem (4.130), from Proposition 4.1, the solution to (4.139) is of rank 1 and unique. If we prove that the optimal solution to (4.275) is also the optimal solution to (4.139), we can infer the tightness of the relaxed problem. From Corollary 4.1, the solution to the relaxed problem is also the solution to the original problem. Hence, the optimal solution to the nonconvex problem (4.137) can be obtained by solving the equivalent convex SDP problem (4.140). For more details, the interested reader is referred to Section 4 in [5].

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Chapter 5
Conclusions and Future Research

In this final chapter, we discuss the potential applications of the material presented in Chapters 1, 2, 3, and 4 to future generations of wireless networks, including fifth generation and beyond. In doing so, our focus is not on open problems in using optimization theory or in developing distributed algorithms for solving such problems; these topics are discussed in [1–9]. Rather, we focus on new trends in robust resource allocation in future wireless networks. We begin by presenting major and important features in future generations of wireless networks that affect resource allocation, then we identify important problems in robust resource allocation in such networks that can be tackled using the material in Chapters 1, 2, 3, and 4.

5.1 Future Wireless Networks

Future wireless networks, including the forthcoming fifth generation, will be increasingly characterized by the following features, as outlined in [10–16]:

- Exponential increase in the number of wireless devices, evolving into connected everything, that is, massive connectivity requiring massive capacity
- Significantly smaller latency (<1 ms)
- Significantly higher data rates per user by way of
  - Carrier aggregation/carrier bonding
  - Cognitive small cells
  - Device-to-device communication
  - Multiple input multiple output (MIMO), including massive MIMO and coordinated multipoint (CoMP)/network MIMO
  - Millimeter-wave communication
  - Hybrid links [radio frequency (RF)/free space optics (FSO)]

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- Massive densification of base stations requiring significantly reduced transmit power levels
- Extensive use of relays and ad hoc networking in locations lacking backhaul access
- Mobile base stations
- Cloud radio access network
- New multiple access techniques, such as
  - Nonorthogonal multiple access (NOMA)
  - Sparse code multiple access (SCMA)
- Energy harvesting as a means of significantly prolonging battery life
- Diverse new applications and services leading to smart societies

Some of these attributes already exist at different stages of implementation, whereas others will be developed and deployed subsequently. In what follows, we provide brief explanations of some important aspects of future wireless networks that will affect resource allocation in such networks.

- **Massive MIMO:** Multiple-input multiple-output is a technique that is used to exploit multipath propagation via multiple antennas to increase radio link capacity, improve energy efficiency, and enhance link security. Exploiting multipath propagation is particularly useful for reducing fast fading and for introducing semi-omnidirectional propagation for narrow-beamwidth millimeter waves. Specifically, MIMO is a method for transmitting and receiving more than one datum simultaneously in the same radio channel and is different from smart antenna techniques that improve the performance of sending and receiving a single datum by way of beamforming and diversity. Single-user MIMO has been developed and implemented in some standards, such as 802.11n, and multiuser MIMO (also known as network MIMO or CoMP) is the focus of current research. In MIMO systems, increasing the number of antennas improves performance but also increases complexity. When the number of antenna elements in MIMO systems is very large, it is called massive MIMO. With this technique, it has been shown that there is a wide gap in the performance of existing schemes when exact channel state information (CSI) is assumed versus when the same schemes are deployed with uncertain CSI [17]. This fact indicates that there is a need to introduce robustness in massive MIMO systems. The problem is further complicated in CoMP due to the fact that there is a need for coordination between multiple and possibly heterogeneous access points, resulting in considerable increase in the number of CSI values and their respective constraints, each with its own uncertainties, in the resource allocation problem.
- **Millimeter Waves:** In future wireless networks, users can expect a substantial increase in their link speeds, which will require proportional increases in the frequency spectrum allocated to the service. However, the scarcity of available spectrum in frequency bands below 4 GHz makes the use of higher bands unavoidable. However, significant attenuation and narrow beamwidths of electromagnetic waves at higher frequency bands pose serious technical challenges for their use. Moreover, CSI fluctuates significantly in higher frequency bands.

Hence, in future wireless networks that use millimeter waves, there will be a need to develop robust and efficient techniques for resource allocation with acceptable computational complexity.

- **Hybrid Links (RF/FSO):** The need for higher data rates and massive capacity in future wireless networks requires the use of new high-speed channels. In situations where the available radio frequency spectrum is insufficient and connections via fiber optic links is not possible due to physical limitations, FSO is a viable candidate to enhance the available capacity. However, FSO is highly fluctuating, depending on many atmospheric and weather conditions, and suffers from unpredictable uncertainties.
- **Multiple Access Techniques:** Existing multiple access techniques, for example, orthogonal frequency division multiple access (OFDMA) and single-carrier frequency division multiple access (SC-FDMA), are relatively simple methods because of their exclusive allocation of channels for intracell communication, but their spectral efficiency is relatively low. To improve on the spectral efficiency of intracell links in future wireless networks, other multiple access techniques that are more complex, such as NOMA and SCMA are being considered, in which resources are shared (i.e., not allocated exclusively) by users. In NOMA, for example, one must use superposition coding on the transmitter side or successive interference cancellation on the receiver side, which are additional blocks in the system. This introduces new constraints to manage intracell interference and complicates the resource allocation problem even further. Hence, there is a need for sensitivity analysis of the respective problems with a view to studying the impact of uncertainty and robustness.
- **Multiple Radio Access:** Future wireless networks will have to utilize all existing and future spectrum allocations, which will include those for Wi-Fi, current cellular bands, and higher-frequency spectrum. This means that end-user devices will be expected to connect to different types of access points, each with its own radio-access technology. This will introduce new complexities and constraints requiring the development of novel robust schemes to deal with the attending uncertainties.
- **Air Interface:** The air interface for existing high-speed links in LTE-based networks consumes significant amounts of power. In future wireless networks, which will be highly populated with new connected things, there will be a serious need to reexamine current air interfaces and to devise new waveforms that will be more energy efficient. If this is done, the impact of the new waveforms on resource allocation schemes must be carefully examined.
- **Battery Life:** Finally, a very important requirement for future wireless networks will be to devise novel techniques to extend battery life. This will be of the utmost importance in the Internet of everything/Internet of things environments as well as in new end-user devices equipped with larger screens. In order to extend the battery life, a number of new concepts, such as energy harvesting, have been proposed. However, such schemes suffer from undesirable fluctuations and uncertainty in the power source that must be countered by devising new robust schemes for efficient power control that ensure steady and reliable communications.

5.2 Future of Resource Allocation

As stated in Chapter 1, the objective of resource allocation in general and power control in particular has been to improve goodputs, for example, the total throughput, revenue, and fairness, and to reduce badputs, for example, the transmit power, consumed energy, and cost, while maintaining the required quality of service (QoS) subject to certain constraints emanating from regulations, hardware and software limitations, and other pertinent restrictions [18]. Note, however, that in future wireless networks, the transmit power levels are expected to be reduced thanks to smaller cells, which will result in reduced interference. In addition, millimeter waves, which have been considered in general to have highly directional, pencil-thin beams that also prevent interference, will see more widespread use [19], with the possibility of more frequency reuse. So the important question is whether the current approaches and objectives for resource allocation in existing wireless networks as stated earlier will continue to be relevant and justified in future wireless networks. In what follows, we will present evidence and arguments demonstrating that resource allocation and interference management will remain important subjects in connection with wireless networks and that devising robust schemes will remain a formidable challenge with greater uncertainty and more practical constraints.

In [20], based on empirical studies and measurements, evidence was presented showing that in densely populated indoor and outdoor environments, there are multipath components in millimeter waves. In general, multi-path components can be used to improve the signal-to-noise ratio but at the same time reduces the benefit of small beamwidths that otherwise could have been considered advantageous in decreasing interference from other sources and users. Hence, one could argue that, although millimeter waves in principle have narrow beams, the existence of multipath components brings about an omnidirectional channel model [21], resulting in interference that needs to be managed and controlled.

One should also recognize that the use of millimeter waves in future wireless networks will not mean that the existing frequency spectrum below 4 GHz will be abandoned. On the contrary, the use of existing bands will be vital in providing coverage over longer distances and in reaching those users that are inside buildings by base stations located outside. The deployment of carrier aggregation in LTE-Advanced needs to be extended to include millimeter waves as well, resulting in complications that will arise because of the differences in the nature of propagation, absorption, and reflection of electromagnetic waves in different and widely dispersed frequency bands.

Moreover, the highly dynamic and time-varying nature of millimeter-wave propagation and channel models, together with the need to reduce the decision-making and processing time, will require novel approaches to resource allocation in future wireless networks that will be simple, efficient, and robust against fast variations in channel state information or other parameter values with a view to providing end users with consistency in their required quality of experience and QoS. In addition, the use of new and advanced techniques in the physical layer, such as massive MIMO, NOMA, and SCMA, in such networks will introduce more complexities in devices and nodes.

Future wireless networks will be increasingly software-based, that is, the use of software-defined radio (SDR) [22], software-defined networks (SDN) [23–29], and network function virtualization (NFV) [30–32] will be on the rise in implementing and deploying such networks. The use of SDR requires a distributed approach to resource allocation, whereas SDN entails centralized schemes. Irrespective of whether SDR, SDN, or even a hybrid approach is used, the techniques in Chapters 1, 2, 3, and 4 will still be applicable, but with additional complexities and uncertainties.

Use of such concepts will introduce new sets of constraints in formulating the resource allocation problem. As discussed earlier in Chapters 1, 2, 3, and 4, resource allocation problems in general are nonconvex and NP-hard, and to achieve the aforementioned objectives, a promising area of research seems to be in the development of robust resource allocation schemes with partial CSI or no CSI feedback. However, such schemes should be kept simple (i.e., low computational complexity), with significant improvements made in their performance compared to existing schemes, with acceptable overhead and message passing, and with relative confidence in their convergence. That said, note that, as is the case with existing and advanced wireless networks, practical realities may be different from requirements [33]. To reduce the complexity of robust resource allocation problems in future wireless networks, sensitivity analysis, as introduced in Chapters 2, and 3, should be performed to examine the impact of different constraints on robust solutions. In this way, only those constraints that have a major impact on the outcome can be retained and considered in the problem.

5.3 Concluding Remarks

The past two decades have witnessed a surge in research on various aspects of resource allocation problems, from theoretical issues, such as converting highly nonconvex and NP-hard optimization problems into convex ones, to implementation aspects, such as developing distributed algorithms. In the above mentioned research, an important practical issue has been to consider uncertainty in parameter values in resource allocation problems for wireless networks.

Traditionally, in applying robust optimization theory to resource allocation in wireless networks, the impact of uncertain parameters on resource allocation problems has been captured using two approaches: the *worst-case* approach, in which error is considered to be bounded and performance is guaranteed for the worst-case condition; and the *Bayesian approach*, in which performance is guaranteed stochastically without consideration of any bound on uncertainty. There is also a middle way that entails combining the worst-case and Bayesian approaches while overcoming their difficulties. Depending on the system model and information regarding errors, each of these approaches can be applied to resource allocation problems in wireless networks.

Our aim in this book was to provide a systematic approach to robust optimization, where different aspects of introducing robustness are considered and the associated costs, including the computational complexity, performance reduction, additional message passing, and equilibrium analysis, are mitigated. In each chapter we presented the state of the art of knowledge and reviewed interesting, promising, novel, and useful examples of recent works in the given context.

In Chapters 2 and 3, we focused on distributed robust resource allocation problems using the decomposition approach and game theory, respectively. In addition, in Chapter 2, we showed how the concept of protection functions could be applied to convert a nonconvex problem into a tractable and convex problem. In Chapter 3, we explained how to utilize some new mathematical concepts, such as variational inequalities and sensitivity analysis, to study the equilibrium point in distributed algorithms via game theory.

Chapter 4 dealt with nonconvex and NP-hard robust resource allocation problems. We presented a taxonomy of various existing approaches to tackling such problems and discussed their respective applications to derive tractable formulations. We also presented recent nonconventional approaches to robust optimization where only partial CSI or no CSI feedback was available.

In Chapter 5, we presented arguments that in future wireless networks, resource allocation problems will continue to dominate and that introducing robustness will become increasingly desirable in significantly more crowded environments where users are heterogeneous and applications and services are diverse. In addition, we offered our views as to how this research area will take shape.

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