Energy and Passivity Based Control of the Double Inverted Pendulum on a Cart

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Abstract— **The paper considers the design of a nonlinear controller for the double inverted pendulum (DIP), a system consisting of two inverted pendulums mounted on a cart. The swingup controller bringing the pendulums from any initial position to the unstable up-up position is designed based on passivity properties and energy shaping. While the swingup controller drives the DIP into a region of attraction around the unstable up-up position, the balance controller designed on the basis of the linearized model stabilizes the DIP at the unstable equilibrium. The simulation results show the effectiveness of the proposed nonlinear design method for the DIP system.**

Keywords— **Double inverted pendulum (DIP), passivity based control (PBC), nonlinear systems, energy based control.**

I. INTRODUCTION

The classical inverted pendulum (cart and pole system) has been widely used in control laboratories to demonstrate the effectiveness of control systems in analogy with the control of many real systems [1,2]. The double inverted pendulum (DIP) is an extension of the inverted pendulum system, it is suitable to investigate and verify different control methods for dynamic systems with higher-order nonlinearities. It is more difficult than the single inverted pendulum (SIP) because there are two linked pendulums on a cart and we should consider to bring both of the pendulums from the unstable hanging position to the stable upright position by only moving the cart on the horizontal plane. The DIP on a cart is also different from the rotating double inverted pendulum often denoted as double inverted pendulum or double pendulum.

It is clear that both SIP and DIP are underactuated mechanical systems that have fewer control inputs than degrees of freedom. There are many similar systems like the DIP or other multiple inverted pendulums, such as Acrobot, Pendubot, three-link gymnast robot, etc. [3] Different from these systems, controlling of the DIP requires to consider not only the pendulums, but also the displacement of the cart and this will certainly increase the design complexity.

For controlling nonlinear underactuated mechanical systems, Spong [3,4], Spong and Praly [5] separate the problem into a swingup control and a balance control strategy. For the implementation of the first strategy, they use the concept of partial feedback linearization and passivity to design the swingup controller. In order to stabilize the pendulums at the desired position an optimal linear quadratic (LQR) or pole placement controller, based upon a linearized plant model, is frequently used. This method has been tested on many typical underactuated mechanical systems such as Acrobot, Pendubot, three-link mechanical Robot and inertia wheel pendulum [6].

Combining Lyapunov theory with passivity properties and energy shaping, the nonlinear controllers for some underactuated mechanical systems have been designed by Fantoni and Lozano applying this idea to SIP [7], Pedubot [8] and to the underactuated hovercraft [9]. In this method, Lyapunov theory takes an important role in controller design and system convergence analysis. The nonlinear underactuated systems often contain feedforward nonlinearities, unstable zero dynamics, and other structural properties that often make it difficult to apply some recent design methodologies, such as complete feedback linearization, backstepping or forwarding [10]. Praly has successfully applied the forwarding technique to swing up the SIP[12] and the spherical inverted pendulum [13] resulting in rather complicated controllers.

Till now, we have not found any application of the above-mentioned technique to the DIP system. The DIP also belongs to the class of underactuated mechanical systems consisting of three interconnected systems (two pendulums, one cart) with only one actuator to move the cart. It is different from all the plants we mentioned before as it has two passive generalized coordinates making it a real challenge for designing swingup controllers. In this paper, we accept the viewpoint of using a switching control strategy and concentrate on the design of a swingup and a balance controller separately. After analyzing the system dynamics and passivity properties, the swingup controller for the DIP system will be developed on the basis of partial feedback linearization and passivity based control (PBC) together with energy shaping. To design the balance controller we apply optimal control theory (LQR) to the linearized system without providing the details. The switching conditions from swingup to balance control were drawn from experiments and are not yet justified by a thorough stability analysis. The simulation results show the effectiveness of the proposed nonlinear controller.
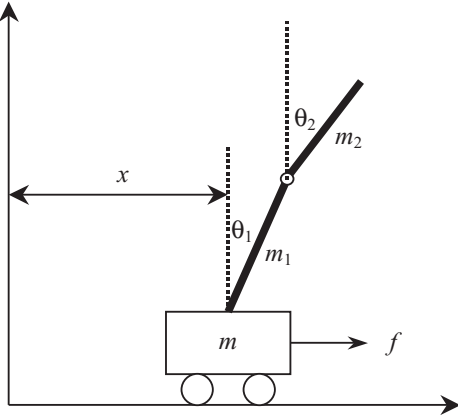
Fig. 1. Double inverted pendulum system

II. ANALYSIS OF SYSTEM DYNAMICS

The DIP system consists of two linked pendulums on a wheeled cart that can move linearly along a horizontal track and a force f to move the cart in order to balance the two linked inverted pendulums on the cart, i.e. to keep θ_1 and θ_2 to be zero, where θ_1 is the angle of the first pendulum from the vertical direction; θ_2 is the angle of the second pendulum from the vertical direction. Fig. 1 shows the illustration of the DIP system. Here, x refers to the position of the cart; m is the mass of the cart; m_1 represents the mass of the first pendulum; m_2 represents the mass of the second pendulum; l_1, l_2 denote the distance between the pivot and the center of mass of respective links. We assume that the masses of the pendulums and the cart are homogeneously distributed and concentrated in their centers of gravity and we neglect frictions. Usually damping in underactuated mechanical plants helps us to relax the conditions for stabilization [14]. In this paper we do not consider internal frictions, so the DIP system with no internal damping represents the worst case for the implementation of various control strategies.

The mathematical model of the DIP can be derived using the Euler-Lagrange equation or the Newtonian approach. Following the first approach involves to determine the kinetic and potential energies of the system's components in terms of generalized coordinates. The form of the Euler-Lagrangian equation used here is:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \frac{\partial L}{\partial q} = Q_q \qquad (1)$$

where:

$L = T - V$
T: kinetic energy
V: potential energy
Q_q: generalized forces not taken into account in T,V
q: generalized coordinates

For the holonomic DIP system, we select q as: $q =$

$[x,\ \theta_1,\ \theta_2]^T$. The system kinetic energy, T is:

$$T = T_{cart} + T_{pendulum1} + T_{pendulum2} = T_1 + T_2 + T_3 \quad (2)$$

where:

$$T_1 = \frac{1}{2}m\dot{x}^2$$

$$T_2 = \frac{1}{2}m_1[(\dot{x} + l_1\dot{\theta}_1\cos\theta_1)^2 + (l_1\dot{\theta}_1\sin\theta_1)^2] + \frac{1}{2}J_1\dot{\theta}_1^2$$

$$T_3 = \frac{1}{2}m_2[(\dot{x} + L_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2)^2 +$$

$$(L_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2)^2] + \frac{1}{2}J_2\dot{\theta}_2^2$$

J_1, J_2 are inertias of the first and second link with respect to the center of mass, L_1 is the length of the first pendulum, here $L_1 = 2l_1$. The potential energy V is given by:

$$\begin{aligned} V &= V_{cart} + V_{pendulum1} + V_{pendulum2} \\ &= 0 + m_1gl_1\cos\theta_1 + m_2g(L_1\cos\theta_1 + l_2cos\theta_2) \end{aligned} \quad (3)$$

The Euler-Lagrangian equations for the DIP system result in:

$$h_1\ddot{x} + h_2\ddot{\theta}_1\cos\theta_1 + h_3\ddot{\theta}_2\cos\theta_2 = h_2\dot{\theta}_1^2\sin\theta_1 + h_3\dot{\theta}_2^2\sin\theta_2 + f$$

$$h_2\cos\theta_1\ddot{x} + h_4\ddot{\theta}_1 + h_5\cos(\theta_1 - \theta_2)\ddot{\theta}_2 = h_7\sin\theta_1 - h_5\dot{\theta}_2^2\sin(\theta_1 - \theta_2)$$

$$h_3\cos\theta_2\ddot{x} + h_5\cos(\theta_1 - \theta_2)\ddot{\theta}_1 + h_6\ddot{\theta}_2 = h_5\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + h_8\sin\theta_2$$

$$(4)$$

where we have used the following parameters to simplify the equations:

$$\begin{aligned} &h_1 = m + m_1 + m_2 & &h_5 = m_2l_2L_1 \\ &h_2 = m_1l_1 + m_2L_1 & &h_6 = m_2l_2^2 + J_2 \\ &h_3 = m_2l_2 & &h_7 = m_1l_1g + m_2L_1g \\ &h_4 = m_1l_1^2 + m_2L_1^2 + J_1 & &h_8 = m_2l_2g \end{aligned}$$

$$(5)$$

The Euler-Lagrange equations can be put into a frequently used compact form [15]:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = Q_q \qquad (6)$$

where:

$$M(q) = \begin{bmatrix} h_1 & h_2\cos\theta_1 & h_3\cos\theta_2 \\ h_2\cos\theta_1 & h_4 & h_5\cos(\theta_1 - \theta_2) \\ h_3\cos\theta_2 & h_5\cos(\theta_1 - \theta_2) & h_6 \end{bmatrix}$$

$$C(q,\dot{q}) = \begin{bmatrix} 0 & -h_2\dot{\theta}_1\sin\theta_1 & -h_3\dot{\theta}_2\sin\theta_2 \\ 0 & 0 & h_5\dot{\theta}_2\sin(\theta_1 - \theta_2) \\ 0 & -h_5\dot{\theta}_1\sin(\theta_1 - \theta_2) & 0 \end{bmatrix}$$

$$q = \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \end{bmatrix}, \quad g(q) = \begin{bmatrix} 0 \\ -h_7\sin\theta_1 \\ -h_8\sin\theta_2 \end{bmatrix}, \quad Q_q = \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}$$

It is clear that $M(q)$ is symmetric with a determinant given by:

$$\begin{aligned} \det(M(q)) = &h_1h_4h_6 - h_1h_5^2\cos^2(\theta_1 - \theta_2) - h_2^2h_6\cos^2\theta_1 + \\ &2h_2h_3h_5\cos\theta_1\cos\theta_2\cos(\theta_1 - \theta_2) - h_3^2h_4\cos^2\theta_2 \end{aligned}$$

$$(7)$$

To show that $\det(M) > 0$ we substitute expressions in (5) for (7) resulting in:

$$\det(M(q)) = \underbrace{h_1 h_5^2 \sin^2(\theta_1 - \theta_2) + h_2^2 h_6 \sin^2 \theta_1 + h_3^2 h_4 \sin^2 \theta_2 + H}_{H_2 > 0}$$
$$+ \underbrace{2h_2 h_3 h_5 (\cos \theta_1 \cos \theta_2 \cos(\theta_1 - \theta_2) - 1)}_{H_1} \tag{8}$$

where $H > 0$ represents a very complicated constant. It is easy to see that the first part of $\det(M)$, H_2, is larger than zero. Next we only consider the H_1 part, which is the uncertainty part of (8). After some algebra it can be put into the form

$$H_1 = h_2 h_3 h_5 (\cos^2 \theta_1 + \cos^2 \theta_2 - \sin^2(\theta_1 - \theta_2) - 2)$$
$$= h_2 h_3 h_5 (-\sin^2 \theta_1 - \sin^2 \theta_2 - \sin^2(\theta_1 - \theta_2)) \tag{9}$$

Comparing the H_2 part in (8) with H_1, we notice that $h_1 h_5^2 - h_2 h_3 h_5 > 0$, $h_2^2 h_6 - h_2 h_3 h_5 > 0$ and $(H + h_3^2 h_4) - h_2 h_3 h_5 > 0$ so we can draw the conclusion that $det(M(q)) > 0$ for all q.

The control objective is to swingup the pendulums from the stable hanging position to the unstable upright position while the cart displacement is brought to zero. There exist four equilibrium points of the DIP system, i.e., up-up, down-down, up-down, down-up, with state variables $[\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2]$ taking the values of $[0,0,0,0]$, $[\pi,0,\pi,0]$, $[0,0,\pi,0]$ and $[\pi,0,0,0]$. The total energy is different for each of the four equilibrium points:

Up-up position: $\quad E_{up-up} = h_7 + h_8$
Down-down position: $\quad E_{down-down} = -h_7 - h_8$
Up-down position: $\quad E_{up-down} = h_7 - h_8$
Down-up position: $\quad E_{down-up} = -h_7 + h_8$

In order to achieve the control objective, i.e. to stabilize the DIP in the up-up position, the following two conditions should be satisfied.

1) $x = 0; \dot{x} = 0$
2) $E = E_{up-up} = h_7 + h_8$

where the second condition refers to the DIP at the desired up-up position with zero displacement and zero cart velocity. According to equations (2), (3), (4) and (5) the explicit energy expression of the DIP, when applying the above conditions, is:

$$E = \frac{1}{2} h_4 \dot{\theta}_1^2 + \frac{1}{2} h_6 \dot{\theta}_2^2 + h_5 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + h_7 \cos \theta_1 + h_8 \cos \theta_2$$
$$= h_7 + h_8 \tag{10}$$

yielding

$$\frac{1}{2} h_4 \dot{\theta}_1^2 + \frac{1}{2} h_6 \dot{\theta}_2^2 + h_5 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) =$$
$$h_7(1 - \cos \theta_1) + h_8(1 - \cos \theta_2) \tag{11}$$

Equation (11) defines a particular manifold when we take the control objective to drive $x, \dot{x}$ and $(E - E_{up-up})$ to zero. As can be seen from (11), $[\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2] = [0,0,0,0]$ i.e. the unstable equilibrium is within the set of all solutions.

III. ENERGY AND PASSIVITY BASED CONTROL

A. Collocated partial feedback linearization

As the DIP system belongs to the class of underactuated mechanical systems, which have fewer control inputs than degrees of freedom, we first consider a general underactuated mechanical system with n generalized coordinates $q_1, \ldots, q_n$, and $m < n$ actuators. By partitioning the vector q, we get $q^T = (q_1^T, q_2^T)$, with q_1 corresponding to the passive and q_2 corresponding to the actuated variables. The Euler-Lagrange equations of the dynamics of an n-degree of freedom mechanical system with q_1 passive coordinates and q_2 actuated coordinates can be written in the following general form [5]:

$$M_{11} \ddot{q}_1 + M_{12} \ddot{q}_2 + C_1(q, \dot{q}) + g_1(q) = 0$$
$$M_{21} \ddot{q}_1 + M_{22} \ddot{q}_2 + C_2(q, \dot{q}) + g_2(q) = f \tag{12}$$

where:

$$M(q) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad C(q, \dot{q})\dot{q} = \begin{bmatrix} C_1(q, \dot{q}) \\ C_2(q, \dot{q}) \end{bmatrix}$$
$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad g(q) = \begin{bmatrix} g_1(q) \\ g_2(q) \end{bmatrix} \tag{13}$$

Here $M(q)$ is the symmetric, positive definite inertia matrix. The vector C includes Coriolis and centrifugal terms, while g contains the terms derived from the potential energy, such as gravitational and elastic generalized forces. The vector f represents the input of the generalized forces produced by the m actuators at q_2. For notational simplicity, we will henceforth not write the explicit dependency on q for all of these coefficients.

As a consequence of the positive definitness of the inertia matrix, an important property that holds for the entire class of underactuated mechanical systems is the so-called *collocated partial feedback linearization* property [3, 4, 5]. The collocated linearization refers to a control that linearizes the equations associated with the actuated degree of freedom q_2. Consider the first formula in (12),

$$M_{11} \ddot{q}_1 + M_{12} \ddot{q}_2 + C_1 + g_1 = 0 \tag{14}$$

As a consequence of the uniform positive definitness of the Matrix $M(q)$, the $l \times l$ matrix M_{11} with $l = n - m$ is invertible. Solving for $\ddot{q}_1$ yields

$$\ddot{q}_1 = -M_{11}^{-1}(M_{12} \ddot{q}_2 + C_1 + g_1) \tag{15}$$

When substituting (15) for the second formula in equation (12) we obtain

$$\overline{M}_{22} \ddot{q}_2 + \overline{C}_2 + \overline{g}_2 = f \tag{16}$$

where:

$$\overline{M}_{22} = M_{22} - M_{21} M_{11}^{-1} M_{12}$$
$$\overline{C}_2 = C_2 - M_{21} M_{11}^{-1} C_1$$
$$\overline{g}_2 = g_2 - M_{21} M_{11}^{-1} g_1 \tag{17}$$

A very simple test shows that the $m \times m$ matrix $\overline{M}_{22}$ is itself symmetric and positive definite. So a partial feedback linearizing controller can be defined according to equation (16) and (17) yielding

$$f = \overline{M}_{22} \cdot u + \overline{C}_2 + \overline{g}_2 \qquad (18)$$

where u was selected as a new control input. So the complete system can be rewritten as

$$M_{11}\ddot{q}_1 + C_1 + g_1 = -M_{12}u$$
$$\ddot{q}_2 = u \qquad (19)$$

Using the collocated linearization method, the original system (12) is feedback equivalent to the system (19). In a first design step, we apply feedback control u to the q_2-subsystem in (19)

$$u = -k_1 q_2 - k_2 \dot{q}_2 + k_3 \bar{u} \qquad (20)$$

rendering the subsystem asymptotically stable for $\bar{u} \equiv 0$. The remaining design problem is to choose the additional control $\bar{u}$.

B. Energy and passivity based control

The collocated linearization approach transfers the original system (12) into a simpler one (19) both in concept and in structure. For the reason of convenience, we rewrite system (19) with control (20) in a more general form

$$\dot{x} = Ax + Bu$$
$$\dot{\xi} = f(\xi) + g(x, \xi) \cdot u \qquad (21)$$

and we refer to the linear subsystem as x-subsystem. Accordingly the nonlinear subsystem in (21) is denoted as ξ-subsystem, where

$$\dot{\xi} = f(\xi) \qquad (22)$$

represents the zero dynamics. Denoting the total energy of the ξ-subsystem as E_ξ, the time derivative yields:

$$\dot{E}_\xi = \frac{\partial E_\xi}{\partial \xi}\dot{\xi} = \frac{\partial E_\xi}{\partial \xi}(f(\xi) + g(x, \xi) \cdot u) = L_f E_\xi + L_g E_\xi \cdot u \qquad (23)$$

As the zero dynamics (22) of the ξ-subsystem is conservative [15], i.e. the total energy E_ξ is constant, we have

$$L_f E_\xi = 0 \qquad (24)$$

Considering equation (23), we get

$$\dot{E}_\xi = L_g E_\xi \cdot u \qquad (25)$$

which implies that (25) defines a passive subsystem with respect to the input u and output y_ξ with

$$y_\xi = L_g E_\xi \qquad (26)$$

If selecting $u = -K_x \cdot x$ for the x-subsystem, we know that if the transfer function $K_x(sI - A)^{-1}B$ of the x-subsystem
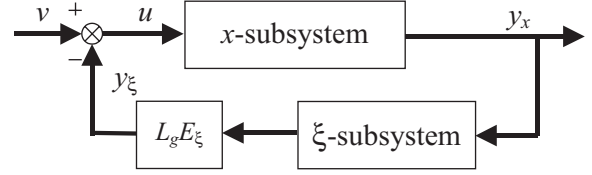

Fig. 2. Rendering the system (21) passive from v to y_x

is strictly positive real, the x-subsystem is passive with respect to the output $y_x = K_x \cdot x$. From the theorem of *feedback passivation design* in [11], we know, that the whole system from the new control input v to the output y_x can be rendered passive with the control $v = -y_x$. Hence global stability will be achieved by

$$u = -L_g E_\xi + v = -L_g E_\xi - K_x \cdot x \qquad (27)$$

Fig. 2 shows the illustration of the passivation design rendering the system passive from the control input v to the control output y_x.

According to equation (19), we rearrange the system equations (6) of the DIP. The actuated variables are given by $[x, \dot{x}]^T$ while the pendulum angles and angle velocities $[\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2]^T$ represent the passive or unactuated variables. In matrix form, the DIP equations read:

$$M_{11}\ddot{q}_1 + N_1 = -M_{12}u$$
$$\ddot{q}_2 = u \qquad (28)$$

where:

$$\ddot{q}_1 = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}, \quad \ddot{q}_2 = \ddot{x}$$

$$M_{11} = \begin{bmatrix} h_4 & h_5\cos(\theta_1 - \theta_2) \\ h_5\cos(\theta_1 - \theta_2) & h_6 \end{bmatrix}$$

$$N_1 = C_1 + g_1 = \begin{bmatrix} h_5\dot{\theta}_2^2\sin(\theta_1 - \theta_2) - h_7\sin\theta_1 \\ -h_5\dot{\theta}_1^2\sin(\theta_1 - \theta_2) - h_8\sin\theta_2 \end{bmatrix}$$

$$M_{12} = M_{21}^T = \begin{bmatrix} h_2\cos\theta_1 \\ h_3\cos\theta_2 \end{bmatrix}, \quad M_{22} = h_1$$

In equation (28) the first subsystem is represented by the pendulums, while the second subsystem refers to the cart. According to equations (16) and (17), we can define the partial feedback linearizing controller u as:

$$f = \overline{M}_{22} \cdot u + \overline{N}_2 \qquad (29)$$

where:

$$\overline{M}_{22} = M_{22} - M_{21}M_{11}^{-1}M_{12}$$
$$\overline{N}_2 = N_2 - M_{21}M_{11}^{-1}N_1$$
$$N_2 = C_2 + g_2 = \begin{bmatrix} -h_2\dot{\theta}_1^2\sin\theta_1 - h_3\dot{\theta}_2^2\sin\theta_2 \end{bmatrix}$$

Introducing

$$u = -k_1 x_2 - k_2 \dot{x}_2 + k_3 \bar{u} \qquad (30)$$

with

$$\bar{u} = -L_g E_p \tag{31}$$

where

$$E_p = \frac{1}{2}h_4\dot{\theta}_1^2 + \frac{1}{2}h_6\dot{\theta}_2^2 + h_5\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) + h_7\cos\theta_1 + h_8\cos\theta_2 \tag{32}$$

denotes the total energy of the pendulums, finally results in a feedback controller

$$u = -k_1 x - k_2 \dot{x} - k_3 L_g E_p \tag{33}$$

To evaluate the formulas we rearrange the equations of the pendulum subsystem and form the ξ-subsystem by introducing

$$\xi = \begin{bmatrix} \theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2 \end{bmatrix}^T$$

Using eq. (28)

$$\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = -M_{11}^{-1}N_1 - M_{11}^{-1}M_{12} \cdot u \tag{34}$$

we finally get

$$f(\xi) = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ -M_{11}^{-1}N_1 \end{bmatrix}, \quad g(\xi) = \begin{bmatrix} 0 \\ 0 \\ -M_{11}^{-1}M_{12} \end{bmatrix} \tag{35}$$

and we derive

$$\begin{aligned} \bar{u} = -L_g E_p &= -\left[\frac{\partial E_p}{\partial \theta_1} \frac{\partial E_p}{\partial \theta_2} \frac{\partial E_p}{\partial \dot{\theta}_1} \frac{\partial E_p}{\partial \dot{\theta}_2} \right] \cdot g(\xi) \\ &= h_2\dot{\theta}_1\cos\theta_1 + h_3\dot{\theta}_2\cos\theta_2 \end{aligned} \tag{36}$$

Combing passivity design with energy shaping applied to the pendulum subsystem, we introduce

$$k_3 \cdot \bar{u} = \widetilde{k}_3 \cdot \widetilde{E} \cdot \bar{u}$$

where $\widetilde{E} = E_p - E_{up-up}$. For $\widetilde{E} = 0$ the total energy of the pendulum subsystem is shaped to the energy of the up-up position. Now the swingup controller for the DIP system reads

$$u = -k_1 x - k_2 \dot{x} + \widetilde{k}_3 \cdot \widetilde{E} \cdot \underbrace{(h_2\dot{\theta}_1\cos\theta_1 + h_3\dot{\theta}_2\cos\theta_2)}_{\bar{u}} \tag{37}$$

From the discussion in section II we conclude that the first two terms in (37) ensure the cart position and velocity to converge to zero, whereas the third term forces the total energy E_p of the pendulum subsystem to converge to E_{up-up} while $[\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2]$ converges to the manifold defined in (11).
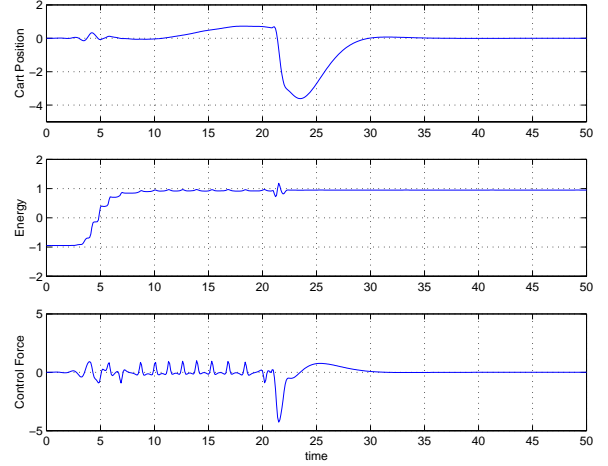

Fig. 3. Responses of cart displacement, total energy and the control force

IV. SIMULATION RESULTS

In order to test the control strategy, we carried out simulations using MATLAB and SIMULINK. By using the controller (37) based on passivity together with energy shaping, the DIP system can achieve closed loop stability. However, the achieved stability is not asymptotic to a fixed point, but only to a manifold, which has been shown in (11). For this reason, the control strategy has to switch from swingup control to balance control, with a LQR balance controller achieving local asymptotic stability at the desired equilibrium. Since the DIP system is linearly controllable in a neighborhood of the up-up position, we can only design a nonlinear swingup controller intersecting the trajectory in the neighborhood of the desired equilibrium.

The difficulties in applying such a control strategy are mainly at the supervisory level, i.e. to set up conditions when to switch the controllers. After doing many experiments, we determined the following conditions to be met before switching from swingup to balance control.

$$\begin{aligned} 0 \leq |\theta_1| &\leq 0.2 \text{ [rad]} \\ 0 \leq |\theta_2| &\leq 0.2 \text{ [rad]} \\ 0 \leq |\dot{\theta}_1| &\leq 0.5 \text{ [rad/sec]} \\ 0 \leq |\dot{\theta}_2| &\leq 0.5 \text{ [rad/sec]} \end{aligned} \tag{38}$$

Fig. 3 shows the responses of cart displacement, total energy (32) and control force f (6). The pendulum angles and velocities are depicted in Fig. 4. We can see that the switching point is at about $t = 21s$. Note that before switching to balance control the energy is increasing from $E_{down-down}$ to E_{up-up}. Fig. 3 and 4 only show the transient responses for bringing the pendulums from the down-down position to the desired up-up position. In order to test the switching strategy, we have done lots of experiments using different initial conditions, i.e. the initial condition for the angular values $[\theta_1, \theta_2]$ were varied from $[0.1, 0.1]$ up to $[\pi, \pi]$ in steps given by
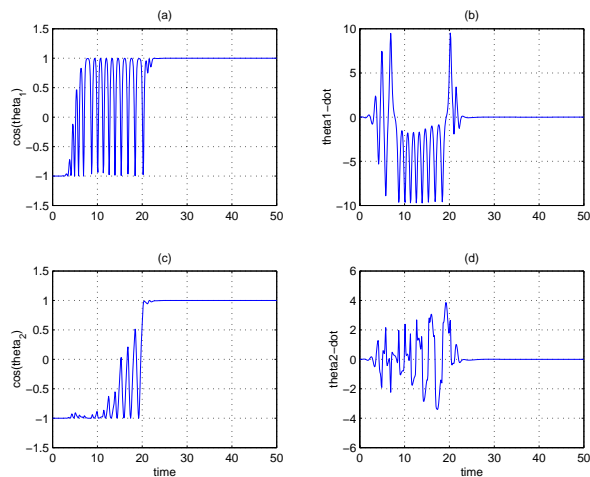
Fig. 4. Simulations of pendulum angles and pendulum velocities
(a) $\cos\theta_1$; (b) $\dot{\theta}_1$; (c) $\cos\theta_2$; (d) $\dot{\theta}_2$

$[\triangle\theta_1, \triangle\theta_2] = [0.2, 0.2]$. By properly adjusting the parameters $\{k_1, k_2, \widetilde{k}_3\}$, satisfying control responses can be achieved for all of the initial conditions mentioned above.

V. Conclusions

On the basis of a switching control strategy, a swingup controller based on passivity and energy shaping was designed. The simulations show that this swingup controller can bring the pendulums to the basin defined by (38) from any initial position, i.e. the initial angels of the pendulums can be chosen arbitrarily. As the balance controller can stabilize the pendulums at the up-up position inside the basin, we achieve global stabilization of the DIP system.

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